CAMBRIDGE

Sixth Term Examination Paper (STEP)

Mathematics 2 (9470)

2025

Examiners' report and mark scheme

STEP Mathematics 2 examiners' report

Paper 2 overview

As is commonly the case, the vast majority of candidates focused on the Pure questions in Section A of the paper, with a good number of attempts made on all of those questions. Candidates that attempted the Mechanics questions in Section B generally answered both questions. More candidates attempted Question 11 in Section C than either Mechanics question, but very few attempted Question 12 in that section.

There were a large number of good responses seen for all the questions, but a significant number of responses lacked sufficient detail in the presentation, particularly when asked to prove a given result or provide an explanation.

Candidates who did well on this paper generally:	Candidates who did less well on this paper generally:
• gave careful explanations of each step within their solutions • indicated all points of interest on graphs and other diagrams clearly • made clear comments about the approach that needed to be taken, particularly when having to explore a number of cases as part of the solution to a question • used mathematical terminology accurately within their solutions.	• made errors with basic algebraic manipulation, such as incorrect processing of indices • produced sketches of graphs in which significant points were difficult to see clearly because of the chosen scale • skipped important lines within lengthy sections of algebraic reasoning.

Section A: Pure Mathematics overview

The Pure questions were considerably more popular than the questions in the other sections of the paper. A significant number of attempts at each of the questions were seen and there were some very good solutions for each of the questions.

Question 1

1 The function Min is defined as

$$\mathrm{Min}(a,b) = \begin{cases} a & \text{if } a \leqslant b \\ b & \text{if } a > b\,. \end{cases}$$

(i) Sketch the graph $y = \mathrm{Min}(x^2, 2x)$.

(ii) Solve the equation $2\,\mathrm{Min}(x^2, 2x) = 5x - 3$.

(iii) Solve the equation $\mathrm{Min}(x^2, 2x) + \mathrm{Min}(x^3, 4x) = mx$ in the cases $m = 2$ and $m = 6$.

(iv) Show that $(1, -3)$ is a local maximum point on the curve $y = 2\,\mathrm{Min}(x^2, x^3) - 5x$ and find the other three local maxima and minima on this curve.

Sketch the curve.

This was a popular question and there were many very good responses seen, with a small number of candidates scoring full marks. Almost all responses included attempts at all parts of the question.

Part (i) was generally answered well, although many candidates did not make clear that the gradient of the quadratic section was zero at the origin. Additionally, while most sketches showed two straight line sections for the parts that should show the line $y = 2x$, it was not always clear that these two straight line sections were parts of the same straight line.

Part (ii) was well answered, but many candidates omitted the coefficient of 2 when solving the equation and therefore were not able to reach the correct points. Additionally, many solutions did not show sufficient evidence of checking that the solutions fell within the required ranges. There was a small, but significant, number of candidates that struggled to factorise their correct quadratic equation.

Part (iii) was well answered, with most candidates able to identify the correct function for each of the ranges and solve the corresponding equations. A common error, however, was to solve the equation $6x = 6x$ either as $x = 0$ or as $x = 0$ or 1, rather than noting that it is valid for any value of x within the relevant range. Some candidates did not combine all of their results from the different ranges correctly but were awarded the marks provided that all of the correct values were seen somewhere within the solution.

Many candidates struggled with the explanation that $(1, -3)$ is a local maximum of the curve, although there were some very good explanations seen. Candidates were generally good at identifying the other maxima and minima on the curve, although numerical errors, particularly in the simplification of the y coordinates, were common. When sketching the graphs, many candidates were able to draw the quadratic and cubic sections, although there were several examples where the symmetry of the curves was not evident. Many candidates tried to smooth the graph around the point where the two sections join, rather than having a clear change of gradient at that point. There were also several cases where points of significance were not marked on the graph. Almost no candidates attempted to justify the relative positioning of the two minimum points on the graph.

Question 2

2 **(i)** **(a)** Show that if the complex number z satisfies the equation

$$z^2 + |z + b| = a,$$

where a and b are real numbers, then z must be either purely real or purely imaginary.

(b) Show that the equation

$$z^2 + \left|z + \tfrac{5}{2}\right| = \tfrac{7}{2}$$

has no purely imaginary roots.

(c) Show that the equation

$$z^2 + \left|z + \tfrac{7}{2}\right| = \tfrac{5}{2}$$

has no purely real roots.

(d) Show that, when $\frac{1}{2} < b < \frac{3}{4}$, the equation

$$z^2 + |z + b| = \tfrac{1}{2}$$

will have at least one purely imaginary root and at least one purely real root.

(ii) Solve the equation

$$z^3 + |z + 2|^2 = 4.$$

This was a popular question and there was a wide variety in the quality of responses seen, with a small proportion of candidates producing perfect, or close to perfect, solutions.

