CAMBRIDGE

Sixth Term Examination Paper (STEP)

Mathematics 3 (9475)

2025

Examiners' report and mark scheme

STEP Mathematics 3 examiners' report

Paper 3 overview

The majority of candidates focused solely on the pure questions, with questions 1, 2 and 8 the most popular. The statistics questions were more popular than the mechanics questions in this exam series.

Candidates who did well on this paper generally:	**Candidates who did less well on this paper generally:**
• were careful to explain and justify the steps in their arguments, explaining what they had done rather than expecting the examiner to infer what had been done from disjointed groups of calculations • paid close attention to what was required by the questions • made fewer unnecessary mistakes with calculations • thought carefully about how to present rigorous arguments involving trig functions and their inverse functions, especially in relation to domain considerations • understood that questions set on the STEP papers require sufficient justification to earn full credit • knew the difference between 'positive' and 'non-negative' • attempted all parts of a question, picking up marks for later parts even when they had not necessarily attempted or completed previous parts.	• did not pay attention to 'Hence' instructions: this means that you must use the previous part • presented explanations that were not precise enough (e.g. in Question 3 describing the transformations but not in the context of the graphs or in Question 8 not explaining use of trigonometric relationships sufficiently well) • made additional assumptions, e.g. that a function was differentiable when this had not been given • tried to present if and only if arguments in a single argument when dealing with each direction separately would have been more appropriate and safer (note that this is not always the case; in general candidates need to consider what is the most appropriate presentation of an if and only if argument) • tried to carry out too many steps in one go, resulting in them not justifying the key steps sufficiently • did not take sufficient care with graphs/curve sketching.

Section A: Pure Mathematics overview

Most candidates focused on the pure section, but generally the more successful candidates selected questions to answer rather than attempting each question in the order given on the paper.

Question 1

1 *You need not consider the convergence of the improper integrals in this question.*

For $p, q > 0$, define

$$\mathrm{b}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x.$$

(i) Show that $\mathrm{b}(p,q) = \mathrm{b}(q,p)$.

(ii) Show that $\mathrm{b}(p+1,q) = \mathrm{b}(p,q) - \mathrm{b}(p,q+1)$ and hence that $\mathrm{b}(p+1,p) = \frac{1}{2}\,\mathrm{b}(p,p)$.

(iii) Show that

$$\mathrm{b}(p,q) = 2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

Hence show that $\mathrm{b}(p,p) = \dfrac{1}{2^{2p-1}}\,\mathrm{b}\big(p,\frac{1}{2}\big)$.

(iv) Show that

$$\mathrm{b}(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t.$$

(v) Evaluate

$$\int_0^\infty \frac{t^{\frac{3}{2}}}{(1+t)^6}\,\mathrm{d}t.$$

This question was the most popular question in terms of the number of attempts, and it was generally well done. Some candidates spent significant time attempting methods involving integration by parts in the early parts of this question which did not work.

In part (iii) the most common method in the 'Hence show that...' involved using the substitution $u = 2\theta$ at some point. After this the integral looks like what is required in the given answer but with the limits 0 to π rather than 0 to $\frac{\pi}{2}$. Candidates needed to point out that the symmetry in the integral enabled the limits to be changed back to 0 to $\frac{\pi}{2}$ with the appearance of a factor of 2.

Part (iv) was generally well done by those who got that far.

In part (v) some marks were given for piecing together the earlier results to get to an integral that is relatively easy to calculate. Some candidates did not provide sufficient justification to be awarded full credit. A good number of candidates got to a final correct value of the integral. Some candidates had success with alternative methods involving more difficult integration having not made so much use of the properties of b to simplify the calculation.

Question 2

2 Let $f(x) = 7 - 2|x|$.

A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = a$ and $u_n = f(u_{n-1})$ for $n > 0$.

(i) **(a)** Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(x))$.

(b) Find all solutions of the equation $f(f(x)) = x$.

(c) Find the values of a for which the sequence $u_0, u_1, u_2, \ldots$ has period 2.

(d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \ldots$ has period 2, but neither u_0 or u_1 is equal to either of u_2 or u_3.

(ii) **(a)** Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(f(x)))$.

(b) Consider the sequence $u_0, u_1, u_2, \ldots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $f(f(f(x))) = x$.

(c) Find a value of a such that the sequence $u_3, u_4, u_5, \ldots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5.

In terms of attempts, this was the third most popular question. Most candidates who attempted this question were able to make good progress with many of the parts. Candidates were generally able to sketch the graph of $y = \mathrm{f}(x)$, but sketches of $y = \mathrm{f}(\mathrm{f}(x))$ often had some features missing or incorrect. Many candidates opted to work out the equation for each of the straight-line segments before sketching the graph and, while this generally resulted in the correct overall shape, important points such as the vertical positioning of the points where the two graphs cross were often incorrect. A number of candidates would have benefitted from making their sketches larger. A small number of candidates did not sketch the two graphs on the same set of axes, which meant that some of the marks for this part of the question were not accessible.

Part (c) was not answered well, with many candidates simply restating their solutions to part (b) without considering the fact that some of the solutions would lead to sequences with a period of 1. Allowance was made here for those candidates who stated that a solution with period 1 also has period 2 and listed all their solutions to (b).

In part (d) candidates successfully calculated the terms of the sequence and many identified the connection with the previous parts to explain that the remainder of the sequence would have a period of 2. A small number of candidates only checked that $u_0 \neq u_2$ and $u_1 \neq u_3$ and therefore did not fully answer this part of the question.

Those who had made good sketches for the graphs in part (i) (a) generally made good attempts at the sketches in part (ii) (a), although similar issues were encountered with the positioning of the intersections of the two graphs.

Almost all candidates who attempted part (ii) (b) were able to calculate the sequences starting with the two given values, although many did not realise that all three values that appeared in each solution would also be solutions. The question was posed using 'Hence' and so this approach was required. Some candidates simply stated their set of solutions without providing any explanation. While some candidates commented on the fact that there must be eight solutions based on their sketch, a significant number of candidates did not realise that the two period 1 values that lead to a constant sequence would also be solutions.

Question 3

3 Let $\text{f}(x)$ be defined and positive for $x > 0$.

Let a and b be real numbers with $0 < a < b$ and define the points $A = (a, \text{f}(a))$ and $B = (b, -\text{f}(b))$.

Let $X = (m, 0)$ be the point of intersection of line AB with the x-axis.

(i) Find an expression for m in terms of a, b, $\text{f}(a)$ and $\text{f}(b)$.

(ii) Show that, if $\text{f}(x) = \sqrt{x}$, then $m = \sqrt{ab}$.

Find, in terms of n, a function $\text{f}(x)$ such that $m = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

(iii) Let $\text{g}_1(x)$ and $\text{g}_2(x)$ be defined and positive for $x > 0$. Let $m = M_1$ when $\text{f}(x) = \text{g}_1(x)$ and let $m = M_2$ when $\text{f}(x) = \text{g}_2(x)$.

Show that if $\dfrac{\text{g}_1(x)}{\text{g}_2(x)}$ is a decreasing function then $M_1 > M_2$.

Hence show that

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

(iv) Let p and c be chosen so that the curve $y = p(c-x)^3$ passes through both A and B. Show that

$$\frac{c-a}{b-c} = \left(\frac{\text{f}(a)}{\text{f}(b)}\right)^{\frac{1}{3}}$$

and hence determine c in terms of a, b, $\text{f}(a)$ and $\text{f}(b)$.

Show that if f is a decreasing function, then $c < m$.

Part (i) was generally well done. However, some candidates simply stated the result without showing sufficient working and full credit could not be given. Sign errors were another common pitfall and usually meant that the accuracy mark could not be awarded.

Part (ii) was frequently attempted, though for many candidates it marked the end of their attempt. A common mistake was overlooking the fact that the function $f(x)$ was defined to be positive for $x > 0$, leading to marks being unavailable. Another frequent issue was providing insufficient justification – some candidates simply stated a function without explaining their reasoning or showing it had the required properties, which prevented them from earning full marks.

Part (iii) was less frequently attempted, especially the first subpart. Some candidates assumed that the given functions were differentiable and attempted to provide arguments involving derivatives which did not gain credit. The second subpart was generally handled better, though again, a lack of justification was common. A few candidates also attempted alternative methods not involving the previous part, thus ignoring 'hence' in the question.

Part (iv) was relatively well done by those who attempted it. The first subpart was accessible to most candidates. The second subpart was more challenging and required careful attention, especially when working with inequalities and avoiding unwarranted assumptions of equality.

Question 4

4 **(i)** x_2 and y_2 are defined in terms of x_1 and y_1 by the equation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

G_1 is the graph with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and G_2 is the graph with equation

$$\frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{4} = 1.$$

Show that, if (x_1, y_1) is a point on G_1, then (x_2, y_2) is a point on G_2.

Show that G_2 is an anti-clockwise rotation of G_1 through $45°$ about the origin.

(ii) **(a)** The matrix

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$

represents a reflection. Find the line of invariant points of this matrix.

(b) Sketch, on the same axes, the graphs with equations

$$y = 2^x \quad \text{and} \quad 0.8x + 0.6y = 2^{-0.6x+0.8y}.$$

(iii) Sketch, on the same axes, for $0 \leqslant x \leqslant 2\pi$, the graphs with equations

$$y = \sin x \quad \text{and} \quad y = \sin(x - 2y).$$

You should determine the exact co-ordinates of the points on the graph with equation $y = \sin(x - 2y)$ where the tangent is horizontal and those where it is vertical.

This question was one of the less popular pure questions but still had a good number of attempts. In the question, candidates are led through an example of how to apply the mathematics they know to a new context and then are expected to apply what they have learnt to other problems.

For part (i) most candidates knew how to approach the problem but showed insufficient detail in their working for full credit, usually by not making the link (x_1, y_1) on $G_1 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$ when trying to show that (x_2, y_2) is on G_2.

A handful did not realise the significance of the indexing on the points and instead tried to show that the equations of the two curves were equivalent.

Almost all candidates recognised that the given matrix was a rotation matrix, but some did not make the link between this and the relationship between the points on the curves clear.

In part (ii) (a) it became evident that a number of candidates do not know the difference between a line of invariant points and an invariant line (in the second case points can move under the transformation but must stay on the same line). This meant some candidates did a lot more working than was necessary and often ended up with an extra answer, meaning that they could not gain full credit for this part.

A few candidates used the general form of a reflection matrix in the line $y = \tan\theta$ and a t substitution to find the required line. This method also required candidates to reject one solution, which was usually done by those taking this route.

In part (ii) (b) only the most successful candidates showed convincingly that if (x_1, y_1) was on the graph of $y = 2^x$ then $(x_2, y_2) = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ was on the other graph, but most realised that the two graphs were reflections of each other and so could make an attempt at the sketch. The most common mistakes here were assuming that the second graph was asymptotic to the *y* axis and not showing the two graphs intersecting twice.

Attempts were variable for part (iii). A lot of candidates found a matrix connecting points on the two curves, but often had the relationship the 'wrong way around' with $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ rather than the correct version $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

Many candidates could differentiate the implicit equation $y = \sin(x - 2y)$ successfully and some successfully went on to find the points where the tangent was horizontal and vertical. Some who had found the correct transformation matrix could use this to find the points with horizontal tangents but struggled to use a similar argument convincingly for the vertical tangents.

Some candidates successfully set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$ to get $x - 2y = \frac{\pi}{2}$ or $x - 2y = \frac{2\pi}{3}$ but were then uncertain how they could use this to find the coordinates of the relevant points.

Many of the candidates who found the points with horizontal or vertical tangents could 'join the dots' to complete the sketch, but some joined them in the wrong order. Many candidates laboured under the misunderstanding that it is not possible for an implicit function to be one-to-many valued which caused a variety of different mistakes.

Question 5

5 Three points, A, B and C, lie in a horizontal plane, but are not collinear. The point O lies above the plane.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

P is a point with $\overrightarrow{OP} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, where α, β and γ are all positive and $\alpha + \beta + \gamma < 1$.

Let $k = 1 - (\alpha + \beta + \gamma)$.

(i) The point L is on OA, the point X is on BC and LX passes through P.

Determine $\overrightarrow{OX}$ in terms of β, γ, $\mathbf{b}$ and $\mathbf{c}$ and show that $\overrightarrow{OL} = \dfrac{\alpha}{k+\alpha}\mathbf{a}$.

(ii) Let M and Y be the unique pair of points on OB and CA respectively such that MY passes through P, and let N and Z be the unique pair of points on OC and AB respectively such that NZ passes through P.

Show that the plane LMN is also horizontal if and only if OP intersects plane ABC at the point G, where $\overrightarrow{OG} \equiv \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Where do points X, Y and Z lie in this case?

(iii) State what the condition $\alpha + \beta + \gamma < 1$ tells you about the position of P relative to the tetrahedron $OABC$.

This question was the least popular pure question by a large margin, and of the attempts made less than half were 'substantial' attempts.

As is often the case with vector questions, a carefully drawn diagram can be very helpful in selecting an appropriate method for solving the question and the most successful candidates made good use of this.

Various methods were used in part (i), but mostly these involved finding vector equations of relevant lines and manipulating these to show the required results. Candidates should be aware that questions on the STEP papers need enough justification to fully support their solutions. Many candidates lost accuracy marks through their argument not being convincing enough or lacking some details.

Part (ii) had some very good solutions, but many candidates found it difficult to understand what it means for LMN to be horizontal. A clear diagram here would have helped candidates to find a solution method.

Some candidates tried to do both directions of the 'if and only if' in one go. They usually did not gain full marks here, either because they did not link one pair of statements with an if and only if symbol or because they did not appreciate that one step needed a different approach for each direction of implication. It is always 'safer' to approach each direction of implication separately.

The most common issues were not using $k \neq 0$ when justifying $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta} \Rightarrow \alpha = \beta$, or for not convincingly explaining why $\alpha = \beta = \gamma$ mean that LMN was horizontal.

Part (iii) required a one-line answer, and some candidates who had taken the time to read the whole question successfully answered this part even if they had not answered the previous parts. Some candidates confused 'positive' with 'non-negative' and stated that point P could be inside or on the faces of the tetrahedron.

Question 6

6 **(i)** Let a, b and c be three non-zero complex numbers with the properties $a+b+c=0$ and $a^2+b^2+c^2=0$.

Show that a, b and c cannot all be real.

Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a+b+c=0$ and $a^3+b^3+c^3=0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a+b+c+d=0$ and $a^3+b^3+c^3+d^3=0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c=1$, $d=-2$ and $e=3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a+b+c+d+e=0$ and $a^3+b^3+c^3+d^3+e^3=0$.

Question 6 proved to be quite challenging for many candidates, with a significant number scoring fewer than 5 marks and only attempting part (i) or part (i) and part (ii).

In part (i), the first mark was easily earned for strict inequalities or stating the only solution is the zero solution but was not earned if it was stated that 'the square of any real is positive' rather than any non-zero real. For the rest of the question, it was very common for candidates to attempt to consider each of a, b, c in the form $x+y$i, and then substitute in to obtain four equations in six variables. Those that tried this invariably made no progress. While it is possible to answer the question using real and imaginary parts, it requires far more work and so no credit was awarded for just writing down these four equations. Those who left the algebra in terms of a, b, c or used the roots of a quadratic tended to answer this part well.

Part (ii) also saw attempts to split a, b, c into real and imaginary parts. This saw no further progress, or credit. The most common way that this question was answered was by writing down an identity relating the sum of cubes to the cube of the sum. This identity could be written in several equivalent forms, and saw many errors in the coefficients and signs, for which candidates were penalised accuracy marks.

Careful thought about presentation was required before commencing the algebraic manipulation to part (iii) to avoid introducing sign and arithmetic errors. Complicated identities were common and often contained errors. Establishing that $abc+bcd+acd+abd = 0$ was a common approach and led to considering the roots of a quartic.

Part (iv) was generally well answered. Most candidates that attempted it were able to identify a quadratic and solve it. Several candidates that could not solve some of parts (i), (ii), (iii) skipped straight to this part and picked up some marks. This is good general exam practice and STEP candidates should remember that subsequent parts of a question can often still be answered even if an early part seems challenging.

Question 7

7 Let $\mathrm{f}(x) = \sqrt{x^2+1} - x$.

(i) Using a binomial series, or otherwise, show that, for large $|x|$, $\sqrt{x^2+1} \approx |x| + \dfrac{1}{2|x|}$.

Sketch the graph $y = \mathrm{f}(x)$.

(ii) Let $\mathrm{g}(x) = \tan^{-1} \mathrm{f}(x)$ and, for $x \neq 0$, let $\mathrm{k}(x) = \frac{1}{2} \tan^{-1} \dfrac{1}{x}$.

(a) Show that $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{1}{2}\pi$.

(b) Show that $\mathrm{k}(x) + \mathrm{k}(-x) = 0$.

(c) Show that $\tan \mathrm{k}(x) = \tan \mathrm{g}(x)$ for $x > 0$.

(d) Sketch the graphs $y = \mathrm{g}(x)$ and $y = \mathrm{k}(x)$ on the same axes.

(e) Evaluate $\displaystyle\int_0^1 \mathrm{k}(x)\,\mathrm{d}x$ and hence write down the value of $\displaystyle\int_{-1}^0 \mathrm{g}(x)\,\mathrm{d}x$.

In part (i), some candidates tried to expand $\sqrt{1+x^2}$ as a series in increasing powers of x^2, not appreciating that they needed $|x|$ to be small for such an expansion to be valid.

A lot of candidates used the expansion to correctly identify the asymptotes of $\mathrm{f}(x)$.

In part (ii) (a), the most common approach was to consider $\tan\big(\mathrm{g}(x)+\mathrm{g}(-x)\big)$, which, using the tan double angle formula, '$=\infty$'. Only a small number justified their answer by using the positivity of $\mathrm{f}(x)$ to get $0<\mathrm{g}(x)+\mathrm{g}(-x)<\pi$. Many candidates then simply stated the answer, or wrote $\mathrm{g}(x)+\mathrm{g}(-x)=\tan^{-1}\infty$, stating that this is $\frac{\pi}{2}$. A common theme in this question was a lack of consideration of ranges/domains of the trigonometric functions which meant there were marks that were unavailable.

Part (ii) (b) was done well in general, with many candidates knowing that $y=\arctan x$ is an odd function. Some overcomplicated it, using the double angle formula again, and not gaining a mark for justifying the range of $k(x)$, i.e. $\tan\big(k(x)+k(-x)\big)=0 \not\Rightarrow k(x)+k(-x)=0$ in general.

In part (ii) (c), the most common approach was again to use the tan double angle formula, realising that $\tan 2k(x)=x^{-1}$, and arriving at a quadratic for $\tan k(x)$. Marks were again unavailable for those candidates that either did not attempt to solve the quadratic or did solve for the two roots but then not explaining why $\tan k(x)=\mathrm{f}(x)$ was the correct root to choose.

There were also some nice geometric arguments for (c), drawing a right-angled triangle with angle $2k(x)$, then bisecting the angle and finding the side lengths of the smaller right-angled triangle with angle $k(x)$ to find $\tan k(x)$.

The sketches in part (ii) (d) were good in general, although some candidates' sketches contradicted the relations for $g(x)$,$k(x)$, given in the question, for example sketching $k(x)$ as an even function, or not using $\tan k(x)=\tan g(x)$ for $x>0$.

Part (ii) (e) was done well by candidates who attempted it. Some overcomplicated the integral by changing variables, but the majority realised they could integrate by parts directly. For the last part, candidates either used their sketches to find the right area or integrated the relation in part (a) directly using part (c).

Question 8

8 **(i)** Show that

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right).$$

Hence prove by induction that, for $n \geqslant 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right).$$

Find similarly $z^{2n} - \frac{1}{z^{2n}}$ as a product of $\left(z + \frac{1}{z}\right)$ and a sum.

(ii) **(a)** By choosing $z = \mathrm{e}^{\mathrm{i}\theta}$, show that

$$\sin 2n\theta = 2\sin\theta\sum_{r=1}^{n}\cos(2r-1)\theta\,.$$

(b) Use this result, with $n = 2$, to show that $\cos\frac{2}{5}\pi = \cos\frac{1}{5}\pi - \frac{1}{2}$.

(c) Use this result, with $n = 7$, to show that $\cos\frac{2}{15}\pi + \cos\frac{4}{15}\pi + \cos\frac{8}{15}\pi + \cos\frac{16}{15}\pi = \frac{1}{2}$.

In terms of the number of attempts, this was the second most popular question. The induction in part (i) was generally done very well with a clearly laid out proof. The fifth (method) mark in part (i) was often gained by finding a relevant identity. However, the final mark in part (i) was missed by a large majority of candidates due to not handling the alternating sign in the summation correctly. The summation is quite tricky in this respect, requiring the notation to be set up so that either the final term in the sum is positive, if terms are kept in the same order as for the previous part, or reversing the order of the sum (in which case the first term is positive).

Candidates often missed out on one or both marks in part (ii) (a) due to forgetting factors of 2 and i.

Most candidates that attempted parts (ii) (b) and (ii) (c) gained two method marks for successfully substituting a valid value of θ in both. However, a significant number of attempts at parts (ii) (b) and (ii) (c) did not gain full credit due to insufficient justification when manipulating trigonometric expressions. In general, candidates who stated the trigonometric identities they were using, or which specific terms were equivalent were successful here. Those that did multiple steps at once without justification often missed out on marks because it was not possible to pick out the results they had used. It is very significant here that the answer was given. Candidates should in general attempt to give more details when proving an answer given in the question to show they understand the intermediate steps between the starting point and given answer.

Most candidates did not attempt the part (iii). The successful attempts were from candidates who had given very clear answers to previous parts. There were a few different choices of n and θ that led to the required result.

Section B: Mechanics overview

Of the small number of candidates that attempted the mechanics section, more marks were generally scored on question 10 than on question 9.

Question 9

9 In this question, $n \geqslant 2$.

(i) A solid, of uniform density, is formed by rotating through $360°$ about the y-axis the region bounded by the part of the curve $r^{n-1}y = r^n - x^n$ with $0 \leqslant x \leqslant r$, and the x- and y-axes.

Show that the y-coordinate of the centre of mass of this solid is $\dfrac{nr}{2(n+1)}$.

(ii) Show that the normal to the curve $r^{n-1}y = r^n - x^n$ at the point $\big(rp, r(1-p^n)\big)$, where $0 < p \leqslant 1$, meets the y-axis at $(0, Y)$, where $Y = r\left(1 - p^n - \dfrac{1}{np^{n-2}}\right)$.

In the case $n = 4$, show that the greatest value of Y is $\frac{1}{4}r$.

(iii) A solid is formed by rotating through $360°$ about the y-axis the region bounded by the curves $r^3y = r^4 - x^4$ and $ry = -(r^2 - x^2)$, both for $0 \leqslant x \leqslant r$.

A and B are the points $(0, -r)$ and $(0, r)$, respectively, on the surface of the solid.

Show that the solid can rest in equilibrium on a horizontal surface with the vector $\overrightarrow{AB}$ at three different, non-zero, angles to the upward vertical. You should not attempt to find these angles.

This question had the least number of attempts with a relatively small number of substantial attempts. Overall, this question was not done well, with only a small number of candidates achieving half marks or more. A significant number of candidates got 0,1, or 2 marks in total for the question, with the main issue being getting started by knowing a suitable formula for the centre of mass.

In part (i), many candidates overlooked fact that the curve was rotated about the y-axis, rather than the usual x-axis, and didn't change their formula to the correct variables.

Often candidates skipped straight to part (ii), which was done generally done well.

Part (iii) did not have many attempts. Of those candidates that did answer this part, most were only able to get the first few marks for finding the centre of mass of the full shape. The final part (showing there are multiple ways to balance the solid) was not attempted enough to observe any patterns but it was clear that candidates struggled to demonstrate clear understanding of how the equilibrium condition relates to the position of the centre of mass.

Question 10

10 A plank AB of length L initially lies horizontally at rest along the x-axis on a flat surface, with A at the origin.

Point C on the plank is such that AC has length sL, where $0 < s < 1$.

End A is then raised vertically along the y-axis so that its height above the horizontal surface at time t is $\mathrm{h}(t)$, while end B remains in contact with the flat surface and on the x-axis.

The function $\mathrm{h}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{h}}{\mathrm{d}t^2} = -\omega^2\mathrm{h}, \quad \text{with } \mathrm{h}(0) = 0 \text{ and } \frac{\mathrm{dh}}{\mathrm{d}t} = \omega L \text{ at } t = 0,$$

where ω is a positive constant.

A particle P of mass m remains in contact with the plank at point C.

(i) Show that the x-coordinate of P is $sL\cos\omega t$, and find a similar expression for its y-coordinate.

(ii) Find expressions for the x- and y-components of the acceleration of the particle.

(iii) N and F are the upward normal and frictional components, respectively, of the force of the plank on the particle. Show that

$$N = mg(1 - k\sin\omega t)\cos\omega t,$$

and that

$$F = mgsk + N\tan\omega t$$

where $k = \dfrac{L\omega^2}{g}$.

(iv) The coefficient of friction between the particle and the plank is $\tan\alpha$, where α is an acute angle.

Show that the particle will not slip initially, provided $sk < \tan\alpha$.

Show further that, in this case, the particle will slip

- while N is still positive,
- when the plank makes an angle less than α to the horizontal.

This question did not receive very many attempts.

Part (i) was generally done well, although a significant portion of candidates did not draw a diagram and didn't correctly calculate y, i.e. not performing the necessary subtraction. This meant errors followed through to subsequent parts.

Part (ii) was generally done well.

A good portion of those that attempted part (iii) realised they had to resolve in two directions (either horizontal and vertical or parallel and perpendicular), and made a good attempt to do this. Choosing to resolve horizontally and vertically proved to be more straightforward. Those that chose to resolve parallel and perpendicular often had some difficulties with calculating the resultant acceleration.

Part (iv) was not answered successfully on the whole. Considering the equivalent problem when the plank is not moving may have led to considering $t = \frac{\alpha}{\omega}$, leading to the key idea that $F < \mu N$ when $t = 0$ and $F > \mu N$ when $t = \frac{\alpha}{w}$.

Section C: Probability and Statistics overview

In general, those candidates that attempted the statistics section appeared to be confident with the content and could apply their knowledge successfully, especially on question 12.

Question 11

11 **(i)** Let $\lambda > 0$. The independent random variables $X_1, X_2, \ldots, X_n$ all have probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

and cumulative distribution function $F(x)$.

The value of random variable Y is the largest of the values $X_1, X_2, \ldots, X_n$. Show that the cumulative distribution function of Y is given, for $y \geqslant 0$, by

$$G(y) = \left(1 - e^{-\lambda y}\right)^n .$$

(ii) The values $L(\alpha)$ and $U(\alpha)$, where $0 < \alpha \leqslant \frac{1}{2}$, are such that

$$P(Y < L(\alpha)) = \alpha \quad \text{and} \quad P(Y > U(\alpha)) = \alpha.$$

Show that

$$L(\alpha) = -\frac{1}{\lambda} \ln\left(1 - \alpha^{\frac{1}{n}}\right)$$

and write down a similar expression for $U(\alpha)$.

(iii) Use the approximation $e^t \approx 1 + t$, for $|t|$ small, to show that, for sufficiently large n,

$$\lambda L(\alpha) \approx \ln(n) - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right).$$

(iv) Hence show that the median of Y tends to infinity as n increases, but that the width of the interval $U(\alpha) - L(\alpha)$ tends to a value which is independent of n.

(v) You are given that, for $|t|$ small, $\ln(1 + t) \approx t$ and that $e^3 \approx 20$.

Show that, for sufficiently large n, there is an interval of width approximately $4\lambda^{-1}$ in which Y lies with probability 0.9 .

Parts (i) and (ii) were generally well done.

Candidates were often able to make good progress with parts (iv) and (v) even if they had found difficulty with part (iii) (since the answer to part (iii) was given in the question).

In part (iii), many candidates incorrectly assumed that $\alpha^{\frac{1}{n}} \to 0$, leading to the incorrect approximation $\ln\left(1-\alpha^{\frac{1}{n}}\right) \approx -\alpha^{\frac{1}{n}}$

A significant number of candidates ignored the word 'hence' in part (iv), either:

- not realising that $L\left(\frac{1}{2}\right)$ was the median
- instead solving $G(m) = \frac{1}{2}$ directly to find the median m.Most candidates who attempted part (v) focused entirely on estimating the size of $U(0.05) - L(0.05)$, without ever stating that $P\big(L(0.05) < Y < U(0.05)\big) = 0.9$).

In parts (iii), (iv) and (v), a number of candidates did not give sufficient precision in the use of approximations/limits, for example writing asymptotic results as equalities which held for all n

Approximately half of the candidates implicitly utilised the identity $U(\alpha) = L(1-\alpha)$. Whilst, formally, the bound $0 < \alpha \leq \frac{1}{2}$ given in the question invalidated this method unless the range of the arguments of L and U were first extended to $0 < \alpha < 1$, the identity allowed candidates to save considerable repetition of work and candidates who employed this method were not penalised on account of this technical ubtlety.

Question 12

12 **(i)** Show that, for any functions f and g, and for any $m \geqslant 0$,

$$\sum_{r=1}^{m+1}\left(\mathrm{f}(r)\sum_{s=r-1}^{m}\mathrm{g}(s)\right)=\sum_{s=0}^{m}\left(\mathrm{g}(s)\sum_{r=1}^{s+1}\mathrm{f}(r)\right).$$

(ii) The random variables $X_0, X_1, X_2, \ldots$ are defined as follows

- X_0 takes the value 0 with probability 1;
- X_{n+1} takes the values $0, 1, \ldots, X_n + 1$ with equal probability, for $n = 0, 1, \ldots$.

(a) Write down $\mathrm{E}(X_1)$.

Find $\mathrm{P}(X_2 = 0)$ and $\mathrm{P}(X_2 = 1)$ and show that $\mathrm{P}(X_2 = 2) = \frac{1}{6}$.

Hence calculate $\mathrm{E}(X_2)$.

(b) For $n \geqslant 1$, show that

$$\mathrm{P}(X_n = 0) = \sum_{s=0}^{n-1}\frac{\mathrm{P}(X_{n-1} = s)}{s+2}$$

and find a similar expression for $\mathrm{P}(X_n = r)$, for $r = 1, 2, \ldots, n$.

(c) Hence show that $\mathrm{E}(X_n) = \frac{1}{2}\left(1 + \mathrm{E}(X_{n-1})\right)$.

Find an expression for $\mathrm{E}(X_n)$ in terms of n, for $n = 1, 2, \ldots$.

Most candidates answered all parts of this question well, with many candidates earning full or close to full marks.

In part (i), a small number of candidates erroneously believed that

$$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m} g(s)\right) = \left(\sum_{r=1}^{m+1} f(r)\right)\times\left(\sum_{s=r-1}^{m} g(s)\right)$$

and likewise for the second sum. Such attempts earned no credit.

In part (ii) (a), a significant number of candidates did not understand the concept of X_{n+1} being uniformly distributed on $\{0,1,\dots,X_n+1\}$, usually leading to the incorrect values.

In part (ii) (b), a number of candidates gave no justification of the written result, simply writing

$$P(X_n=0)=\frac{1}{2}P(X_{n-1}=0)+\frac{1}{3}P(X_{n-1}=1)+\cdots+\frac{1}{n+1}P(X_{n-1}=n-1)=\sum_{s=0}^{n-1}\frac{P(X_{n-1}=\ s)}{s=2};$$

such attempts earned no credit.

In part (ii) (c), most candidates solved this part either by inductively proving that $E(X_n)=1-2^{-n}$ or by noting that $E(X_n)-1=\frac{1}{2}[E(X_{n-1})-1]$ and applying recursion. A smaller number of candidates applied recursion directly to the formula $E(X_n)=\frac{1}{2}[E(X_{n-1})+1]$ leading to a correct solution via geometric series.

STEP Mathematics 3 mark scheme

Question		Answer	Marks	Guidance
1	(i)	$b(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$. The substitution $u=1-x$ leads to	M1	For identifying suitable substitution and making some progress with it.
		$b(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx=\int_1^0 -(1-u)^{p-1}u^{q-1}du$		For the M1 it is sufficient to see either the limit swap or the presence of factor of (-1).
		$=\int_0^1 (1-u)^{p-1}u^{q-1}du=b(q,p)$.	A1	Fully correct working leading to result (AG)
			[2]	
	(ii)	$\int_0^1 x^p(1-x)^{q-1}dx+\int_0^1 x^{p-1}(1-x)^q dx$		
		$=\int_0^1 x^{p-1}(1-x)^{q-1}(x+(1-x))dx+\int_0^1 x^{p-1}(1-x)^{q-1}dx$	M1	Method involving common factor in two integrands

Question	Answer	Marks	Guidance
	$b(p+1,q)+b(p,q+1)=b(p,q)$.	A1	Fully correct working and conclusion.
	Setting $p = q$ then gives $b(p+1,p)+b(p,p+1)=b(p,p)$. Since $b(p+1,p)=b(p,p+1)$ by part (i) this gives $b(p+1,p)=\frac{1}{2}b(p,p)$	B1 [3]	Need both parts of this argument (namely setting $p = q$ and use of $b(p,q) = b(q,p)$) The use of $b(p,q)=b(q,p)$ can be implied by presence of $2b(p+1,p)$ term.
(iii)	Starting with the original $b(p,q)$, the substitution $x=\sin^2\theta$ leads to $b(p,q)=\int_0^{\pi/2}(\sin\theta)^{2(p-1)}\,(\cos\theta)^{2(q-1)}2\sin\theta\cos\theta\,d\theta=$ $2\int_0^{\pi/2}(\sin\theta)^{2p-1}\,(\cos\theta)^{2q-1}d\theta$.	M1 A1	For identifying suitable substitution and making some progress with it. Fully correct working leading to result (AG)

Question	Answer	Marks	Guidance
	This gives that $b(p,p) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2p-1}\,d\theta$ $= 2\int_0^{\pi/2} \frac{(\sin 2\theta)^{2p-1}}{2^{2p-1}}\,d\theta$	M1	For setting $q = p$ and use of $2\sin\theta\cos\theta = \sin 2\theta$.
	The substitution $t = 2\theta$ then gives $b(p,p) = 2\times\frac{1}{2}\int_0^{\pi} \frac{(\sin t)^{2p-1}}{2^{2p-1}}\,dt$	M1	For identifying suitable substitution and carrying it out correctly. Allow one incorrect limit for M1. Also can give M1 if fully correct but with no working (since substitution is particularly straightforward).
*	by the symmetry of integrand about $\frac{\pi}{2}$. $= 2\int_0^{\pi/2} \frac{(\sin t)^{2p-1}}{2^{2p-1}}\,dt =$	B1	Use of symmetry of sin to deal with a factor of 2. Need to see word "symmetry" (or equivalent e.g. diagram or identity) and change of upper limit to $\frac{\pi}{2}$.