Part (i) (a) was generally completed well, with most candidates explaining the reasoning carefully in their responses. Part (i) (b) was also completed well by many candidates, with numerical errors being the main area where marks were lost. In part (i) (c), however, a large number of candidates did not realise there were two cases to be considered and instead only analysed one case. In part (i) (d) there was a lot of variation in the quality of responses seen. Those who had successfully completed part (i) (b) were often able to demonstrate that there was a purely imaginary root. Those who had successfully completed part (i) (c) were often able to make good progress in showing that there must be a purely real root, although many stopped at the point of showing that the discriminant was positive and did not show that at least one of the roots of the quadratic equation lay within the appropriate range. As with part (i) (c), there were a large number of candidates who did not recognise that there were two cases to be explored in this part of the question.

Part (ii) was completed well by many candidates, including those who had lost marks in previous parts of the question. Several candidates lost marks through numerical errors or by not explaining clearly enough that all of the solutions had been found in each case.

Question 3

3 **(i)** Sketch a graph of $y = \dfrac{\ln x}{x}$ for $x > 0$.

(ii) Use your graph to show the following.

(a) $3^{\pi} > \pi^3$

(b) $\left(\frac{9}{4}\right)^{\sqrt{5}} > \sqrt{5}^{\frac{9}{4}}$

(iii) Given that $1 < x < 2$, decide, with justification, which is the larger of x^{x+2} or $(x+2)^x$.

(iv) Show that the inequalities $9^{\sqrt{2}} > \sqrt{2}^9$ and $3^{2\sqrt{2}} > \left(2\sqrt{2}\right)^3$ are equivalent. Given that $e^2 < 8$, decide, with justification, which is the larger of $9^{\sqrt{2}}$ and $\sqrt{2}^9$.

(v) Decide, with justification, which is the larger of $8^{\sqrt[3]{3}}$ and $\sqrt[3]{3}^8$.

This was a popular question with many candidates able to make good progress through most parts of the question.

Part (i) was answered well by the vast majority of candidates, with the maximum point clearly labelled in most cases. In a small number of cases the asymptotes were not sufficiently clear, although in a small number of cases the behaviour near $x = 0$ was not correct.

Part (ii) (a) was generally completed well, with most candidates recognising the relationship between the graph and the required results. In some cases, solutions were presented showing that the given result implied the correct ordering of the numbers, but did not present the logic correctly to show that the ordering of the numbers implies the given result. In part (ii) (b) a significant number of candidates did not justify the order of the three numbers within their solution.

Many candidates were able to produce good solutions to part (iii). Some chose to consider a translation of the curve from part (i) and looked for the point of intersection between $y = \frac{\ln x}{x}$ and $y = \frac{\ln(x+2)}{x+2}$ as the method to justify the inequality.

Part (iv) was well answered by most candidates who attempted it. Most were able to show the equivalence of the two inequalities, but some only showed the logic in one direction. A large proportion of candidates were then able to see how to apply the equivalence between the two inequalities to determine which of the given values was the larger.

Part (v) was found to be very challenging, with some candidates making no written attempt. Those candidates that made progress deduced that there was a need to find an equivalent inequality to allow a similar process to part (iv) to be carried out. Several candidates made mistakes with their manipulation of indices within this part of the question. A good proportion of those who identified the equivalent inequality were then able to recognise that an approach similar to part (iii) was required to reach the final answer.

Question 4

4 Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \leqslant x$.
For example, if $x = -4.2$, then $\lfloor x \rfloor = -5$.

(i) Show that, if n is an integer, then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

(ii) Let n be a positive integer and define function f_n by

$$\mathrm{f}_n(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \ldots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor$$

(a) Show that $\mathrm{f}_n\left(x + \frac{1}{n}\right) = \mathrm{f}_n(x)$.

(b) Evaluate $\mathrm{f}_n(t)$ for $0 \leqslant t < \frac{1}{n}$.

(c) Hence show that $\mathrm{f}_n(x) \equiv 0$.

(iii) **(a)** Show that $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$.

(b) Hence, or otherwise, simplify

$$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor + \ldots .$$

This was a popular question, but candidates often struggled to explain their reasoning with sufficient clarity in many parts.

Almost all candidates seemed to understand why the result in part (i) is true, but many were not precise enough in their explanation. The most common approach was to split x as the sum of two values, one of which was an integer, but many candidates did not state that the other part of the number was greater than or equal to 0 and strictly less than 1.

Part (ii) (a) was answered well in general, with almost all candidates realising that part (i) could be applied here. Solutions were often very well presented for this part. Part (ii) (b) also had many good responses, but in some cases the justification that one or more of the terms must be equal to 0 was missing. Many candidates realised that they could combine the results from the previous two parts to obtain this result, but many arguments were incomplete. In particular, some only showed that the result applied for $x \geq 0$.

Part (iii) (a) was answered well, with most candidates choosing to split the argument into two cases. A significant number realised that this is also a special case of the result in part (ii) (c) and obtained the result from the fact that $\mathrm{f}_2\left(\frac{x}{2}\right) = 0$. Many candidates realised in part (iii) (b) that the result from (iii) (a) could be applied so that the sum could be expressed in a form where most terms cancelled. A common mistake was simply to claim that all but the first term in the sum would cancel and ignore the final term of the partial sum. A few candidates successfully managed to find the complete solution by considering the cases for the different signs of x.

Question 5

5 *You need not consider the convergence of the improper integrals in this question.*

(i) Use the substitution $x = u^{-1}$ to show that

$$\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(x^3+1)}}\,\mathrm{d}x = 0\,.$$

(ii) Use the substitution $x = u^{-2}$ to show that

$$\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = 2\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x\,.$$

(iii) Find, in terms of p and s, a value of r for which

$$\int_0^\infty \frac{x^r-1}{\sqrt{x^s(x^p+1)}}\,\mathrm{d}x = 0,$$

given that p and s are fixed values for which the required integrals converge.

(iv) Show that, for any positive value of k, it is possible to find values of p and q for which

$$\int_0^\infty \frac{1}{\sqrt{x^p+1}}\,\mathrm{d}x = k\int_0^\infty \frac{1}{\sqrt{x^q+1}}\,\mathrm{d}x\,.$$

This was the second most popular question after Question 1 and was attempted by the vast majority of candidates. In general, solutions were very good, particularly for the first two parts.

Parts (i) and (ii) were answered well, with candidates generally showing a good level of proficiency with completing the given substitutions. In part (i) a small number of candidates did not give enough detail in their method when dealing with the limits of the integral following the substitution. Most recognised that the substitution could be used to show that $I = -I$ and produced clear explanations of this. Solutions to part (ii) were often fully correct.

Those candidates who were able to identify the correct substitution were often successful in solving part (iii), although in some cases errors were made with the indices when simplifying the expression. Some candidates attempted substitutions which did not allow them to make any significant progress on solving this part of the question.

Part (iv) was generally answered more successfully than part (iii) with most candidates able to identify the correct substitution to be made. Some candidates started with a more general substitution, from which the form that was needed was deduced. The substitution was again completed successfully by most candidates who reached this part, and the most complete responses noted that the change to the limits would be valid for any of the appropriate values for k. Having completed the substitution, many were able to identify a possible pair of values for p and q. Those who tried to argue that such a pair must exist often did not explain their reasoning clearly enough.

Question 6

6 **(i)** The circle $x^2+(y-a)^2=r^2$ touches the parabola $2ky=x^2$, where $k>0$, tangentially at two points. Show that $r^2=k(2a-k)$.

Show further that if $r^2=k(2a-k)$ and $a>k>0$, then the circle $x^2+(y-a)^2=r^2$ touches the parabola $2ky=x^2$ tangentially at two points.

(ii) The lines $y=c\pm x$ are tangents to the circle $x^2+(y-a)^2=r^2$. Find r^2, and the coordinates of the points of contact, in terms of a and c.

(iii) C_1 and C_2 are circles with equations $x^2+(y-a_1)^2=r_1^2$ and $x^2+(y-a_2)^2=r_2^2$ respectively, where $a_1\neq a_2$ and $r_1\neq r_2$.

Each circle touches the parabola $2ky=x^2$ tangentially at two points and the lines $y=c\pm x$ are tangents to both circles.

(a) Show that $a_1+a_2=2c+4k$ and that $a_1^2+a_2^2=2c^2+16kc+12k^2$.

(b) The circle $x^2+(y-d)^2=p^2$ passes through the four points of tangency of the lines $y=c\pm x$ to the two circles, C_1 and C_2. Find d and p^2 in terms of k and c.

(c) Show that the circle $x^2+(y-d)^2=p^2$ also touches the parabola $2ky=x^2$ tangentially at two points.

This was one of the less popular questions from the Pure section of the paper, but still received many attempts. This question was found to be challenging and few candidates gained many of the marks.

In part (i) almost all solutions attempted to use a calculus or discriminant argument. When arguing based on the discriminant, candidates often did not explain sufficiently clearly how the value of the discriminant related to tangency. Many candidates only solved this part of the question in one direction, not realising that the converse required a separate argument.

Part (ii) was generally completed more successfully than part (i), but responses frequently included algebraic mistakes or did not appreciate the number of solutions that needed to be found.

Very few candidates attempted part (iii). Part (iii) (a) was done well by many of those who attempted it. In part (iii) (b) there were again a number of algebraic mistakes seen. Most candidates who attempted part (iii) (c) related it back to part (i), but most did not make any justification beyond the algebraic manipulation.

Question 7

7 The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

describes the motion of a particle with position $x(t)$ at time t. At $t = 0$, $x = a$, where $a > 0$.