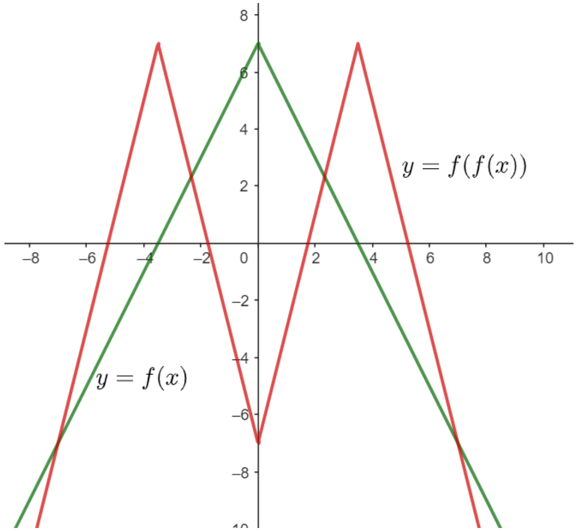
Question	Answer	Marks	Guidance
**	$2\int_0^{\pi/2} \frac{(\sin t)^{2p-1}(\cos t)^{2\times\frac{1}{2}-1}}{2^{2p-1}}dt = \frac{b(p,\frac{1}{2})}{2^{2p-1}}$	A1	Bring together for fully correct result (given), noting use of $(\cos t)^0 = 1$ Writing out $b\left(p,\frac{1}{2}\right)$ independently, with the absence of the $\cos t$ term, is to be considered equivalent to the above.
		[6]	
	Alternative for last two marks of (iv)		
*	$b(p,p) = 2\times\frac{1}{2}\int_0^{\pi}\frac{(\sin t)^{2p-1}}{2^{2p-1}}dt = 4\int_0^{\frac{\pi}{4}}\frac{(\sin 2\theta)^{2p-2}}{2^{2p-1}}\frac{4\sin 2\theta\cos 2\theta d\theta}{4\cos 2\theta}$		
	$= \frac{1}{2^{2p-1}}\int_0^1 \frac{u^{p-1}}{(1-u)^{\frac{1}{2}}}du$	M1	Substitution $u = \sin^2 2\theta$
**	$= \frac{1}{2^{2p-1}}b\left(p,\frac{1}{2}\right)$	A1	

Question		Answer	Marks	Guidance
	(iv)	Using the substitution $t = \frac{x}{1-x}$ (or equivalently $x = \frac{t}{1+t}$)	M1	For identifying suitable substitution(s). Other possible substitutions: $t = \tan^2(\theta)$ $x = 1/(1+t)$ $x = t/(t-1)$ Approaches involving a series of two substitutions $t = \tan(\theta)$ then $u = t^2$ $x = \frac{1}{t}$ then $u = t - 1$ $x = (t-1)/t$ then $u = 1 + t$ $u = \ln(1/x)$ then $t = e^u - 1$ $x = \tanh^2(\theta)$ then $u = \sinh^2(\theta)$
		$b(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx =$ $\int_0^\infty \left(\frac{t}{1+t}\right)^{p-1}\left(1-\frac{t}{1+t}\right)^{q-1}\frac{1}{(1+t)^2}dt$	M1	For working in method of substitution

Question		Answer	Marks	Guidance
		$= \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt$	**A1** **[3]**	Fully correct working leading to result (AG). (Note that if arrives at same integral but with power of $q - 1$ in numerator need to quote (i) to switch that to $p - 1$ and gain this mark.)
	(v)	By part (iv) the integral given is $b\left(\frac{5}{2},\frac{7}{2}\right)$	**B1**	Use of previous answers. May attempt direct integration of this see ALT below.
		By part (i) this is equal to $b\left(\frac{7}{2},\frac{5}{2}\right)$.	**B1**	Use of previous answers
		and then by part(ii) equal to $\frac{1}{2}b\left(\frac{5}{2},\frac{5}{2}\right)$. By part (iii) this is equal to $\frac{1}{2^5}b\left(\frac{5}{2},\frac{1}{2}\right)$	**B1** **B1**	Use of previous answers This can be given if seen, without link to previous step.

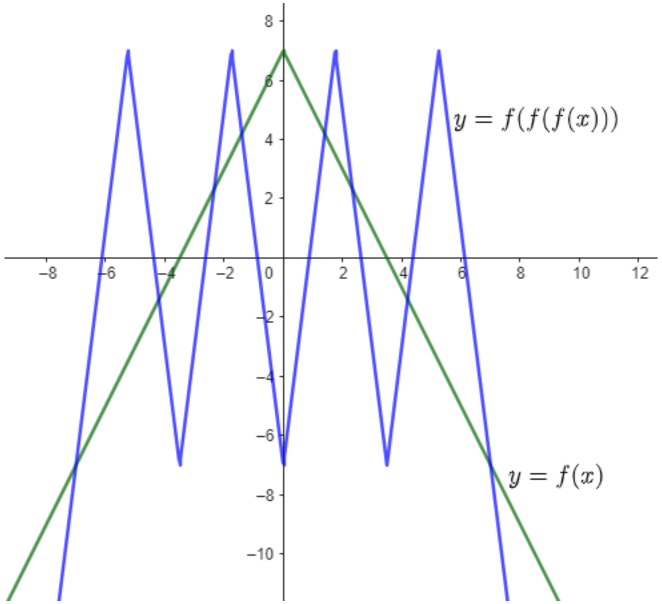
Question		Answer	Marks	Guidance
		$\frac{1}{2^4}\int_0^{\pi/2}(\sin\theta)^4\,d\theta = \frac{1}{2^6}\int_0^{\pi/2}(1-\cos 2\theta)^2 d\theta =$ $= \frac{1}{2^6}\int_0^{\pi/2} 1 - 2\cos 2\theta + \frac{\cos 4\theta + 1}{2}\,d\theta$ $= \frac{1}{64}\left[\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}} =$	**M1**	Need to have got to an expression with multiple angles only, using the double angle formulae.
		$\frac{3\pi}{256}$	**A1**	Fully correct answer
			[6]	

Question		Answer	Marks	Guidance
		Alternative (v)		
		By part (iv) the integral given is $b\left(\frac{5}{2},\frac{7}{2}\right)$.	**B1**	
		Using the result in (iii) this is		
		$2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta$	**M1**	For stating switching to this form of $b(p,q)$
		de Moivre, with $z = \cos\theta + i\sin\theta$	**M1**	For use of de Moivre in this way $(z + z^{-1})$ etc.
		$2^{10}\sin^4\theta\cos^6\theta = (z - z^{-1})^4(z + z^{-1})^6$		
		$= (z^8 - 4z^4 + 6 - 4z^{-4} + z^{-8})(z^2 + 2 + z^{-2}) =$		
		$z^{10} + z^{-10} + 2(z^8 + z^{-8}) - 3(z^6 + z^{-6}) - 8(z^4 + z^{-4}) + 2(z^2 + z^{-2}) + 12$		
		$= 2\cos 10\theta + 4\cos 8\theta - 6\cos 6\theta - 16\cos 4\theta + 2\cos 2\theta + 12$		
		Therefore		
		$2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta =$	**M1**	For arriving at an easily integrable expression.
		$\frac{1}{256}\int_0^{\frac{\pi}{2}} \cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + \cos 2\theta + 6\, d\theta =$		
		$\frac{1}{256}\left[\frac{\sin(10\theta)}{10} + \frac{\sin 8\theta}{4} - \frac{\sin 6\theta}{2} - 2\sin 4\theta + \frac{\sin 2\theta}{2} + 6\theta\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{256}$	**M1** **A1**	

Question	Answer	Marks	Guidance
2 (i) (a)		G1 G1 G1 G1 [4]	Two straight lines for y = f(x), roughly correct gradients, meeting on y-axis Four straight lines for y = f(f(x)) with roughly correct gradients (twice as steep than those for y = f(x)) Symmetry between (0,7) on y = f(x) and (0,-7) on y = f(f(x)) 'Peaks' on both graphs have same y-coordinates (y = 7). Intersections at $(\pm 7, -7)$ correspond to minimum point on y-axis.
(b)	Using graph, need to find the intersections of $y = x$ with each of $y = 4x - 7, y = -4x - 7, y = 4x + 21, y = -4x + 21.$	M1	Method which involves solving for intersections of the relevant lines, guided by graph, (or equivalent such as attempting to solve $7 - 2\|7 - 2\|x\|\| = x$ algebraically). For attempts to solve using quadratics, award M1 for getting

Question	Answer	Marks	Guidance
			any one quadratic: $15x^2 + 126x + 147 = 0$ or $15x^2 - 98x + 147 = 0$ $5x^2 + 14x - 147 = 0$ or $15x^2 - 14x - 49 = 0$
	This gives values of $\frac{7}{3}, \frac{-7}{5}, -7, \frac{21}{5}$	**A2**	Award **A1** for 3 correct solutions or if any incorrect values are included.
		[3]	
(c)	For this we need a such that $f(a) \neq a$ but $f\big(f(a)\big) = a$.	**M1**	For property of a point with period 2.
	Candidates are solutions of $f\big(f(x)\big) = x$, from above, namely: $\frac{7}{3}, \frac{-7}{5}, -7, \frac{21}{5}$.		
	We have $f\left(\frac{7}{3}\right) = \frac{7}{3}, f(-7) = -7$ and $f\left(\frac{-7}{5}\right) = \frac{21}{5}, f\left(\frac{21}{5}\right) = -\frac{7}{5}$. Therefore sequence has period 2 only when $a = \frac{-7}{5}, \frac{21}{5}$.	**A1FT**	Checking leading to correct values (could be via graph) Follow through, their answer to (i)(b) with 7/3 and –7 removed.
		[2]	

Question	Answer	Marks	Guidance
(c)	**Alternative: Some candidates may consider a sequence with period 1 to also have period 2. In this case marks as follows**		
	A sequence which has period 1 also has period 2.	**M1**	
	Such a are $\frac{7}{3}, \frac{-7}{5}, -7, \frac{21}{5}$, from answer to (b)	**A1**	
		[2]	
(d)	With $a = \frac{28}{5}$ we get $u_0 = \frac{28}{5}, u_1 = \frac{-21}{5}, u_2 = \frac{-7}{5}$, and (by part (c)) the sequence is periodic from there.	**M1**	For calculation of values necessary for checking and comment that sequence is periodic.
	$u_3 = \frac{21}{5}$, ($u_4 = \frac{-7}{5}, u_5 = \frac{21}{5}$ and will then repeat $-\frac{7}{5}, \frac{21}{5} - \frac{7}{5}, \frac{21}{5}, ..$)	**A1**	Value of u_3 and check that other condition is satisfied.
	Neither of u_0, u_1 is equal to either of u_2, u_3.		
		[2]	

Question	Answer	Marks	Guidance
(ii) (a)		G1 G1 G1 [3]	8 straight lines for $y = f(f(f(x))$, looking roughly correct (position, gradients etc) 4 peaks for $y = f(f(f(x))$ at same y-coordinate (y = 7) as the peak of $y = f(x)$. Three mins with same negative y-coordinate (y = -7) and the intersections at ± 7 at this y-coordinate.
(b)	(We can see from the graph, by adding $y = x$, that there are a total of 8 solutions.) Any solution of $f(x) = x$ is also a solution of $f(f(f(x)) = x$. From earlier, these are $\frac{7}{3}, -7$.	B1 FT	Identifying both of these values . Follow through from their solutions to f(x)=x identified in part (i)(c).

Question	Answer	Marks	Guidance
	We have that $f(f(f(1)) = 1$. We also that $f(f\left(f\left(\frac{-7}{9}\right)\right) = \frac{-7}{9}$. Therefore $x = 1$ $x = -\frac{7}{9}$ are both solutions of $f(f(f(x)) = x$.	M1	For checking these two values
	The other solutions must be other values in the 3-cycles which include $1, -\frac{7}{9}$, namely $f\left(\frac{-7}{9}\right), f\left(f\left(\frac{-7}{9}\right)\right), f(1), f(f(1))$.	E1	For commenting that the other values in the cycles will also be solutions.
	These values are $\frac{49}{9}, \frac{-35}{9}, 5, -3$.	A1	Identifying all four of these values
		[4]	
(c)	u_3 has to be one of $1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3$.	B1FT	Follow through using of their answers to (ii)(b) as a value for u_3.
	So we are looking for solutions, x, of $f(f(f(x)) = 1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3$. which are not 'values above' and so that, when taking them as u_0, neither of u_1, u_2 are included in the values above.		
	The full list of values is $$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{11}{2}, \pm\frac{13}{2}, \pm\frac{7}{18}, \pm\frac{35}{18}, \pm\frac{49}{18}, \pm\frac{77}{18}, \pm\frac{91}{18}, \pm\frac{119}{18}$$	B1	Any one will do
		[2]	

Question	Answer	Marks	Guidance
(c)	**Alternative: Some candidates may consider a sequence with period 1 to also have period 3. In this case marks as follows, this must be consistent with their approach to 2(i)(c)**		
	u_3 has to be one of $1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3, -7, \frac{7}{3}$	B1	
	So we are looking for solutions, x, of $f\left(f\left(f(x)\right)\right) = 1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3, -7, \frac{7}{3}$ which are not 'values above' and so that, when taking them as u_0, neither of u_1, u_2 are included in the values above.		
	The full list of such values is		
	$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{11}{2}, \pm\frac{13}{2}, \pm\frac{7}{18}, \pm\frac{35}{18},$		
	$\pm\frac{49}{18}, \pm\frac{77}{18}, \pm\frac{91}{18}, \pm\frac{119}{18}, \pm\frac{7}{2}, \pm\frac{7}{6}, \pm\frac{35}{6}$	B1	Any one will do
		[2]	

Question		Answer	Marks	Guidance
3	(i)	The equation of the line AB is $\left(y - f(a)\right) = \left(\frac{-(f(a)+f(b))}{b-a}\right)(x-a)$.	M1	Finding equation of line and
		Substituting $y = 0, x = m$ gives $-f(a) = \left(\frac{-(f(a)+f(b))}{b-a}\right)(m-a)$ leading to	M1	Sub $y = 0$, $x = m$. No need to set these in a single step.
		$\frac{f(a)(b-a)}{f(a)+f(b)} + a = m \Leftrightarrow m = \frac{f(a)(b-a) + a\left(f(a)+f(b)\right)}{f(a)+f(b)} \Leftrightarrow$ $m = \frac{f(a)b + af(b)}{f(a)+f(b)}$.	A1	Fully correct working
			[3]	

Question	Answer	Marks	Guidance
(ii)	Setting $f(x)=\sqrt{x}$ gives $m=\frac{b\sqrt{a}+a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. Noting that $b\sqrt{a}+a\sqrt{b}=\left(\sqrt{a}+\sqrt{b}\right)\sqrt{ab}$, we see that $m=\sqrt{ab}$.	B1	Use of correct relationship that leads to result. Or use difference of squares. No FT as AG.
*	We need to find a function $f(x)$ such that $m=\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{f(a)b+f(b)a}{f(a)+f(b)}$ $\Leftrightarrow f(a)a^nb+f(a)b^{n+1}+f(b)a^{n+1}+f(b)ab^n$ $=f(a)a^{n+1}+f(a)b^{n+1}+f(b)a^{n+1}+f(b)b^{n+1}$	M1	
**	$\Leftrightarrow\ [b-a]\cdot(f(a)a^n-f(b)b^n)\ =\ 0$ $\Leftrightarrow\ f(a)a^n=f(b)b^n$	A1	Equate the two fractions and at least get to the second line of calculation for the M1.
***	Finding any function $f(x)$ that satisfies the above e.g. $f(x)=\frac{1}{x^n}$ or $f(x)=\frac{a^nb^n}{x^n}$. *(Alternatively, if the candidate recognises the need for switching a and b and/or writes the function of a form similar to $f(x)=(a+b-x)^n$ then the E1 is awarded.)*	E1	Award for giving a function that satisfies relationship above.
		[4]	