(i) Solve the differential equation in the case where $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2$ when $t = 0$.

What happens to the particle as t increases from 0?

(ii) Solve the differential equation in the case where $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2 + p^2$ when $t = 0$, where $p > 0$.

What happens to the particle as t increases from 0?

(iii) Solve the differential equation in the case where $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2 - q^2$ when $t = 0$, where $q > 0$.

What happens to the particle as t increases from 0? Give conditions on a and q for the different cases which arise.

While there were several very good responses to this question, there were also a significant number of candidates who did not recognise that the first differential equation could easily be turned into a first-order differential equation. Where the correct solution method was not identified, responses often did not make any further progress with the question, and many responses were very brief before the candidate opted to move on to a different question.

Those who recognised the method that was needed were often able to solve part (i) well although many candidates appeared to assume that the information $\frac{dx}{dt} = a^2$ at $t = 0$ would mean that $\frac{dx}{dt} = x^2$ for all t.

In part (ii) many candidates appeared to be familiar with the form of the required integral and so were able to reach a solution of the differential equation, although many struggled to explain the initial motion sufficiently clearly.

Part (iii) was generally answered well by those that attempted it, but in many cases the absolute value signs within the integrals was not dealt with sufficiently clearly within the solution. Many responses to this part of the question did not consider all of the possible cases, with the case where $a = q$ being the most commonly omitted.

Throughout the question, responses often attempted to explain the motion of the particle as $t \to \infty$, rather than the motion as t increases from 0.

Question 8

8 If we split a set S of integers into two subsets A and B whose intersection is empty and whose union is the whole of S, and such that

- the sum of the elements of A is equal to the sum of the elements of B
- and the sum of the squares of the elements of A is equal to the sum of the squares of the elements of B,

then we say that we have found a *balanced partition* of S into two subsets.

(i) Find a balanced partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into two subsets A and B, each of size 4.

(ii) Given that $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ are sequences with

$$\sum_{k=1}^{m} a_k = \sum_{k=1}^{m} b_k \quad \text{and} \quad \sum_{k=1}^{m} a_k^2 = \sum_{k=1}^{m} b_k^2,$$

show that

$$\sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} (c + b_k)^3 = \sum_{k=1}^{m} b_k^3 + \sum_{k=1}^{m} (c + a_k)^3$$

for any real number c.

(iii) Find, with justification, a balanced partition of the set $\{1, 2, 3, \ldots, 16\}$ into two subsets A and B, each of size 8, which also has the property that

- the sum of the cubes of the elements of A is equal to the sum of the cubes of the elements of B.

(iv) You are given that the sets $A = \{1, 3, 4, 5, 9, 11\}$ and $B = \{2, 6, 7, 8, 10\}$ form a balanced partition of the set $\{1, 2, 3, \ldots, 11\}$.

Let $S = \{n^2, (n+1)^2, (n+2)^2, \ldots, (n+11)^2\}$, where n is any positive integer. Find, with justification, two subsets C and D of S whose intersection is empty and whose union is the whole of S, and such that

- the sum of the elements of C is equal to the sum of the elements of D.

Of the Pure questions, this attracted a relatively small number of responses, although many good solutions were seen.

Many candidates were able to write down the partition of the set as their answer to part (i) without much supporting working and this was awarded full marks.

Part (ii) was answered well by most candidates, but many responses did not give sufficiently clear explanations. In particular, some simply produced the two binomial expansions and then claimed that the result would be true.

A small number of candidates attempted to solve part (iii) without using the result from part (ii). Such attempts were rarely successful. Of those who applied the result from part (ii), many did not show that the properties of a balanced partition would be satisfied by their solution. Some candidates simply showed that these were true by calculating the values for the specific case rather than showing a more general result.

In part (iv) many candidates recognised the need to include 0 in the set and then deduced a correct partition. However, in many cases there was insufficient justification that the two sets would have the required property.

Section B: Mechanics overview

Only a small proportion of candidates attempted questions from this section of the paper, although many of those candidates attempted both Mechanics questions.

Question 9

9 Points A and B are at the same height and a distance $\sqrt{2}r$ apart. Two small, spherical particles of equal mass, P and Q, are suspended from A and B, respectively, by light inextensible strings of length r. Each particle individually may move freely around and inside a circle centred at the point of suspension.

The particles are projected simultaneously from points which are a distance r vertically below their points of suspension, directly towards each other and each with speed u. When the particles collide, the coefficient of restitution in the collision is e.

(i) Show that, immediately after the collision, the horizontal component of each particle's velocity has magnitude $\frac{1}{2}ev\sqrt{2}$, where $v^2 = u^2 - gr\left(2-\sqrt{2}\right)$ and write down the vertical component in terms of v.