Question	Answer	Marks	Guidance
	Alternatively, for last three marks in (ii), candidates could		
***	state an appropriate function $f(x)$.	**E1**	
	and then go on to show that it meets the requirements in the question that $f(x)$ is defined and positive for $x > 0$, and that		
*;**	$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{f(a)b+f(b)a}{f(a)+f(b)}$.	**M1A1**	
(iii)	We have $M_1 = \frac{g_1(a)b+ag_1(b)}{g_1(a)+g_1(b)}$ and $M_2 = \frac{g_2(a)b+ag_2(b)}{g_2(a)+g_2(b)}$.		
	So $M_1 > M_2 \Leftrightarrow \frac{g_1(a)b+ag_1(b)}{g_1(a)+g_1(b)} > \frac{g_2(a)b+ag_2(b)}{g_2(a)+g_2(b)}$ $\Leftrightarrow \big(g_1(a)b + ag_1(b)\big)\big(g_2(a) + g_2(b)\big) > \big(g_2(a)b + ag_2(b)\big)\big(g_1(a) + g_1(b)\big)$ $\Leftrightarrow bg_1(a)g_2(b) + ag_1(b)g_2(a) > bg_2(a)g_1(b) + ag_2(b)g_1(a)$ $\Leftrightarrow b\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) + a(g_1(b)g_2(a) - g_2(b)g_1(a)) > 0$ $\Leftrightarrow (b-a)\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) > 0$	**M1**	For simplifying $M_1 > M_2$ to "useable form", need at least one line of simplifying
	We know that $\frac{g_1(x)}{g_2(x)}$ is a decreasing function so that, since $b > a$, $\frac{g_1(b)}{g_2(b)} < \frac{g_1(a)}{g_2(a)}$ giving $g_1(b)g_2(a) < g_1(a)g_2(b)$ and $0 < g_1(a)g_2(b) - g_1(b)g_2(a)$	**M1**	Correct use of decreasing function

Question	Answer	Marks	Guidance
	Therefore, since $b > a$ we have $(b-a)\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) > 0$ and so $M_1 > M_2$ by the working above.	A1	Fully correct explanation.
	Take $g_1(x) = 1$. By formula in (i) $M_1 = \frac{a+b}{2}$. As seen earlier, taking $g_2(x) = \sqrt{x}$, gives $M_2 = \sqrt{ab}$. Alternatively taking $g_1(x) = \sqrt{x}$ and $g_2(x) = x$ is also valid for proving the right side of the inequality first.	M1	Correct choice of g_1 and g_2.
	Since $\frac{g_1(x)}{g_2(x)} = \frac{1}{\sqrt{x}}$ is a decreasing function the result just established gives us that $\frac{a+b}{2} > \sqrt{ab}$. Same can be used to prove $\sqrt{ab} > \frac{2ab}{a+b}$.	A1	Decreasing function statement required
	Then $\frac{a+b}{2} > \sqrt{ab} \Leftrightarrow 1 > \frac{2\sqrt{ab}}{a+b} \Leftrightarrow \sqrt{ab} > \frac{2ab}{a+b}$ where the last iff holds by multiplying both sides of the previous inequality by $\sqrt{ab} > 0$. Similarly, for $\frac{a+b}{2} > \sqrt{ab}$.	B1	Correct working of first inequality. This could also be given by taking a direct approach to other inequality with correct choice of g_1, g_2
		[6]	
(iv)	$f(a) = p(c-a)^3$ and $-f(b) = p(c-b)^3$.	M1	Equations for $f(a), f(b)$
	Dividing one by the other we get $\frac{(c-a)^3}{(b-c)^3} = \frac{f(a)}{f(b)}$ giving $\frac{c-a}{b-c} = \frac{f(a)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}}$	A1	Fully correct working

Question	Answer	Marks	Guidance
	We then have $f(b)^{\frac{1}{3}}c - f(b)^{\frac{1}{3}}a = f(a)^{\frac{1}{3}}b - f(a)^{\frac{1}{3}}c \Leftrightarrow (f(b)^{\frac{1}{3}} + f(a)^{\frac{1}{3}})c = f(a)^{\frac{1}{3}}b + f(b)^{\frac{1}{3}}a$ So we have $c = \frac{f(a)^{\frac{1}{3}}b+f(b)^{\frac{1}{3}}a}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}}$	**B1**	For correct c. FT if e.g. in M1 missed a minus sign or MR (as method is valid and c not given in Q3)
*	Now $c < m \Leftrightarrow \frac{f(a)^{\frac{1}{3}}b+f(b)^{\frac{1}{3}}a}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}} < \frac{f(a)b+af(b)}{f(a)+f(b)} \Leftrightarrow$		
	$\left(f(a)^{\frac{1}{3}}b + f(b)^{\frac{1}{3}}a\right)\left(f(a) + f(b)\right) < \left(f(b)^{\frac{1}{3}} + f(a)^{\frac{1}{3}}\right)\left(f(a)b + af(b)\right) \Leftrightarrow$ $f(a)^{\frac{4}{3}}b + f(a)^{\frac{1}{3}}bf(b) + f(b)^{\frac{1}{3}}af(a) + f(b)^{\frac{4}{3}}a$ $< f(b)^{\frac{1}{3}}f(a)b + f(b)^{\frac{4}{3}}a + f(a)^{\frac{4}{3}}b + f(a)^{\frac{1}{3}}f(b)a$ $\Leftrightarrow f(a)^{\frac{1}{3}}bf(b) + f(b)^{\frac{1}{3}}af(a) < f(b)^{\frac{1}{3}}f(a)b + f(a)^{\frac{1}{3}}f(b)a$ $\Leftrightarrow (b-a)\left(f(a)^{\frac{1}{3}}f(b) - f(b)^{\frac{1}{3}}f(a)\right) < 0 \Leftrightarrow f(a)^{\frac{1}{3}}f(b) - f(b)^{\frac{1}{3}}f(a) < 0$ $\Leftrightarrow \frac{f(a)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}} < \frac{f(a)}{f(b)} \Leftrightarrow f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$	**M1**	For working and at least one line of simplifying

Question	Answer	Marks	Guidance
**	So $c < m \Leftrightarrow f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$	**A1**	Equivalence of inequalities.
***	If x and y are positive, then $x < y \Leftrightarrow x^{\frac{1}{3}} < y^{\frac{1}{3}} \Leftrightarrow x^{\frac{2}{3}} < y^{\frac{2}{3}}$. Applying this, since $f(x)$ is decreasing and positive, so is $f(x)^{\frac{2}{3}}$. *(Note that using derivates here is not appropriate as the function may not be differentiable, however, if the method carries along in the right spirit the M1 is awarded but not the A1 for accuracy of the argument.)*	**M1**	Statement, with justification, about $f(x)^{\frac{2}{3}}$ being decreasing, FT their inequality c < m. Also award when the candidate states that any decreasing function raised to a positive power is also decreasing – note the generality of the statement.
****	Therefore $f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$ giving that $c < m$.	**A1**	CSO
		[7]	

Question	Answer	Marks	Guidance
	Alternative for last four marks in (iv)		
*	Bringing c and m in the form such as $c = ka + (1-k)b$ and $m = qa + (1-q)b$, where $k = \frac{f(b)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}} = \frac{1}{1+\left(\frac{f(a)}{f(b)}\right)^{\frac{1}{3}}}$ and $q = \frac{f(b)}{f(b)+f(a)} = \frac{1}{1+\frac{f(a)}{f(b)}}$.	M1	For bringing the expression for c and m in a form that can be compared and splitting the expressions into a form weight_1*a + weight_2*b.
**	Hence as both k and q positive and $a < b$, we need to prove that $k > q$ and hence $(1-k) < (1-q)$ to show that $c < m$. *(Note that as both k and q are in $(0,1)$ we may think of c and m as expectation values weighting the values of a and b and hence the expression which has "less a and more b " in it will be bigger.)*	A1	For setting up correct, simplified inequalities that can be used to find $c < m$.
***	As $f(x)$ is positive and decreasing we have $\frac{f(a)}{f(b)} > 1$ and hence $\frac{f(a)}{f(b)} > \left(\frac{f(a)}{f(b)}\right)^{\frac{1}{3}} > 1$. Using this one finds $k > q$ and $(1-k) < (1-q)$. *(Note that using derivates here is not appropriate as the function may not be differentiable, however, if the method carries, assuming differentiability, is correct, the M1 is awarded but not the A1 for accuracy of the argument.)*	M1	Statement, with justification, of the form for $x > 1$ one finds $x > x^{\frac{1}{3}} > 1$, or similar, that can be used to show how k and q can be related.
****	Therefore as $k > q$ and $(1-k) < (1-q)$ one finds $m > c$ as $a < b$.	A1	Note that showing $\frac{f(a)}{f(b)} > 1$ here is important.

Question			Answer	Marks	Guidance
4	**(i)**		(x_1, y_1) on $G_1 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{4} = 1.$	**B1**	For statement about (x_1, y_1) being on G_1
		*	Then $(x_2, y_2) = \left(\frac{x_1 - y_1}{\sqrt{2}}, \frac{x_1 + y_1}{\sqrt{2}}\right)$ and, using the inverse matrix, $(x_1, y_1) = \left(\frac{x_2 + y_2}{\sqrt{2}}, \frac{-x_2 + y_2}{\sqrt{2}}\right)$	**M1**	For getting (x_1, y_1) in terms of (x_2, y_2) Allow for $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$
		**	Therefore $$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$$ $$\Rightarrow \frac{\left(\frac{x_2}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x_2}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right)^2}{4} = 1$$ So that (x_2, y_2) is on G_2.	**A1**	Fully correct working to show that if (x_1, y_1) on G_1 then (x_2, y_2) is on G_2 Allow **BOD** for **A1** if **B0** If **B0** earlier then can allow **B1** here if they state something like "since (x_1, y_1) is on G_1 this is true" If the implication is shown in the "wrong direction" then allow **SC**

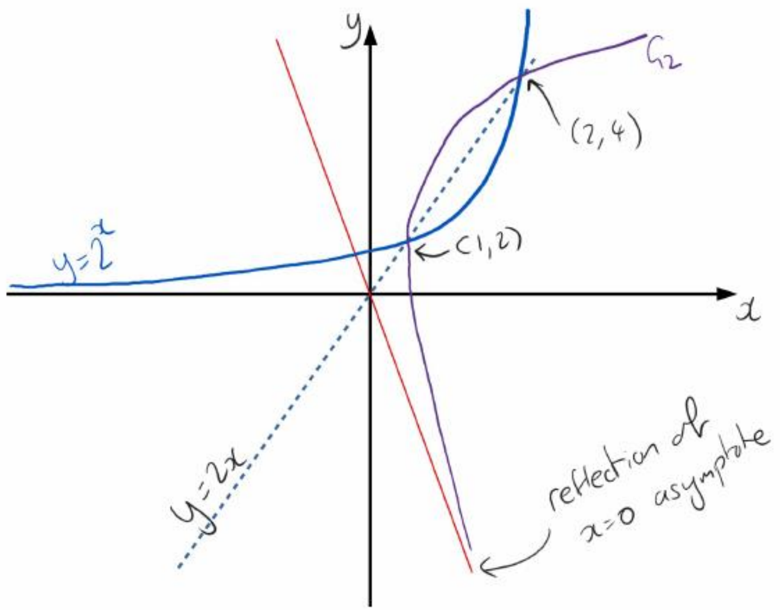
Question			Answer	Marks	Guidance
					B2 for using (x_1, y_1) on G_1 and (x_2, y_2) on G_2 to obtain given relationship for (x_2, y_2) in terms of (x_1, y_1). If they convincingly argue that the implication can be reversed then allow **SC B3**

ALT		*	We have $$\frac{\left(\frac{x_2}{\sqrt{2}}+\frac{y_2}{\sqrt{2}}\right)^2}{9}+\frac{\left(-\frac{x_2}{\sqrt{2}}+\frac{y_2}{\sqrt{2}}\right)^2}{4}$$ $$=\frac{\left(\frac{x_1-y_1}{2}+\frac{x_1+y_1}{2}\right)^2}{9}+\frac{\left(\frac{y_1-x_1}{2}+\frac{x_1+y_1}{2}\right)^2}{4}$$	**M1**	For taking G_2 equation and replacing (x_2, y_2) with expressions in (x_1, y_1)
		**	$$=\frac{(x_1)^2}{9}+\frac{(y_1)^2}{4}$$ $$=1$$	**A1**	For completing argument
			We know that the relationship $\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}$ means that the point (x_2, y_2) is an anticlockwise rotation of (x_1, y_1) by 45° about the origin. Therefore, every point on G_2 is a rotation of a point on G_1 and so G_2 is an anticlockwise rotation of G_1 about the origin through 45 degrees.	**B1**	Don't need to see "every point" Identifying matrix as relevant rotation matrix (no working needed) **and** stating conclusion for the graphs. If general rotation matrix seen must have cos / sin in correct places.
				[4]	

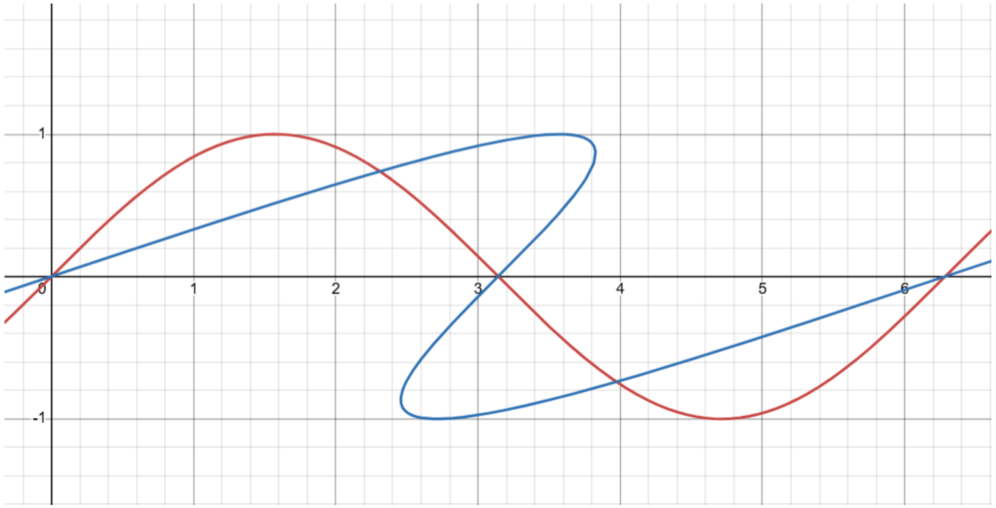
	(ii)	**(a)**	Need points (x, y) such that $\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$	**M1**	Setting up method for finding LOIP If Invariant lines need to see one more step of working before **M1**
			$\Leftrightarrow 0.8y = 1.6x \Leftrightarrow y = 2x.$	**A1**	Might see set up for invariant lines. Can have the **M1** but for **A1** need to see incorrect lines rejected If using general reflection matrix allow: **M1:** Setting up a quadratic equation in $t = \tan\theta$ **A1:** Correctly solving quadratic equation to obtain $y = 2x$ only (i.e. must reject one solution of quadratic equation)
				[2]	
		(b) *	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$	**M1**	Relationship between (x_1, y_1) and (x_2, y_2)

		**	If (x_1, y_1) is on $y = 2^x$ then we have: $$y_1 = 2^{x_1}$$ $$0.8x_2 + 0.6y_2 = 2^{-0.6x_2+0.8y_2}$$ and so (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$	**A1**	Do not need to see "if and only if" stated M1 A1 for convincingly establishing relationship between the two graphs Must show that (x_1, y_1) on $y = 2^x$ implies that (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$

ALT		*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$	**M1**	
		**	Consider $$0.8x_2 + 0.6y_2 = 2^{-0.6x_2+0.8y_2}$$ This is equivalent to: $$0.8(-0.6\,x_1 + 0.8y_1) + 0.6(0.8x_1 + 0.6y_1) = 2^{-0.6(-0.6\,x_1+0.8y_1)+0.8(0.8x_1+0.6y_1)}$$ i.e. $y = 2^x$ And so if (x_1, y_1) is on $y = 2^x$ then (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$	**A1**	M1 A1 for convincingly establishing relationship between the two graphs Must show that (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$ implies (x_1, y_1) on $y = 2^x$

			Therefore $y = 2^x$ is a reflection of $0.8x + 0.6y = 2^{-0.6x+0.8y}$ in $y = 2x$. 	**B3**	B1: $y = 2^x$ and what looks like a reflection in an oblique line with positive gradient (reflection line might not appear) Award even if looks like has been reflected in $y = x$ B1: Roughly correct asymptote as $y \to -\infty$ for G_2 shown and used. No need to see a line for asymptote but if shown must be through origin and looks like a reflection of x-axis B1: Two points of intersection shown between G_1 and G_2. Not necessary to find the coordinates but if labelled then they must be correct. Follow through for a maximum of **B2** possible if equation of line of reflection is wrong (as long as line of reflection is oblique). Must be using a reflection between the graphs (not rotation etc)
				[5]	

	(iii)	*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.	**M1**	Relationship between (x_1, y_1) and (x_2, y_2) Could see for inverse relationship $$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$ Allow **M1 BOD** for sight of one of the two correct matrices even if points are backwards
		**	If (x_1, y_1) is on $y = \sin x$ then we have: $$y_1 = \sin x_1$$ $$y_2 = \sin(x_2 - 2y_2)$$. And so (x_2, y_2) is on $y = \sin(x - 2y)$	**A1**	Do not need to see "if and only if" If **M1** awarded for inverse relationship then this part must be carefully explained. Must show that (x_1, y_1) is on $y = \sin x$ implies that (x_2, y_2) is on $y = \sin(x - 2y)$

ALT		*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.		
		**	Then consider $y_2 = \sin(x_2 - 2y_2)$ This is equivalent to $y_1 = \sin(x_1 + 2y_1 - 2y_2)$ i.e. $y_1 = \sin x_1$ And so if (x_1, y_1) is on $y = \sin x$ then (x_2, y_2) is on $y = \sin(x - 2y)$		Must show that (x_2, y_2) is on $y = \sin(x - 2y)$ implies that (x_1, y_1) is on $y = \sin x$
			 Note: only need range [0, 2π]	**B2**	Generosity needed. *y* = sin(*x*-2*y*) needs to be between -1 and 1 (or at least intended to be!), includes some *x* with multiple *y*, some vertical tangents in roughly the right places. **B1** for smooth, continuous graph with correct *x* intercepts and looks like it had been pushed to the right, and at some point it must "escape" *y* = sin *x*. **SC B1 for shear in wrong direction, must be between ±1 and have correct *x* intercepts**

			Differentiating $y = \sin(x - 2y)$ with respect to x gives $\frac{dy}{dx} = \left(1 - 2\frac{dy}{dx}\right)\cos(x - 2y)$ Therefore $\frac{dy}{dx}(1 + 2\cos(x - 2y)) = \cos(x - 2y) \Rightarrow \frac{dy}{dx} = \frac{\cos(x-2y)}{1+2\cos(x-2y)}$	**M1**	Recognisable attempt at differentiation using implicit and chain rules. Might be incorrect. Do not need to rearrange to $\frac{dy}{dx} =$ for the M mark
		*	So $\frac{dy}{dx} = 0 \Leftrightarrow x - 2y = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.	**M1**	Using $\frac{dy}{dx} = 0$ to get an equation in x and y Might see $2n\pi \pm \frac{\pi}{2}$ instead of two values
		**	When $x - 2y = \frac{\pi}{2}, y = \sin(x - 2y) = 1$ and $x = \frac{\pi}{2} + 2$ When $x - 2y = \frac{3\pi}{2}, y = \sin(x - 2y) = -1$ and $x = \frac{3\pi}{2} - 2$ Therefore points where $\frac{dy}{dx} = 0$ have form $\left(\frac{\pi}{2} + 2, 1\right)$ and $\left(\frac{3\pi}{2} - 2, -1\right)$.	**A1**	CAO Note: ONLY need the points within the given range

ALT		*	For previous two marks only $$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{\pi}{2}\\1\end{pmatrix}=\begin{pmatrix}\frac{\pi}{2}+2\\1\end{pmatrix}$$	**B1**	For correct point found via transformation of maximum of sin x
		**	$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{3\pi}{2}\\-1\end{pmatrix}=\begin{pmatrix}\frac{3\pi}{2}-2\\-1\end{pmatrix}$$	**B1**	For second correct point found via transformation **NOTE:** you can either award via **M1 A1** scheme or via **B1 B1** scheme.
			$y=\sin(2x-y)$ has infinite gradient when $\cos(x-2y)=-\frac{1}{2}\Leftrightarrow x-2y=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$	**M1**	Using $\frac{dy}{dx}=\infty$ to get an equation in x and y Might see more general cases
			When $x-2y=\frac{2\pi}{3}$, $y=\sin(x-2y)=\frac{\sqrt{3}}{2}$, so $x=\sqrt{3}+\frac{2\pi}{3}$ When $x-2y=\frac{4\pi}{3}$, $y=\sin(x-2y)=-\frac{\sqrt{3}}{2}$, so $x=-\sqrt{3}+\frac{4\pi}{3}$. So we have vertical tangent at points of form $\left(-\sqrt{3}+\frac{4\pi}{3},-\frac{\sqrt{3}}{2}\right)$ and $\left(\sqrt{3}+\frac{2\pi}{3},\frac{\sqrt{3}}{2}\right)$	**A1**	CAO
				[9]	

Question		Answer	Marks	Guidance
5	(i) *	Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. Then $\overrightarrow{OX} = \overrightarrow{OL} + \overrightarrow{LX} = \overrightarrow{OL} + \lambda\overrightarrow{LP} = \delta\mathbf{a} + \lambda((\alpha - \delta)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c})$ for some λ.	M1	Equation for $\overrightarrow{OX}$ Sight of correct line equation in **a, b, c** gets M1
	**	Also $\overrightarrow{OX} = \overrightarrow{OC} + \mu\overrightarrow{CB} = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ for some μ.	M1	Second equation for $\overrightarrow{OX}$ Could be $\mathbf{b} + \mu(\mathbf{c} - \mathbf{b})$, $(1 - \mu)\mathbf{b} + \mu\mathbf{c}$, $(1 - \mu)\mathbf{c} + \mu\mathbf{b}$
ALT	*	**First two marks** can be awarded as: Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. Then $\overrightarrow{LX} = \lambda\overrightarrow{LP} = \lambda(\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} - \delta\boldsymbol{a})$ for some λ.	M1	Can only award first two marks via first scheme or this scheme. i.e. cannot award one **M1** from each scheme.
	**	Also $\overrightarrow{LX} = \overrightarrow{LO} + \overrightarrow{OX} = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c}) - \delta\boldsymbol{a}$ for some μ	M1	Sight of correct line equation in **a, b, c** gets **M1**

Question		Answer	Marks	Guidance
		Equating coefficients gives the equations $\delta + \lambda(\alpha - \delta) = 0$, $\lambda\beta = \mu$ and $\lambda\gamma = 1 - \mu$. Adding the last two together leads to $\lambda = \frac{1}{\beta+\gamma}$ and then also $\mu = \frac{\beta}{\beta+\gamma}$.	**M1**	Attempt to solve simultaneous equations found by equating coefficients. Enough to have attempted just one of λ or μ in terms of β and γ
		So we have $\overrightarrow{OX} = \mathbf{c} + \frac{\beta}{\beta+\gamma}(\mathbf{b} - \mathbf{c}) = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$.	**A1**	For correct $\overrightarrow{OX}$ aef
		From the first of the three equations earlier $\delta = \frac{\lambda\alpha}{\lambda-1}$	**M1**	For valid method to attempt to find $\overrightarrow{OL}$
		$= \frac{\frac{\alpha}{\beta+\gamma}}{\frac{1}{\beta+\gamma}-1} = \frac{\alpha}{1-\beta-\gamma} = \frac{\alpha}{k+\alpha}$ as required.	**A1**	Fully correct **(AG)**

ALT		**All 6 marks in (i)** $\overrightarrow{OX} = [\overrightarrow{OC} + \mu\overrightarrow{CB}] = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ for some μ.	**M1**	Equation for $\overrightarrow{OX}$ any of $\mathbf{b} + \mu(\mathbf{c} - \mathbf{b})$, $(1-\mu)\mathbf{b} + \mu\mathbf{c}$, $(1-\mu)\mathbf{c} + \mu\mathbf{b}$
		Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. $\overrightarrow{LX} = -\delta\mathbf{a} + \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \lambda\overrightarrow{LX} = \delta\boldsymbol{a} + \lambda[-\delta\mathbf{a} + \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})]$	**M1**	Equation for $\overrightarrow{OP}$ Correct equation for $\overrightarrow{OP}$ implies previous method mark as well
		Equating coefficients gives: $\alpha = \delta(1-\lambda)$ $\beta = \lambda\mu$ $\gamma = \lambda(1-\mu)$ Adding the last two gives $\lambda = \beta + \gamma$ and so $\mu = \frac{\beta}{\lambda} = \frac{\beta}{\beta+\gamma}$	**M1**	Attempt to solve simultaneous equations found by equating coefficients. Enough to have attempted just one of λ or μ in terms of β and γ

		So we have $\overrightarrow{OX} = \mathbf{c} + \frac{\beta}{\beta+\gamma}(\mathbf{b} - \mathbf{c}) = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$.	**A1**	For correct $\overrightarrow{OX}$ aef
		From the first of the three equations earlier $\delta = \frac{\alpha}{1-\lambda}$	**M1**	For valid method to attempt to find $\overrightarrow{OL}$
		$= \frac{\alpha}{1-(\beta+\gamma)} = \frac{\alpha}{1-\beta-\gamma} = \frac{\alpha}{k+\alpha}$ as required.	**A1**	Fully correct **(AG)**
			[6]	

	(ii) *	LMN is horizontal if and only if the vectors $\overrightarrow{LM}$ and $\overrightarrow{LN}$ are both linear combinations of the horizontal vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$	**E1**	Characterisation of "*LMN* is horizontal" aef
	**	Suppose $\overrightarrow{LM}$ is a linear combination of the horizontal vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. Then $\overrightarrow{LM}=r(\mathbf{b}-\mathbf{a})+s(\mathbf{c}-\mathbf{a})$ for some real numbers r,s. Also $\overrightarrow{LM}=\overrightarrow{LO}+\overrightarrow{OM}=-\frac{\alpha}{k+\alpha}\mathbf{a}+\frac{\beta}{k+\beta}\mathbf{b}$.	**M1**	Setting up equations for either $\overrightarrow{LM}$ or $\overrightarrow{LN}$
	***	Equating coefficients gives $s=0$ and then $r=\frac{\alpha}{k+\alpha}=\frac{\beta}{k+\beta}$ so that $\frac{\alpha}{k+\alpha}=\frac{\beta}{k+\beta}$.	**M1**	Equating coefficients
ALT	*	**First 3 marks** *L* lies on *OA*, *M* on *OB* and *N* on *OC* so *LMN* will be parallel to *ABC* if and only if the ratios *OL* : *OA*; *OM* : *OB* and *ON* :*OC* are the same	**E1**	Must mention **all three** ratios for this mark. Withhold if explanation not convincing
	**	Therefore we need $\frac{\alpha}{k+\alpha}$: 1 and $\frac{\beta}{k+\beta}$:1 to be the same	**M1**	Only 2 ratios needed here
	***	Therefore $\frac{\alpha}{k+\alpha}=\frac{\beta}{k+\beta}$	**M1**	

ALT	*	**First 3 marks** *L* lies on *OA*, *M* on *OB* and *N* on *OC* so *LMN* will be parallel to *ABC* if and only if the line *LM* is parallel to *AB* and the line *LN* is parallel to *AC*	**E1**	Need **two** different lines considered here for E1. Withhold if explanation not convincing
	**	$$\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM} = \frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a} = \lambda\overrightarrow{AB}$$ And $\lambda\overrightarrow{AB} = \lambda(\mathbf{b} - \mathbf{a})$	**M1**	
	***	Therefore $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$	**M1**	
	*	$$\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$$ $$\alpha(k+\beta) = \beta(k+\alpha)$$ $$k\alpha = k\beta$$ And since $k \neq 0$ we have $\alpha = \beta$.	**A1FT**	Deducing $\alpha = \beta$ from $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$. Must have the previous M mark, but **M0 M1 A1FT** is OK Need to see $k \neq 0$ or other convincing argument for this mark

	**	A symmetrical argument with $\overrightarrow{LN} = \overrightarrow{LO} + \overrightarrow{ON} = -\frac{\alpha}{k+\alpha}\mathbf{a} + \frac{\gamma}{k+\gamma}\mathbf{c}$ shows that $\alpha = \gamma$ so that $\alpha = \beta = \gamma$ [and vector equation of the line *OP* can be written as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$]	**A1FT**	BOD "By symmetry we also have $\alpha = \gamma$" Allow if **A0** for previous mark through lack of $k \neq 0$ Deducing by symmetry $\alpha = \beta = \gamma$ enough for **A1** here **M0 M1 A0 A1 FT** is fine
ALT	*	**Previous two marks** $\mathrm{f}(x) = \frac{x}{k+x}$ has derivative $\mathrm{f}'(x) = \frac{k}{(k+x)^2}$ so as k > 0 this is a strictly increasing function	**A1FT**	For showing that this function is strictly increasing
	**	Therefore if $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta} = \frac{\gamma}{k+\gamma}$ then we must have $\alpha = \beta = \gamma$	**A1FT**	Allow even if did not earn previous A mark though lack of k > 0

		The equation of the plane ABC is $s = \mathbf{a} + x(\mathbf{b} - \mathbf{a}) + y(\mathbf{c} - \mathbf{a}) = (1 - x - y)\mathbf{a} + x\mathbf{b} + y\mathbf{c}$. The point of intersection of OP with the plane ABC is when all these coefficients are equal, i.e. $x = y = 1 - x - y$ (which has solution $x = y = \frac{1}{3}$) therefore has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	Solving for intersection of OP and ABC to show given result for $\overrightarrow{OG}$ Award for convincing argument that OP in the form as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$ intersects ABC at G BOD mention of centroid
		Conversely if OP intersects with the plane ABC at the point G with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ then we must have that $\alpha = \beta = \gamma$.	**M1**	Deducing $\alpha = \beta = \gamma$ in converse argument
	*	This means that $\overrightarrow{LM} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\beta}{k+\beta}\boldsymbol{b} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\alpha}{k+\alpha}\boldsymbol{b} = \frac{\alpha}{k+\alpha}(\boldsymbol{b} - \boldsymbol{a})$.	**M1**	For $\overrightarrow{LM}$ as a multiple of $\mathbf{b} - \mathbf{a}$.
	**	Similarly, $\overrightarrow{LN}$ is a multiple of $\boldsymbol{c} - \boldsymbol{a}$.	**M1**	For $\overrightarrow{LN}$ is a multiple of $\mathbf{c} - \mathbf{a}$.
	***	and so LMN is horizontal	**A1**	Conclusion

ALT	*	**Previous 3 marks** This means that $\overrightarrow{OL} = \frac{\alpha}{k+\alpha}\boldsymbol{a}$, $\overrightarrow{OM} = \frac{\alpha}{k+\alpha}\mathbf{b}$ and $\overrightarrow{ON} = \frac{\alpha}{k+\alpha}\mathbf{c}$	**M1**	Showing that the coefficients of the three vectors are the same
	**	So the ratio of the lengths *OL*:*OA*, *OL*:*OA* and *OL*:*OA* are all equal	**M1**	Attempt to show that proportions of distance travelled along OA etc are the same
	***	Therefore *LMN* is parallel to *ABC* and so *LMN* is horizontal	**A1**	Withhold if argument not convincing. Must be clear that *L* lies on line *OA*

		We have $\overrightarrow{OX} = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$, when $\beta = \gamma$, this gives $\overrightarrow{OX} = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{b}$,.	**M1**	Calculating $\overrightarrow{OX}$
		so *X* is the midpoint of *BC*.	**A1**	Allow M1 A1 for just stating X is the midpoint of BC (or equivalent for Y/Z)
		Similarly, *Y* is the midpoint of *AC* and *Z* is the midpoint of *AB*.	**B1**	
			[13]	
	(iii)	It's on the interior of the tetrahedron *OABC*.	**B1**	
			[1]	
		Additional Alternative for (ii)		
	(ii)	Plane *ABC* has normal $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})$ and so plane *LMN* will have to have a normal parallel to this	**E1**	Characterisation of "*LMN* is horizontal" aef
		LMN normal is given by $\overrightarrow{LM} \times \overrightarrow{LN} = \left(\frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a}\right) \times \left(\frac{\gamma}{k+\gamma}\mathbf{c} - \frac{\alpha}{k+\alpha}\mathbf{a}\right)$	**M1**	Attempt to find normal for LMN

		Expanding gives the normal as: $$n_1 = \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}$$ $$n_2 = \frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma}\mathbf{b}\times\mathbf{c} - \frac{\alpha}{k+\alpha}\frac{\gamma}{k+\gamma}\mathbf{a}\times\mathbf{c} - \frac{\alpha}{k+\alpha}\frac{\beta}{k+\beta}\mathbf{b}\times\mathbf{a}$$ Since normal are parallel we have $\frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma} = \frac{\alpha}{k+\alpha}\frac{\gamma}{k+\gamma}$	**M1**	Equating coefficients
		$$\frac{\alpha\gamma}{(k+\alpha)(k+\gamma)} = \frac{\beta\gamma}{(k+\beta)(k+\gamma)}$$ $$\alpha(k+\beta) = \beta(k+\alpha)$$ $$k\alpha = k\beta$$ And since $k \neq 0$ we have $\alpha = \beta$.	**A1**	Deducing $\alpha = \beta$ Need to see $k \neq 0$ or other convincing argument for this mark
		A symmetrical argument with $\frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma} = \frac{\alpha}{k+\alpha}\frac{\beta}{k+\beta}$ shows that $\alpha = \gamma$ so that $\alpha = \beta = \gamma$ [and vector equation of the line OP can be written as $\lambda(\mathbf{a}+\mathbf{b}+\mathbf{c})$]	**A1**	BOD "By symmetry we also have $\alpha = \gamma$" Independent to previous mark Deducing by symmetry $\alpha = \beta = \gamma$ enough for **A1** here

	*	The equation of the plane ABC is $s = \mathbf{a} + x(\mathbf{b} - \mathbf{a}) + y(\mathbf{c} - \mathbf{a}) = (1 - x - y)\mathbf{a} + x\mathbf{b} + y\mathbf{c}$. The point of intersection of OP with the plane ABC is when all these coefficients are equal, i.e. $x = y = 1 - x - y$ (which has solution $x = y = \frac{1}{3}$) therefore has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	Solving for intersection of OP and ABC to show given result for $\overrightarrow{OG}$ Award for convincing argument that OP in the form as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$ intersects ABC at G
ALT for prev B1	*	Equation of plane is given by **r.n** = **a.n** $[\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})] \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a})$ $\lambda[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $3\lambda[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $\lambda = \frac{1}{3}$ Therefore the point of intersection of OP with the plane ABC has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	
		Conversely if OP intersects with the plane ABC at the point G with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ then we must have that $\alpha = \beta = \gamma$.	**M1**	Deducing $\alpha = \beta = \gamma$ in converse argument
		This means that $\overrightarrow{LM} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\beta}{k+\beta}\boldsymbol{b} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\alpha}{k+\alpha}\boldsymbol{b} = \frac{\alpha}{k+\alpha}(\boldsymbol{b} - \boldsymbol{a})$.	**M1**	For $\overrightarrow{LM}$ as a multiple of $\mathbf{b} - \mathbf{a}$.