(ii) Show that the strings will become taut again at a time t after the collision, where t is a non-zero root of the equation

$$(r-evt)^2 + \left(-r + vt - \tfrac{1}{2}\sqrt{2}gt^2\right)^2 = 2r^2.$$

(iii) Show that, in terms of the dimensionless variables

$$z = \frac{vt}{r} \quad \text{and} \quad c = \frac{\sqrt{2}v^2}{rg}$$

this equation becomes

$$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1+e) = 0.$$

(iv) Show that, if this equation has three equal non-zero roots, $e = \frac{1}{3}$ and $v^2 = \frac{9}{2}\sqrt{2}rg$. Explain briefly why, in this case, no energy is lost when the string becomes taut.

(v) In the case described in **(iv)**, the particles have speed U when they again reach the points of their motion vertically below their points of suspension. Find U^2 in terms of r and g.

Only a small number of candidates attempted this question and many were not able to set up the problem sufficiently well to make good progress. Those who recognised that conservation of energy could be used in part (i) were often able to reach the given results successfully, although many assumed that the vertical component of the velocity would also change during the collision.

Both parts (ii) and (iii) were well answered by those that attempted them, with errors in the algebra being the main cause of marks being lost.

Very few of those who attempted part (iv) were able to explain why no energy is lost when the string becomes taut again, but most were then able to produce good solutions to part (v).

Question 10

10 The lower end of a rigid uniform rod of mass m and length a rests at point M on rough horizontal ground. Each of two elastic strings, of natural length ℓ and modulus of elasticity λ, is attached at one end to the top of the rod. Their lower ends are attached to points A and B on the ground, which are a distance $2a$ apart. M is the midpoint of AB.

P is the point at the top of the rod and lies in the vertical plane through AMB.

Suppose that the rod is in equilibrium with angle $PMB = 2\theta$, where $\theta < 45°$ and ℓ is such that both strings are in tension.

(i) Show that angle APB is a right angle.

Show that that the force exerted on the rod by the elastic strings can be written as the sum of

- a force of magnitude $\dfrac{2a\lambda}{\ell}$ parallel to the rod
- and a force of magnitude $\sqrt{2}\,\lambda$ acting along the bisector of angle APB.

(ii) By taking moments about point M, or otherwise, show that $\cos\theta + \sin\theta = \dfrac{2\lambda}{mg}$.

Deduce that it is necessary that $\frac{1}{2}mg < \lambda < \frac{1}{2}\sqrt{2}\,mg$.

(iii) N and F are the magnitudes of the normal and frictional forces, respectively, exerted on the rod by the ground at M.

Show, by taking moments about an appropriate point, or otherwise, that

$$N - F\tan 2\theta = \tfrac{1}{2}mg.$$

As with Question 9, very few candidates attempted this question, and those that did were often unable to set up the problem sufficiently well to make much progress.

A significant number of candidates were able to produce a geometric argument to show that the angle is a right angle in part (i), but many then did not produce correct expressions for the tension in the two strings. Many candidates struggled with the concept of resolving forces in two non-orthogonal directions and so struggled to make any progress beyond this point.

The small number of candidates who were able to complete part (i) often managed to solve the remaining two parts of the question well.

Section C: Probability and Statistics overview

Like Section B, there were significantly fewer responses to the questions in this section than for Section A. Question 11 was the most popular from either Section B or C, while Question 12 was the least popular.

Question 11

11 **(i)** By considering the sum of a geometric series, or otherwise, show that

$$\sum_{r=1}^{\infty} rx^{r-1} = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1.$$

(ii) Ali plays a game with a fair $2k$-sided die. He rolls the die until the first $2k$ appears. Ali wins if all the numbers he rolls are even.

(a) Find the probability that Ali wins the game.

If Ali wins the game, he earns £1 for each roll, including the final one. If he loses, he earns nothing.

(b) Find Ali's expected earnings from playing the game.

(iii) Find a simplified expression for

$$1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \ldots + (n+1)x^n,$$

where n is a positive integer.

(iv) Zen plays a different game with a fair $2k$-sided die. She rolls the die until the first $2k$ appears, and wins if the numbers rolled are strictly increasing in size. For example, if $k = 3$, she wins if she rolls 2, 6 or 1, 4, 5, 6, but not if she rolls 1, 4, 2, 6 or 1, 3, 3, 6.

If Zen wins the game, she earns £1 for each roll, including the final one. If she loses, she earns nothing.

Find Zen's expected earnings from playing the game.

(v) Using the approximation

$$\left(1 + \frac{1}{n}\right)^n \approx \mathrm{e} \quad \text{for large } n,$$

show that, when k is large, Zen's expected earnings are a little over 35% more than Ali's expected earnings.

A large number of very good solutions to this question were seen.