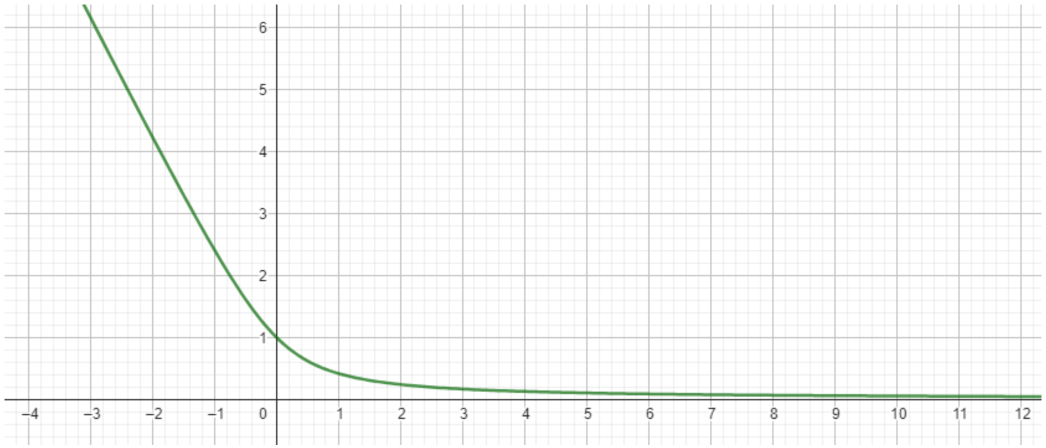
		Similarly, $\overrightarrow{LN}$ is a multiple of $\boldsymbol{c} - \boldsymbol{a}$.	**M1**	For $\overrightarrow{LN}$ is a multiple of $\mathbf{c} - \mathbf{a}$.
		and so *LMN* is horizontal as the normal to the plane *LMN* is parallel to the normal to plane *ABC*	**A1**	Conclusion
		We have $\overrightarrow{OX} = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$, when $\beta = \gamma$, this gives $\overrightarrow{OX} = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{b}$,.	**M1**	Calculating $\overrightarrow{OX}$
		so X is the midpoint of *BC.*	**A1**	
		Similarly, *Y* is the midpoint of *AC* and *Z* is the midpoint of *AB*.	**B1**	
			[13]	

Question		Answer	Marks	Guidance
6	(i)	If a, b, c are real and non-zero then we would have $a^2 + b^2 + c^2 > 0$ since each of $a^2, b^2, c^2 > 0$.	B1	
		We have that $2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 0$.	M1	For identity
		Therefore $ab + bc + ca = 0$.	A1	
	*	A polynomial with roots a, b, c is $z^3 - (a + b + c)z^2 + (ab + bc + ca)z - abc = 0$. In this case, this is $z^3 - abc = 0$. Each of a, b, c is a root of this so $a^3 = abc, b^3 = abc, c^3 = abc$.	M1	Common values of cubes
	**	Therefore $\|a\| = \|b\| = \|c\| = \sqrt[3]{\|abc\|}$.	A1	
			[5]	

Question	Answer	Marks	Guidance
	Alternative for last two marks of (i)		
*	$a^2 = a(-b-c) = bc, b^2 = ca, c^2 = ab$	**M1**	Or equivalent expression. May appeal to symmetry. One expression of this form only is insufficient for M1,
**	So e.g. $$\frac{\lvert a\rvert^2}{\lvert b\rvert} = \frac{\lvert b\rvert^2}{\lvert a\rvert} \Longrightarrow \lvert a\rvert = \lvert b\rvert = \lvert c\rvert$$	**A1**	Convincingly arriving at the given statement by rearranging. Must conclude about all three magnitudes being equal, not just 2 of 3.
(ii)	Suppose $a+b+c=0$ and $a^3+b^3+c^3=0$.		
	Since $a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2) - ab^2 - ac^2 - ba^2 - bc^2 - ca^2 - cb^2$	**M1**	For identity
	$0 = ab^2+ac^2+ba^2+bc^2+ca^2+cb^2$	**M1**	
	$= ab(b+a)+bc(c+b)+ac(a+c) = -abc-abc-abc = -3abc.$	**M1**	Use of $a+b+c=0$
	This means that $abc=0$,	**A1**	

Question		Answer	Marks	Guidance
		which cannot be the case as all of a, b, c are non-zero.	A1 [5]	
		Alternative for (ii)		
		$c = -(a+b)$, $so\ (a+b)^3 = a^3 + b^3$	M1	Using $a+b+c=0$ to simplify this
		$a^3 + b^3 + 3ab^2 + 3ab^2 = a^3 + b^3$ $ab(a+b) = 0$	M1	Expanding, $(a+b)^3$ expanded correctly (condone sign error $-(a+b)^3 = a^3 + b^3$) Factorised, = 0
		so $abc = 0$	A1	Correctly justified
		So one of $a, b, c = 0$, contradiction	A1	
	(iii)	$a + b = -(c + d)$. Therefore $(a+b)^3 = -(c+d)^3$ giving $a^3 + b^3 + 3ab(a+b) = -\left(c^3 + d^3 + 3cd(c+d)\right)$	M1 M1	Use of this

Question		Answer	Marks	Guidance
		Using $a^3+b^3=-c^3-d^3$ and $a+b=-(c+d)$ gives $(a+b)(ab-cd)=0$.	**A1**	For $(a+b)(ab-cd)=0$.
		If $a=-b$ then a and b have the same modulus and we are done.	**M1**	For this case
		Else $ab=cd$.	**M1**	
		The roots of the equation $x^2-(a+b)+ac=0$ are a and b.		
		But this equation is the same as $x^2+(c+d)x+cd=0$ which has roots $-c$ and $-d$		
		Therefore a is either $-c$ or $-d$ and we are done.	**A1**	
			[6]	
	(iv)	We need $a+b=-2$ and $a^3+b^3=-20$.	**M1**	
		Since $a^3+b^3=(a+b)(a^2-ab+b^2)$ we must also have that $a^2-ab+b^2=10$.	**M1**	
		Substituting $b=-2-a$ into this gives $a^2+a(a+2)+(a+2)^2=10$ or $3a^2+6a-6=0$	**M1**	
		Solving $a^2+2a-2=0$ give $a=-1\pm\sqrt{3}$ which leads to $b=-1\mp\sqrt{3}$	**A1**	Correct pair
			[4]	

Question			Answer	Marks	Guidance
7	(i)	*	$\frac{\sqrt{x^2+1}}{\lvert x\rvert}\sqrt{\frac{x^2+1}{x^2}} = \sqrt{1+\frac{1}{x^2}}$.	M1	Rearrange to expression with a 'small' term that can be expanded.
		**	Using the binomial expansion, where $\lvert x\rvert$ is large: $$\frac{\sqrt{x^2+1}}{\lvert x\rvert} = \sqrt{\frac{x^2+1}{x^2}} = \sqrt{1+\frac{1}{x^2}} \approx 1+\frac{1}{2}\times\frac{1}{x^2} = 1+\frac{1}{2\lvert x\rvert^2}$$. Multiplying both sides by $\lvert x\rvert$ gives the required result.	A1	
				G1	For generally correct (allow 'inflection points' but must have no minima/maxima) shape $x \geq -C, C \geq 0$, correct y-intercept, and $y \to 0$ as $x \to +\infty$
				G1	Looks like a straight line as $x \to -\infty$. (No need to explicitly say it looks like $y \approx -2x$, but if the asymptotic behaviour is

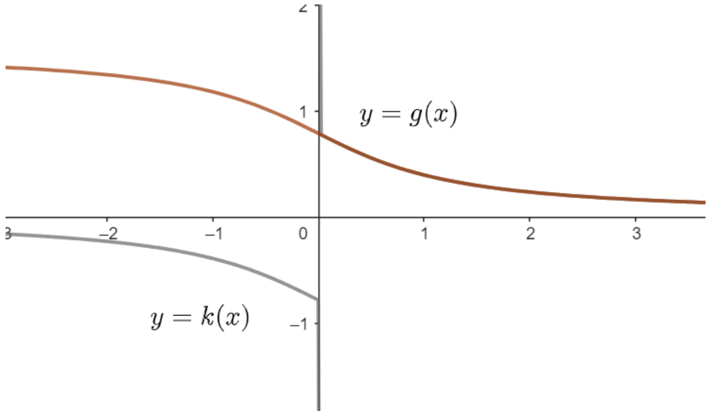
Question			Answer	Marks	Guidance
				[4]	explicitly incorrect, or written down wrong in working, say $y \approx -x$, then G0.)
			Alternative for first two marks		
		*	Note that $\sqrt{x^2+1} - \lvert x \rvert = \frac{1}{\sqrt{x^2+1}+\lvert x \rvert}$. Thus, $\sqrt{x^2+1} = \lvert x \rvert + \frac{1}{\sqrt{x^2+1}+\lvert x \rvert}$.	**M1**	
		**	For large $\lvert x \rvert$, we see $\sqrt{x^2+1} \approx \sqrt{x^2} = \lvert x \rvert$, and so $\sqrt{x^2+1} \approx \lvert x \rvert + \frac{1}{2\lvert x \rvert}$.	**A1**	
	(ii)	**(a)**	We have $f(x)f(-x) = \left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right) = 1$. For $0 < A, B < \frac{\pi}{2}$, we have $\tan A \tan B = 1 \Rightarrow \cos A \cos B - \sin A \sin B = 0 \Rightarrow \cos(\mathrm{A} + \mathrm{B}) = 0 \Rightarrow A + B = \frac{\pi}{2}$.	**M1**	Simplifying $f(x)f(-x)$.
			Since $g(x) \in (0, \frac{\pi}{2})$ for all x, $f(x)f(-x) = \tan g(x) \tan g(-x) = 1$ implies $g(x) + g(-x) = \frac{\pi}{2}$.	**A1** **[2]**	Correct use of suitable trig identity.

Question	Answer	Marks	Guidance
	Alternative(ii)		
(ii) (a)	Using the tangent addition formula, we have $\tan\big(g(x)+g(-x)\big) = \frac{\tan(g(x))+\tan(g(-x))}{1-\tan(g(x))\tan(g(-x))}$. The numerator is equal to $f(x)+f(-x) = 2\,(x^2+1)^{1/2} > 0$. But the denominator is equal to $1-\left((x^2+1)^{\frac{1}{2}}-x\right)\left((x^2+1)^{\frac{1}{2}}+x\right) = 0$. Therefore, $\tan\big(g(x)+g(-x)\big) = \frac{a}{b}$ with $a>0, b=0$.	**M1**	Any attempt at showing the denominator is 0 i.e. evaluating $f(x)f(-x)$. Allow the use of the 'arctan addition formula' for this M1.
	We also have $f(x) > 0 \Rightarrow 0 < g(x) < \frac{\pi}{2}$, and similarly $0 < g(-x) < \frac{\pi}{2}$. Hence, we have $0 < g(x)+g(-x) < \pi$, and therefore $g(x)+g(-x) = \frac{\pi}{2}$.	**A1**	Correct expression for $\tan\,(g(x)+g(-x))$ and correct use of the range of $g(x)+g(-x)$. A0 for no justification. Allow "$\tan\big(g(x)+g(-x)\big) = \infty$" $\Rightarrow g(x)+g(-x) = (2n+1)\frac{\pi}{2}$. If using the arctan addition formula, allow 'the argument of arctan is infinite, so it must be

Question		Answer	Marks	Guidance
				$(2n+1)\frac{\pi}{2}$ for some n' or equivalent.
		Second alternative(ii)		
(ii)	**(a)**	$f(x)f(-x) = (\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x) = 1$ For $u > 0$, we have $\tan^{-1} u + \tan^{-1}\frac{1}{u} = \frac{\pi}{2}$,	**M1**	Any attempt to show $f(-x) = f(x)^{-1}$ May be stated without proof
		And so, since $f(x) > 0$, $g(x) + g(-x) = \frac{\pi}{2}$	**A1**	Must state that $f(x) > 0$ if using the identity. If identity is "proven" by drawing a suitable right-angled triangle and summing the angles, $f(x) > 0$ need not be explicitly stated and A1 can be given
	(b)	$k(x) + k(-x) = \frac{1}{2}\tan^{-1}\frac{1}{x} + \frac{1}{2}\tan^{-1}\left(-\frac{1}{x}\right) = \frac{1}{2}\tan^{-1}\frac{1}{x} - \frac{1}{2}\tan^{-1}\left(\frac{1}{x}\right) = 0$	**E1** **[1]**	Must see at least this line of working. If differentiating or using trig identities, must be fully correct.

Question		Answer	Marks	Guidance
	(c)	For $x > 0$, $\tan 2k(x) = x^{-1}$.	**M1**	
		Using the tan double angle formula, $\frac{2\tan k(x)}{1-(\tan k(x))^2} = \tan 2k(x) = x^{-1}$, we then have a quadratic for $k(x)$: $(\tan k(x))^2 + 2x \tan k(x) - 1 = 0$, so that $\tan k(x) = -x \pm \sqrt{1+x^2}$.	**M1**	Use of double angle formula.
		For $x > 0$, $\tan k(x) > 0$, so $\tan k(x) = \sqrt{1+x^2} - x = f(x) = \tan(\tan^{-1} f(x)) = \tan g(x)$.	**M1** **A1** **[4]**	Attempt to solve for general solution to quadratic. (Completing the square is enough.) Correct justification.
		Alternative for (ii)(c)		
(ii)	**(c)**	$\tan k(x) = \frac{\sin \frac{1}{2} \tan^{-1} \frac{1}{x}}{\cos \frac{1}{2} \tan^{-1} \frac{1}{x}}$ $= \frac{\sin \tan^{-1} \frac{1}{x}}{\cos \tan^{-1} \frac{1}{x} + 1}$	**M1**	Use of double (or half) angle formulae

Question	Answer	Marks	Guidance
	$= \dfrac{\frac{1}{\sqrt{x^2+1}}}{\frac{x}{\sqrt{x^2+1}}+1}$ $= \dfrac{1}{\sqrt{x^2+1}+x}$	M1	Attempt to evaluate $\sin\tan^{-1}\frac{1}{x}$ or $\cos\tan^{-1}\frac{1}{x}$ e.g. using a right-angled triangle
	$= \sqrt{x^2+1} - x$	M1	Rationalisation of fraction.
	$= \tan g(x)$	A1	Fully correct argument; withhold if half-angle formulae are used without justification of signs, i.e. $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

Question	Answer	Marks	Guidance
(d)		G1	Generally correct shape (as in guidance for (i)) for $g(x)$, correct asymptotes.
		G1	$k(x) = g(x)$ for $x > 0$.
		G1	Correct k(x) when $x < 0$, must be $g(x)$ translated by $\frac{\pi}{2}$, i.e. $g(x)$ for $x < 0$ must also be correct. (Using $g(x) = k(x) + \frac{\pi}{2}$ for $x < 0$.) (Special case: if $k(x)$ correct but sketch gets 0 marks as above for incorrect $g(x)$, give G1 G0 G0.)
		[3]	

Question			Answer	Marks	Guidance
		(e)	$\int_0^1 k(x)dx = \frac{1}{2}\int_0^1 \tan^{-1}\frac{1}{x}dx = \frac{1}{2}\left(\left[x\tan^{-1}\frac{1}{x}\right]_0^1 + \int_0^1 \frac{x}{x^2+1}dx\right)$	M1	Use of integration by parts
			$= \frac{1}{2}\left(\frac{\pi}{4} + \left[\frac{1}{2}\ln(x^2+1)\right]_0^1\right)$	M1	Integrating using log.
			$= \frac{\pi + 2\ln 2}{8}$	A1	Fully correct
			We have that $k(x) = g(x)$ for $x > 0$. From (ii)(a), we have, for $x > 0$, $g(-x) = \frac{\pi}{2} - k(x)$.	M1	Use of (ii)(a) with $k(x) = g(x)$ for $x > 0$, or use of sketch in (d).
			Integrating, we get $\int_{-1}^0 g(x)dx = \int_0^1 g(-x)\,dx = \frac{\pi}{2} - \int_0^1 k(x)\,dx$	A1	Correct integral expression, dependent only on previous M.
			$= \frac{\pi}{2} - \frac{\pi + 2\ln 2}{8} = \frac{3\pi - 2\ln 2}{8}$	A1	Allow FT for $\int_0^1 k(x)\,dx$.
				[6]	Since the question says 'write down the value', $\int_{-1}^0 g(x)dx = \frac{\pi}{2} - I$

Question			Answer	Marks	Guidance
					where I is the computed (possibly incorrect) value for $\int_0^1 k(x)\, dx$, gets M1 BOD A1 A1 (FT).