Part (i) was completed well by the majority of candidates. However, many candidates did not identify the correct probabilities to use in the calculations for part (ii). Attempts at part (ii) (b) often applied a correct method for calculating the expected value, but used the incorrect value for the probability that had been used in part (ii) (a) and so gained the method mark for part (ii) (b).

Most candidates who attempted part (iii) were able to complete it successfully, usually by applying a similar approach to the one used in part (i).

Combinatorial errors were common in part (iv), with candidates often confusing $2k$ and $2k-1$ in their calculations or incorrectly accounting for the requirements of the order.

Part (v) was usually completed well by those candidates that had previously obtained the correct expressions for the expected values.

Question 12

12 Let X be a Poisson random variable with mean λ and let $p_r = \mathrm{P}(X = r)$, for $r = 0, 1, 2, \ldots$. Neither λ nor $\lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$ is an integer.

(i) Show, by considering the sequence $d_r \equiv p_r - p_{r-1}$ for $r = 1, 2, \ldots$, that there is a unique integer m such that $\mathrm{P}(X = r) \leqslant \mathrm{P}(X = m)$ for all $r = 0, 1, 2, \ldots$, and that $\lambda - 1 < m < \lambda$.

(ii) Show that the minimum value of d_r occurs at $r = k$, where k is such that

$$k < \lambda + \tfrac{1}{2} + \sqrt{\lambda + \tfrac{1}{4}} < k + 1.$$

(iii) Show that the condition for the maximum value of d_r to occur at $r = 1$ is

$$1 < \lambda < 2 + \sqrt{2}.$$

(iv) In the case $\lambda = 3.36$, sketch a graph of p_r against r for $r = 0, 1, 2, \ldots, 6, 7$.

This was the least popular question on the paper. Most candidates were able to make good progress with parts (i) and (ii), but many then struggled with the remaining parts.

In part (i) many candidates did not justify their handling of the inequalities or to deal properly with the fact that λ was not an integer.

Part (ii) similarly involved a number of attempts that did not justify the handling of the inequalities sufficiently well. Additionally, some showed that a minimum would satisfy the given conditions if it exists, but did not show that there must be a minimum.

Most of the candidates who reached part (iii) were able to derive the bound $\lambda < 2 + \sqrt{2}$, but almost none were able to prove that $\lambda > 1$.

A number of good sketches of the graph were produced for part (iv), but a significant number sketched it as a continuous curve.

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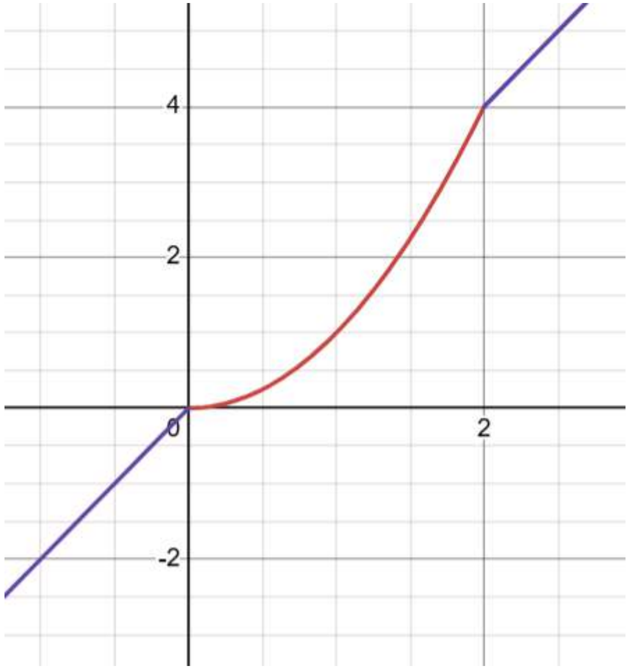
STEP Mathematics 2 mark scheme

Question		Answer	Marks	Guidance
1	i	$\text{Min}(x^2, 2x) = \begin{matrix} x^2 & 0 \le x \le 2 \\ 2x & \text{otherwise} \end{matrix}$	B1	

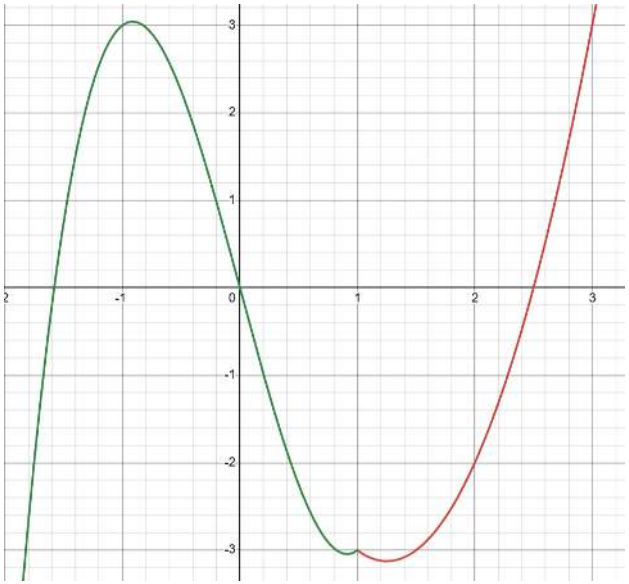
Question		Answer	Marks	Guidance
			G1 G1 [3]	Two straight line sections part of the same line through the origin. Quadratic portion correct and (2,4) indicated.

Question		Answer	Marks	Guidance
	ii	For $0 \leq x \leq 2$, $2x^2 = 5x - 3$ $(x-1)(2x-3) = 0$ $x = 1, x = \frac{3}{2}$ Otherwise, $4x = 5x - 3$ $x = 3$ $x = 1, x = \frac{3}{2}, x = 3$	**B1** **B1** **E1** **[3]**	Solutions must be seen as part of final answer. Solution must be seen as part of final answer. Must be clear that there has been a check for at least one of the solutions to see if it is valid. Can be awarded for a correct judgement from an incorrect solution.
	iii	$\text{Min}(x^3, 4x) = x^3$ when $x^3 \leq 4x$ $x(x-2)(x+2) \leq 0$ $x \leq -2$ or $0 \leq x \leq 2$ For $x \leq -2$: $x^3 + 2x = mx$	**B1**	Award this mark if the equations are solved and then each solution is checked by evaluating the Min function at that point. If checking after solving then all **four** possible equations need to be considered (i.e. $x^2 + 4x = mx$ needs to be included as well).

Question		Answer	Marks	Guidance
		If $m = 2$, this is $x^3 = 0$, so no solution within this range. If $m = 6$, this is $x^3 - 4x = 0$, the only solution within the range is $x = -2$. For $-2 < x < 0$: $$2x + 4x = mx$$ If $m = 2$, this is $4x = 0$, so no solution within this range. If $m = 6$, this is true for all values within the range. For $0 \leq x \leq 2$: $$x^3 + x^2 = mx$$ If $m = 2$, this is $x(x + 2)(x - 1) = 0$, so the only solutions within the range are $x = 0$ and $x = 1$. If $m = 6$, this is $x(x + 3)(x - 2) = 0$, so the only solution within the ranges are $x = 0$ and $x = 2$.		

Question		Answer	Marks	Guidance
		For $x > 2$: $$2x + 4x = mx$$ If $m = 2$, this is $4x = 0$, so no solution within this range. If $m = 6$, this is true for all values within the range.	**B1**	Equation stated correctly for all four cases either for both values of m or written in terms of m.
		$m = 2$: $x = 0, x = 1$	**B1**	
		$m = 6$: $-2 \leq x \leq 0$, $x \geq 2$	**B1** **B1**	
			[5]	
	iv	$$x^2 \leq x^3$$ $$x^3 - x^2 \geq 0$$ $$\text{Min}(x^2, x^3) = \begin{matrix} x^3 & x \leq 1 \\ x^2 & x > 1 \end{matrix}$$	**B1**	

Question		Answer	Marks	Guidance
		If $y = 2x^3 - 5x$, $\frac{dy}{dx} = 6x^2 - 5$, so $y = 2x^3 - 5x$ has a positive gradient at the point $(1, -3)$. If $y = 2x^2 - 5x$, $\frac{dy}{dx} = 4x - 5$, so $y = 2x^2 - 5x$ has a negative gradient at the point $(1, -3)$. Therefore $y = 2Min(x^2, x^3) - 5x$ has a local maximum at $(-1, 3)$.	**E1**	
		Stationary points where $x < 1$: $6x^2 - 5 = 0$, so $x = \pm\sqrt{\frac{5}{6}}$, both of which satisfy $x < 1$. Points are $\left(\pm\sqrt{\frac{5}{6}}, \mp\frac{10}{3}\sqrt{\frac{5}{6}}\right)$	**B1**	
		Stationary points where $x > 1$: $4x - 5 = 0$, so $x = \frac{5}{4}$, which satisfies $x > 1$.	**E1**	Confirmation that all values are valid.

Question		Answer	Marks	Guidance
		Point is $\left(\frac{5}{4}, -\frac{25}{8}\right)$ $\left(\frac{10}{3}\sqrt{\frac{5}{6}}\right)^2 = \frac{250}{27} = \frac{1250}{135}$ and $\left(\frac{25}{8}\right)^2 = \frac{625}{64} = \frac{1250}{128}$ Therefore $-\frac{10}{3}\sqrt{\frac{5}{6}} > -\frac{25}{8}$ and $\left(\sqrt{\frac{5}{6}}, -\frac{10}{3}\sqrt{\frac{5}{6}}\right)$ is higher than $\left(\frac{5}{4}, -\frac{25}{8}\right)$	**B1**	
		 	**G1** **G1** **G1** **[9]**	Graph through origin and correct shape for $x < 1$. Ignore slight inaccuracies in the vicinity of $(1, -3)$. The stationary points should be similar distances above/below the x -axis (allow no more than a ratio of 2 between the distances). Correct shape for $x > 1$. Ignore slight inaccuracies in the vicinity of $(1, -3)$ $\left(-\sqrt{\frac{5}{2}}, 0\right)$ and $\left(\frac{5}{2}, 0\right)$ and correct behaviour at $(1, -3)$Correct relative positions for the two local minima but only if some justification is seen. Only awarded if the coordinates were calculated correctly.