Question		Answer	Marks	Guidance
8	(i)	$\left(z-\frac{1}{z}\right)\left(z^m+\frac{1}{z^m}\right)+\left(z^{m-1}-\frac{1}{z^{m-1}}\right)=z^{m+1}+\frac{1}{z^{m-1}}-z^{m-1}-\frac{1}{z^{m+1}}+z^{m-1}-\frac{1}{z^{m-1}}$	B1	Correct either side and correct six terms.
		$=z^{m+1}-\frac{1}{z^{m+1}}$		
		When $n=1$ the LHS is $z^2-\frac{1}{z^2}$. The RHS is $\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}\right)$ and these are equal.	M1	Base case
		We then have		
		$z^{2k+2}-\frac{1}{z^{2k+2}}=\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z^{2k}-\frac{1}{z^{2k}}\right)$		
		By induction this equals		
		$\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z-\frac{1}{z}\right)\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)$		
		$=\left(z-\frac{1}{z}\right)\left[z^{2k+1}+\frac{1}{z^{2k+1}}++\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)\right]$		

Question	Answer	Marks	Guidance
	$= \left(z - \frac{1}{z}\right)\left[\sum_{r=1}^{k+1}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right)\right]$	**M1**	Induction hypothesis (seen or implied)
	So the result is true by induction.	**A1**	Fully correct working
	For other result, first, it can be seen that		
	$\left(z + \frac{1}{z}\right)\left(z^m - \frac{1}{z^m}\right) + \left(\frac{1}{z^{m-1}} - z^{m-1}\right) = z^{m+1} - \frac{1}{z^{m+1}}$	**M1**	
	Base case is easy to check. Then we have $z^{2k+2} - \frac{1}{z^{2k+2}} = \left(z + \frac{1}{z}\right)\left(z^{2k+1} - \frac{1}{z^{2k+1}}\right) + \left(z^{2k} - \frac{1}{z^{2k}}\right)$ By induction this is		

Question		Answer	Marks	Guidance
		$\left(z+\frac{1}{z}\right)\left(z^{2k+1}-\frac{1}{z^{2k+1}}\right)+\left(z+\frac{1}{z}\right)\sum_{r=1}^{k}(-1)^{r+k}\left(z^{2r-1}-\frac{1}{z^{2r-1}}\right)$ $=\left(z+\frac{1}{z}\right)\left[\sum_{r=1}^{k+1}(-1)^{r+k+1}\left(z^{2r-1}-\frac{1}{z^{2r-1}}\right)\right]$	A1 [6]	Check that the alternating sign is handled correctly
(ii)	(a)	Taking $z=e^{i\theta}$ in the first identity in (i) gives $2i\sin 2n\theta = 2i\sin\theta\sum_{r=1}^{n}2\cos(2r-1)\theta$ $\sin(2n\theta)=2\sin(\theta)\sum_{r=1}^{n}\cos\big((2r-1)\theta\big)$	M1 A1 [2]	
	(b)	Taking $n=2$ and $\theta=\frac{\pi}{5}$ in this gives $\sin\frac{4\pi}{5}=2\sin\frac{\pi}{5}(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})$. Since $\sin\frac{4\pi}{5}=\sin\frac{\pi}{5}$ this gives $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$	M1 M1	Implied by disappearance of factor involving sin from each

Question	Answer	Marks	Guidance
			side as the only change.
	and the result follows because $\cos\frac{2\pi}{5} = -\cos\frac{3\pi}{5}$	**A1**	Need to state (AG so strict here). Need to give cos justification.
		[3]	
(c)	Taking $n = 7$ and $\theta = \frac{\pi}{15}$ gives $\sin\frac{14\pi}{15} = 2\sin\frac{\pi}{15}(\cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cos\frac{5\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{9\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15})$	**M1**	Using $\theta = \frac{2\pi}{15}$ can also work – award M1 here but check other values as subsequent working is more involved
	So, since $\sin\frac{14\pi}{15} = \sin\frac{\pi}{15}$,	**M1**	Implied by disappearance of factor involving sin from each

Question	Answer	Marks	Guidance
	$\cos\frac{\pi}{15}+\cos\frac{3\pi}{15}+\cos\frac{5\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{9\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=\frac{1}{2}$		side as the only change.
	Since $\cos\frac{3\pi}{15}=\cos\frac{6\pi}{15}+\frac{1}{2}$ from previous answer and $\cos\frac{5\pi}{15}=\frac{1}{2}$, $\cos\frac{\pi}{15}+\frac{1}{2}+\cos\frac{6\pi}{15}+\frac{1}{2}+\cos\frac{7\pi}{15}+\cos\frac{9\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=\frac{1}{2}$	**M1**	Needs to be clear.
	Now $\cos\frac{6\pi}{15}+\cos\frac{9\pi}{15}=0$ so that $\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=-\frac{1}{2}$	**M1**	Needs to be clear.
	So $-\cos\frac{16\pi}{15}-\cos\frac{8\pi}{15}-\cos\frac{4\pi}{15}-\cos\frac{2\pi}{15}=-\frac{1}{2}$ giving the result	**A1**	Need to explain use of identities.
		[5]	

Question			Answer	Marks	Guidance
	(iii)		Taking $z = e^{i\theta}$ in the other identity: $z^{2n} - \frac{1}{z^{2n}} = \left(z + \frac{1}{z}\right)\sum_{r=1}^{n}(-1)^{r+n}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$ Gives $2i\sin 2n\theta \ = \ 2\ \cos\theta\,(2i\sin(2n-1)\,\theta - 2i\sin(2n-3)\,\theta + \cdots + (-1)^{n+1}\,2i\sin\theta\ )$	M1	Note FT allowed for three M marks but their incorrect identity must be sufficiently similar.
			So $\sin 2n\theta \ = \ 2\ \cos\theta\,(\sin(2n-1)\,\theta - \sin(2n-3)\,\theta + \cdots + (-1)^{n+1}\sin\theta\ )$	M1	
			Now taking $n = 3$ and $\theta = \frac{\pi}{14}$ gives $\sin\frac{6\pi}{14} = 2\cos\frac{\pi}{14}(\sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14})$	M1	Taking $n = 3$ and $\theta = \frac{\pi}{7}$ can also work.
			Since $\sin\frac{6\pi}{14} = \cos\frac{\pi}{14}$ this gives the result.	A1	
				[4]	

Question		Answer	Marks	Guidance
9	(i)	y-coordinate of centre of mass is given by $\frac{A}{B}$ where $A = \pi\int_0^r x^2 y dy$ and $B = \pi\int_0^r x^2 dy$	M1	Correct A and B, including ratio – must be x^2dy not y^2dx
		$r^{n-1}y = r^n - x^n \Leftrightarrow x^2 = r^{\frac{2n-2}{n}}(r-y)^{\frac{2}{n}}$ So $\frac{A}{\pi r^{\frac{2n-2}{n}}} = \int_0^r (r-y)^{\frac{2}{n}} y dy = \left[-\frac{ny}{2+n}(r-y)^{\frac{2+n}{n}}\right]_0^r + \frac{n}{2+n}\int_0^r (r-y)^{\frac{2+n}{n}} dy$ $= 0 + \frac{n}{2+n} \times \left[-\frac{n}{2+2n}(r-y)^{\frac{2+2n}{n}}\right]_0^r = \frac{n^2 r^{\frac{2+2n}{n}}}{(2+n)(2+2n)}$	M1	Correct integral and by parts attempted
		Giving $A = \frac{\pi r^{\frac{2n-2}{n}} n^2 r^{\frac{2+2n}{n}}}{(2+n)(2+2n)} = \frac{\pi r^4 n^2}{(2+n)(2+2n)}$	A1	
		Then $\frac{B}{\pi r^{\frac{2n-2}{n}}} = \int_0^r (r-y)^{\frac{2}{n}} dy = \left[-\frac{n}{2+n}(r-y)^{\frac{2+n}{n}}\right]_0^r = \frac{nr^{\frac{2+n}{n}}}{2+n}$	M1	Integral attempted using a method that works

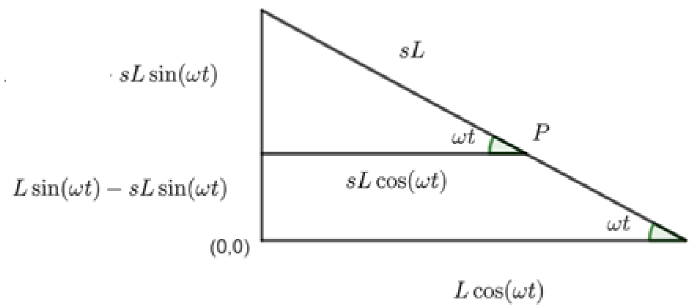
Question	Answer	Marks	Guidance
	So that $B = \frac{nr^{\frac{2+n}{n}}\pi r^{\frac{2n-2}{n}}}{2+n} = \frac{nr^3\pi}{2+n}$	**A1**	
	Then $\frac{A}{B} = \frac{nr}{2(n+1)}$	**A1** **[6]**	Allow FT in the special case where they cancel incorrect minus signs in the ratio, provided they have been penalised already.
	Alternative for (i), using shell formulae.		
(i)	y-coordinate of centre of mass is given by $\frac{A}{B}$ where $A = 2\pi\int_0^r yx \cdot \frac{y}{2}\,dx$ and $B = 2\pi\int_0^r yx\,dx$ (shell formula)	**M1**	Correct A and B, including ratio
	$\frac{A}{\pi r^{2n-2}} = \int_0^r r^{2n}x - 2r^n x^{n+1} + x^{2n+1}\,dy = r^{2n+2}\left[\frac{1}{2} - \frac{2}{n+2} + \frac{1}{2n+2}\right]$	**M1**	Square and integrate term by term

Question	Answer	Marks	Guidance
	$A = \dfrac{\pi r^4 n^2}{(n+2)(2n+2)}$	A1	
	$B = \pi \int_0^r rx - \dfrac{x^{n+1}}{r^{n-1}} dx = \pi r^3 \left[\dfrac{1}{2} - \dfrac{1}{n+2}\right]$	M1	Substitute in and integrate term by term
	$B = \dfrac{\pi n r^3}{n+2}$	A1	
	Then $\dfrac{A}{B} = \dfrac{nr}{2(n+1)}$	A1	
(ii)	$y = r - \dfrac{x^n}{r^{n-1}}$ So that $\frac{dy}{dx} = -\frac{nx^{n-1}}{r^{n-1}}$, at $(rp, r(1-p^n)$ we have $\frac{dy}{dx} = -\frac{n(rp)^{n-1}}{r^{n-1}} = -np^{n-1}$	M1	Calculate and try to plug in $x = rp$
	So the equation of the normal is $y - r(1-p^n) = \frac{x-rp}{np^{n-1}}$ Setting $x = 0$ gives $y = \frac{-rp}{np^{n-1}} + r(1-p^n) = r\left(1 - p^n - \frac{1}{np^{n-2}}\right)$	A1	Correctly finding Y

Question	Answer	Marks	Guidance
	Need the max value of this when $n = 4$ as p varies with $0 \leq p < 1$. When $n = 4$ we have $r\left(1 - p^4 - \frac{1}{4p^2}\right)$ Differentiating with respect to p gives $-4rp^3 + \frac{r}{2p^3}$	M1	Differentiating and setting equal to zero
	Setting this equal to zero gives $p^6 = \frac{1}{8}$ so that $p^2 = \frac{1}{2}$	A1	Finding p at the maximum ***and*** getting $Y = \frac{r}{4}$
	Substituting this back into the expression for Y gives $Y = r\left(1 - \frac{1}{4} - \frac{1}{2}\right) = \frac{r}{4}$ (nearby gradient shows this will correspond to a max, derivative decreases as positive p increases).	E1	Need to justify it gives a max, sketch provided for info, not required by question.
		[5]	
(iii)	"Mass" is given by $\frac{nr^3\pi}{2+n}$. So mass of "top part" is $\frac{2r^3\pi}{3}$. Mass of "bottom part" is $\frac{r^3\pi}{2}$.	B1	
	Centre of mass of top part is at $\frac{2r}{5}$, bottom part is at $-\frac{r}{3}$	B1	Needs to be a negative value or "below x-axis"

Question	Answer	Marks	Guidance
	So centre of mass of whole solid is at $$\frac{\left(\frac{2r^3\pi}{3}\times\frac{2r}{5}+\frac{1r^3\pi}{2}\times-\frac{r}{3}\right)}{\frac{2r^3\pi}{3}+\frac{r^3\pi}{2}}$$	**M1**	Attempting to average and get the centre of mass
	$=\frac{6r}{70}$.	**A1**	
	Need normal to meets y-axis there. For top part need $r\left(1-p^4-\frac{1}{4p^2}\right)=\frac{6r}{70}$. Note that RHS is less than maximum value of LHS, so we can expect solutions.	**M1**	
	For this we need $1-p^4-\frac{1}{4p^2}=\frac{6}{70}\Leftrightarrow 128p^2-140p^6-35=0\Leftrightarrow 0=140p^6-128p^2+35$.	**A1**	Must multiply up by p^2
	There are two such positive values of p^2 leading to two possible positive values of p. e.g. it's a cubic in p^2, and when $p=0$ or $p=1$, $140p^6-128p^2+35>0$, but when $p=\frac{1}{\sqrt{2}}$ we have $140p^6-128p^2+35<0$, meaning there are two roots in $0<p<1$.	**E1**	Any justification of this
	For the bottom part need $-r\left(1-p^2-\frac{1}{2}\right)=\frac{6r}{70}$.	**M1**	Must include the minus sign for the method mark

Question	Answer	Marks	Guidance
	For this we need $p^2+\frac{1}{2}-1=\frac{6}{70} \Leftrightarrow 70p^2-41=0 \Leftrightarrow p=\sqrt{\frac{41}{70}}$.	A1	Need to make clear the only possible value is the positive root (can just ignore the negative one)
		[9]	

Question		Answer	Marks	Guidance
10	(i)	$\frac{d^2h}{dt^2} = -\omega^2 h$ gives general solution of $h(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}$ where α, β are complex numbers leading general real-valued solution of $h(t) = A\cos\omega t + B\sin\omega t$.	M1	
		Using $h(0) = 0$ and $\frac{dh}{dt} = \omega L$ gives $A = 0, B = L$ so that $h(t) = L\sin\omega t$	A1	
		 The x coordinate of P is $sL\cos\omega t$.	M1	Needs diagram, or equivalent (AG) - can include mention of similar triangles. Can be implied if both x and y coordinates found correctly.
		The y coordinate of P is $(1-s)L\sin\omega t$.	A1	
			[4]	
	(ii)	$x(t) = sL\cos\omega t \Rightarrow \ddot{x}(t) = -sL\omega^2\cos\omega t$	B1	

Question	Answer	Marks	Guidance
	$y(t) = (1-s)L\sin\omega t \Rightarrow \ddot{x}(t) = (s-1)L\omega^2\sin\omega t$	**B1**	
		[2]	
(iii)	Resolving horizontally, taking right as positive, gives	**M1**	Attempt to resolve horizontally, which must include N, F and an acceleration. Trig functions can be wrong.
	$N\sin\omega t - F\cos\omega t = -msL\omega^2\cos\omega t$ (1)	**A1**	
	Resolving vertically, taking up as positive, gives	**M1**	Attempt to resolve vertically, which must include N, F and an acceleration. Trig functions can be wrong.
	$N\cos\omega t - mg + F\sin\omega t = m(s-1)L\omega^2\sin\omega t$ (2)	**A1**	

Question	Answer	Marks	Guidance
	$(1) \times \sin wt + (2) \times \cos \omega t$ gives $N - mg\cos\omega t = -msL\omega^2\cos\omega t\sin\omega t - m(1-s)L\omega^2\cos\omega t\sin\omega t$	**M1**	Taking a correct combination that would eliminate F, but can be algebra errors. If F was found first, this mark is gained by substituting F into (2) provided they rearrange so that N appears on its own and not multiplied by anything trig.
	So that $N = mg\cos\omega t - mL\omega^2\cos\omega t\sin\omega t = mg\left(\cos\omega t - \frac{L\omega^2}{g}\cos\omega t\sin\omega t\right)$ $= mg\left(1 - \frac{L\omega^2}{g}\sin\omega t\right)\cos\omega t = mg(1 - k\sin\omega t)\cos\omega t$ where $k = \frac{L\omega^2}{g}$.	**A1**	

Question	Answer	Marks	Guidance
	Then $(2) \times \sin wt - (1) \times \cos \omega t$ gives $-m(1-s)Lw^2 \sin^2 \omega t + msL\omega^2 \cos^2 \omega t = F - mg \sin \omega t$	**M1**	Taking the correct combination to eliminate N. Alternatively, divide (1) by cos ωt to get $F = \ldots$
	Therefore $F = mg \sin \omega t - mL\omega^2 \sin^2 \omega t + msL\omega^2 = msgk + mg \sin \omega t\,(1 - k \sin \omega t)$ $= mgsk + mg \cos \omega t \tan \omega t\,(1 - k \sin \omega t) = mgsk + N \tan \omega t$	**A1**	
		[8]	

Question	Answer	Marks	Guidance
	Alternative (iii), resolving perpendicular and parallel to slope		
(iii)	Perpendicular (away from plank) component of acceleration is $a_{perp} = -sL\omega^2 \cos\omega t \sin\omega t + (s-1)L\omega^2 \sin\omega t \cos\omega t = -L\omega^2 \sin\omega t \cos\omega t$	**M1** **A1**	Attempts to resolve each component of acceleration found earlier into perp direction – e.g. could have trig functions mixed up but still get M1
	Parallel (down the plank) component of acceleration is $a_{par} = -sL\omega^2 \cos^2\omega t - (s-1)L\omega^2 \sin^2\omega t = -L\omega^2(s - \sin^2\omega t)$	**M1** **A1**	Similar to the above comment
	Resolving perpendicular to the plank gives $N - mg\cos\omega t = -mL\omega^2 \cos\omega t \sin\omega t$ $N = mg\left(1 - \frac{L\omega^2}{g}\sin\omega t\right)\cos\omega t$ which produces the desired result when $k = \frac{L\omega^2}{g}$.	**M1** **A1**	Must have N and gravity component equal to an acceleration for the M1

Question	Answer	Marks	Guidance
	Resolving parallel gives $mg\sin\omega t - F = -mL\omega^2(s - \sin^2\omega t)$ $F = mgsk + mg(1 - k\sin\omega t)\sin\omega t = mgsk + N\tan\omega t$	M1 A1	Must have F and gravity component equal to an acceleration for the M1
(iv)	Since does not slip initially we have $F < \mu N$ when $t = 0$.	M1	
	This is if and only if $mgsk < mg\tan\alpha \Leftrightarrow sk < \tan\alpha$	A1	
	NTS that $F = \mu N$ and $N > 0$ at some time t when the angle the plank makes with the horizontal (hereafter known as the 'angle of the plank') is less than α.		
	Consider $t = \frac{\alpha}{\omega}$.	B1	Identifying $t = \frac{\alpha}{\omega}$.
	When $t = \frac{\alpha}{\omega}$ we have $F = mgsk + N\tan\alpha$, so that $F - \mu N = mgsk + N\tan\alpha - N\tan\alpha = mgsk > 0$	M1	Calculate F at the relevant t and compare it to μN

Question	Answer	Marks	Guidance
	So, the required frictional force is greater than the available frictional force and the particle must have slipped at some earlier time, and therefore at some angle smaller than α to the horizontal.	**A1**	Correctly shown the above and a clear statement showing the particle slips and second bullet point is shown

Question	Answer	Marks	Guidance
	So, if $sk < \tan\alpha$ at time $t = 0$ we have $F < \mu N$ and at time $t = \frac{\alpha}{\omega}$ we have $F > \mu N$. Therefore there is a time t^* with $0 < t^* < \frac{\alpha}{\omega}$ such that $F = \mu N$. We have $0 < \omega t^* < \alpha$ and so $0 < \tan\omega t^* < \tan\alpha$. Earlier equation: $F(t) = mgsk + N(t)\tan\omega t$ Therefore $\frac{F(t) - \mu N(t)}{mg} = \frac{F(t) - \tan\alpha\, N(t)}{mg} = sk + \frac{N(t)(\tan\omega t - \tan\alpha)}{mg}$ And at $t = t^*$ this gives $0 = sk + \frac{N(t^*)(\tan\omega t^* - \tan\alpha)}{mg}$ leading to $N(t^*) = \frac{-skmg}{\tan\omega t^* - \tan\alpha} > 0.$ (Numerator and denominator both negative.)	**E1** **[6]**	Justification of second bullet point in question.

Question		Answer	Marks	Guidance
11	(i)	$F(x) = \int_0^x \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}$	B1	
		$G(y) = (P(X_1) \leq y) \times (P(X_2) \leq y) \times \ldots \times (P(X_n) \leq y)$	M1	Must see either the given product or $G(y) = P(X_i \leq y \text{ for all } i)$ (OE)
		$= F(y)^n = \left(1 - e^{-\lambda y}\right)^n$	A1	**AG**
			[3]	
	(ii)	$P\big(Y < L(\alpha)\big) = \alpha \Leftrightarrow G\big(L(\alpha)\big) = \alpha \Leftrightarrow \left(1 - e^{-\lambda L(\alpha)}\right)^n = \alpha \Leftrightarrow e^{-\lambda L(\alpha)} = 1 - \alpha^{\frac{1}{n}}$	M1	For $e^{-\lambda L(\alpha)} = 1 - \alpha^{\frac{1}{n}}$ OE (must take n^{th} root)
		$\Leftrightarrow L(\alpha) = -\frac{1}{\lambda} \ln\left(1 - \alpha^{\frac{1}{n}}\right)$	A1	**AG**
		$\mathrm{P}\big(Y > U(\alpha)\big) = \alpha \Leftrightarrow 1 - \mathrm{G}\big(U(\alpha)\big) = \alpha \Leftrightarrow 1 - \left(1 - e^{-\lambda U(\alpha)}\right)^n = \alpha \Leftrightarrow e^{-\lambda U(\alpha)} = 1 - (1 - \alpha)^{\frac{1}{n}}$	M1	For $e^{-\lambda U(\alpha)} = 1 - (1-\alpha)^{\frac{1}{n}}$ OE (must take n^{th} root)

Question		Answer	Marks	Guidance
		$\Leftrightarrow U(\alpha) = -\frac{1}{\lambda}\ln\left(1-(1-\alpha)^{\frac{1}{n}}\right)$	**A1** **[4]**	Correct statement without working earns M1 A1
	(iii)	$\alpha^{\frac{1}{n}} = e^{\frac{\ln\alpha}{n}} \approx 1 + \frac{\ln\alpha}{n}.$ This gives $-\lambda L(\alpha) = \ln\left(1-\alpha^{\frac{1}{n}}\right) \approx \ln\left(-\frac{\ln\alpha}{n}\right) = \ln(-\ln\alpha) - \ln n$	**M1** **M1**	For correct use of given approximation Substituting approx for $\alpha^{\frac{1}{n}}$ into expression for $L(\alpha)$ Can be awarded for those who work backwards, provided that it is clearly stated that logic is reversible

Question		Answer	Marks	Guidance
		which gives $\lambda L(\alpha) \approx \ln n - \ln(-\ln\alpha) = \ln n - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right)$	A1 **[3]**	**AG** A0 if candidates use $=$ instead of $\approx$ either here or in working
	(iv)	Median of Y is $L\left(\frac{1}{2}\right)$ $\approx \frac{1}{\lambda}\left[\ln n - \ln\left(\ln\frac{1}{\alpha}\right)\right]$, which tends to infinity as n increases.	**M1** **A1**	Stating Median $\geq L(\alpha)$ is sufficient. Must relate to the $L(\alpha)$ **AG**

Question		Answer	Marks	Guidance
		$(1-\alpha)^{\frac{1}{n}} = e^{\frac{\ln(1-\alpha)}{n}} \approx 1 + \frac{\ln(1-\alpha)}{n}$.	**M1**	Use of approximation given Mark can be inferred as indicated below, but do not accept alternative methods

Question		Answer	Marks	Guidance
		This gives $\lambda U(\alpha) = \ln\left(1-(1-\alpha)^{\frac{1}{n}}\right) \approx \ln\left(-\frac{\ln(1-\alpha)}{n}\right) = \ln n - \ln(-\ln\{(1-\alpha\})) = \ln n - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right\}\right)\right).$	 **A1**	 A0 if candidates use = instead of ≈ either here or in working Correct statement without working earns M1 A1 Correct statement except for incorrect use of = earns M1 A0

Question		Answer	Marks	Guidance
		So we have $\lambda\big(\mathrm{U}(\alpha)-\mathrm{L}(\alpha)\big) \approx \ln \mathrm{n} - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right) - \left(\ln \mathrm{n} - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right)\right) = \ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right)$		
		In fact $\mathrm{U}(\alpha)-\mathrm{L}(\alpha) \to \frac{1}{\lambda}\left[\ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left(\frac{1}{1-\alpha}\right)\right)\right]$ as n increases.	**B1** **[5]**	Failing to note (e.g. via $\to, \approx, \sim$ etc.) that this is an asymptotic formula earns B0
	(v)	By definition of $L(\alpha)$ and $U(\alpha)$, the probability Y is between $L(0.05)$ and $U(0.05)$ is 0.9.	**B1**	Must see $\boldsymbol{P}[L(\alpha) < Y < U(\alpha)] = 0.9$ OE
		For large n, $\mathrm{U}(\alpha)-\mathrm{L}(\alpha) \approx \frac{1}{\lambda}\left(\ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right)\right)$		

Question		Answer	Marks	Guidance
		Taking $\alpha = 0.05$ in this:		
		$U(\alpha) - L(\alpha) \approx \frac{1}{\lambda}\left(\ln\left(\ln\left(\frac{1}{0.05}\right)\right) - \ln\left(\ln\frac{1}{0.95}\right)\right)$	**M1**	Substituting $\alpha = 0.05$ Can be awarded for substituting later in solutions which work backwards from answer
		$= \frac{1}{\lambda}\left(\ln(\ln(20)) - \ln\left(\ln\frac{20}{19}\right)\right) \approx \frac{1}{\lambda}\left(\ln 3 - \ln\left(ln\frac{20}{19}\right)\right)$	**M1**	Use of $\ln(20) \approx 3$
		$\approx \frac{1}{\lambda}\left(\ln 3 - \ln\frac{1}{19}\right) = \frac{1}{\lambda}\ln 57$	**M1**	Use of $\ln\frac{20}{19} \approx \frac{1}{19}$ or $\ln\frac{19}{20} \approx -\frac{1}{20}$

Question		Answer	Marks	Guidance
		$\approx \frac{1}{\lambda}\left(3 + \ln\frac{57}{20}\right) \approx 4\lambda^{-1}$, since $\frac{57}{20} = 2.85 \approx e$	**A1** **[5]**	**AG** A0 if candidates use = instead of ≈ either here or in above working If candidate works backwards from answer, withhold unless it is clearly stated that logic is reversible

Question			Answer	Marks	Guidance
12	(i)		$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m}g(s)\right)$ $= f(1)(g(0)+g(1)+\cdots+g(m))$ $+f(2)(g(1)+g(2)+\cdots+g(m))$ $+f(3)(g(2)+g(3)+\cdots+g(m))$ $\ldots+f(m+1)g(m)$	M1	Writing out a representative sample of first sum in full
			$\sum_{s=0}^{m}\left(g(s)\sum_{r=1}^{s+1}f(r)\right)$ $= g(0)f(1)$ $+g(1)(f(1)+f(2))$ $+g(2)(f(1)+f(2)+f(3))$ $\ldots+g(m)(f(1)+f(2)\ldots+f(m+1))$	M1	Writing out a representative sample of second sum in full
			$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m}g(s)\right)=\sum_{s=0}^{m}\left(g(s)\sum_{r=1}^{s+1}f(r)\right)$	A1 **[3]**	AG

Question			Answer	Marks	Guidance
			Alternative (i), by induction		
		(i)	Proceed by induction. $m = 0$: LHS $= f(1)g(0) =$ RHS Assume true for $m = k$. $m = k+1$: $\sum_{r=1}^{k+2}\left(f(r)\sum_{s=r-1}^{k+1}g(s)\right) = \sum_{r=1}^{k+1}\left(f(r)\sum_{s=r-1}^{k}g(s)\right) + g(k+1)\sum_{r=1}^{k+2}f(r)$	**M1**	Correctly relating the LHS (or RHS) sum for $m = k+1$ to the same sum for $m = k$
			$= \sum_{s=0}^{k}\left(g(s)\sum_{r=1}^{s+1}f(r)\right) + g(k+1)\sum_{r=1}^{k+2}f(r)$	**M1**	Using the induction hypothesis
			$= \sum_{s=0}^{k+1}\left(g(s)\sum_{r=1}^{s+1}f(r)\right)$		
			Thus the claimed identity holds.	**A1**	**AG** Correctly completing the induction

Question			Answer	Marks	Guidance
					(including consideration of the base case)
				[3]	
	(ii)	**(a)**	X_1 is 0 with probability 0.5 and 1 with probability 0.5 so its expected value is 0.5	**B1**	No justification required
			$P(X_2 = 0) = P(X_1 = 0) \times P(X_2 = 0 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 0 \mid X_1 = 1)$	**M1**	Use of Law of Total Probability
			$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}$	**A1**	
			$P(X_2 = 1) = P(X_1 = 0) \times P(X_2 = 1 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 1 \mid X_1 = 1)$	**M1**	Use of Law of Total Probability
			$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}$	**A1**	Similarly = 2
			$P(X_2 = 2) = P(X_1 = 0) \times P(X_2 = 2 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 2 \mid X_1 = 1)$ $= \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	**B1** **CSO**	**AG** Any valid justification (e.g. using that

Question			Answer	Marks	Guidance
					probabilities sum to 1)
			Therefore $E(X_2) = 0 \times P(X_2 = 0) + 1 \times P(X_2 = 1) + 2 \times P(X_2 = 2) = \frac{5}{12} + \frac{2}{6} = \frac{9}{12} = \frac{3}{4}$	B1	
				[7]	
		(b)	Possible values of X_n are $0,1,2,\dots,n$ (so possible values of X_{n-1} are $0,1,2,\dots,n-1$		
			$P(X_n = 0)$ $= P(X_{n-1} = 0) \times P(X_n = 0 \mid X_{n-1} = 0) + P(X_{n-1} = 1) \times P(X_n = 0 \mid X_{n-1} = 1)+\dots$ $P(X_{n-1} = n-1) \times P(X_n = 0 \mid X_{n-1} = n-1)$	M1	Total probability with correct upper limit $[n-1]$ for X_{n-1}
			$= \frac{1}{2} P(X_{n-1} = 0) + \frac{1}{3} P(X_{n-1} = 1) + \cdots + \frac{1}{n+1} P(X_{n-1} = n-1) + \cdots = \sum_{s=0}^{n-1} \frac{P(X_{n-1} = s)}{s+2}$	A1	AG
			$P(X_n = r) = P(X_{n-1} = 0) \times P(X_n = r \mid X_{n-1} = 0) + P(X_{n-1} = 1) \times P(X_n = r \mid X_{n-1} = 1)+\dots$ $P(X_{n-1} = n-1) \times P(X_n = r \mid X_{n-1} = n-1)$	M1	Total probability with correct upper limit $[n-1]$ for X_{n-1}
			$= \frac{1}{r+1} P(X_{n-1} = r-1) + \frac{1}{r+2} P(X_{n-1} = r) + \cdots + \frac{1}{n+1} P(X_{n-1} = n-1)$	M1	Correct lower limit $[r-1]$ for X_{n-1}

Question			Answer	Marks	Guidance
			$= \sum_{s=r-1}^{n-1} \frac{P(X_{n-1}=s)}{s+2}$	A1 [5]	
		(c)	Using the first part of this question, $E(X_n) = \sum_{r=0}^{n} r \sum_{s=r-1}^{n-1} \frac{P(X_{n-1}=s)}{s+2} = \sum_{s=0}^{n-1} \frac{P(X_{n-1}=s)}{s+2} \sum_{r=1}^{s+1} r$	M1	Correct formula for expectation (FT errors in calculated probabilities in (b)) + use of (a)
			$= \sum_{s=0}^{n-1} \frac{P(X_{n-1}=s)}{s+2} \frac{(s+2)(s+1)}{2}$	M1	Sum over r
			$= \sum_{s=0}^{n-1} (X_{n-1}=s) \frac{(s+1)}{2} = \frac{1}{2}\left(\sum_{s=0}^{n-1} (X_{n-1}=s)\, s + \sum_{s=0}^{n-1} (X_{n-1}=s)\right) = \frac{1}{2}(E(X_{n-1})+1)$	A1	**AG**
			This recurrence can be solved $x_{n+1} - 0.5x_n = 0$ has solution $x_n = 0.5^n$. Particular solution is $x_n = 1$ giving general solution of $x_n = 1 + A0.5^n$	M1	Some evidence of appropriate method for solving the

Question			Answer	Marks	Guidance
			With $E(X_0) = 0,\ E(X_n) = 1 - \left(\frac{1}{2}\right)^n$	**A1** **[5]**	recurrence being applied correctly.

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