Question			Answer	Marks	Guidance
2	i	a	$\lvert z+b\rvert$ and a are both real, so z^2 must be real. If $z=x+iy$, then $z^2=x^2-y^2+2ixy$ $2xy=0$ $x=0$ (z is purely imaginary) or $y=0$ (z is purely real)	**B1** **M1** **E1** **[3]**	Also award if equating imaginary parts of a *correct* equation. If they have squared the equation, do not award for finding $4xy(x^2-y^2-a)=0$ If using arguments, M1 for stating possible values for the argument of z^2. Also award for stating that $z^2=\pm k$ for a *positive* real number k . Both cases explained. Allow follow through if a small algebraic error beforehand still allows the same analysis to be done. If using the alternative method, award for saying that if $z^2=-k$ then $z=\pm\sqrt{k}i$, and if $z^2=k$ then $z=\pm\sqrt{k}$.
		b	Suppose $z=iy$ for some real value of y: $-y^2+\sqrt{\left(\frac{5}{2}\right)^2+y^2}=\frac{7}{2}$ $\frac{25}{4}+y^2=\frac{49}{4}+7y^2+y^4$ $y^4+6y^2+6=0$	**M1** **A1**	Can award with BOD if numbers are incorrect. For writing down the correct quartic equation (brackets expanded, does not have to have terms grouped).

Question			Answer	Marks	Guidance
			$y^2 = -\frac{6\pm\sqrt{6^2-4(6)}}{2}$ Both values of y^2 are negative, so there are no real solutions for y. (Therefore there are no purely imaginary roots.)	**E1** **[3]**	For establishing that the quartic equation does not have real roots and understanding what this implies for solutions to the original problem (this may be implicit). If the equation has an algebraic error, only award if analysed using methods that would be correct for the correct equation (computing roots and realising they are negative or noting that everything on the LHS is positive).

Question			Answer	Marks	Guidance
		c	Suppose $z = x$ for some real value of x:		
			$x^2 + \sqrt{\left(x + \frac{7}{2}\right)^2} = \frac{5}{2}$	M1	For writing down the equation correctly having used that $y = 0$. If sticking with z variable, then enough to have some indication that the modulus is of a real number. Can award with BOD if the numbers are incorrect.
			$x^2 + \left\|x + \frac{7}{2}\right\| = \frac{5}{2}$		
			If $x \geq -\frac{7}{2}$:	M1	Identification of two cases and the conditions on x for each. Can also award if the conditions are not stated, but the candidate has checked that the solutions fall into the correct range or solve the relevant equations.
			$x^2 + x + 1 = 0$	E1	For correctly analysing the quadratic equation. May be awarded if the equation is wrong but the difficulty is unchanged (i.e. still a quadratic equation) and the analysis is correct. Do not award if there is no evidence of having recognised that this is one of two cases.
			Discriminant = -3, so no roots.		
			If $x < -\frac{7}{2}$:		
			$x^2 - x - 6 = 0$		
			$(x - 3)(x + 2) = 0$		
			Roots are 3 and -2, neither of which is valid.	E1	Same as above.
			(Therefore there are no purely real roots.)		
				[4]	

Question			Answer	Marks	Guidance
		d	Suppose $z = iy$ for some real value of y:		
			$-y^2 + \sqrt{b^2 + y^2} = \frac{1}{2}$		
			$y^2 + b^2 = \frac{1}{4} + y^2 + y^4$		
			$y^4 - \left(b^2 - \frac{1}{4}\right) = 0$	**M1**	For writing down the quartic equation; OR for identifying values which can be used to show existence of a root via the IVT (if it is otherwise clear that this is the method they are using).
			There will be a positive value for y^2 if $b^2 > \frac{1}{4}$, which is satisfied if $\frac{1}{2} < b < \frac{3}{4}$, so there must be at least one purely imaginary root.	**E1**	For establishing that the correct equation has at least one real root, using any method (using the intermediate value theorem is included). If the equation is incorrect, only award marks if analysis is practically unchanged compared to the correct equation.
			Suppose $z = x$ for some real value of x:		
			$x^2 + \sqrt{(x+b)^2} = \frac{1}{2}$		
			$x^2 + \lvert x + b \rvert = \frac{1}{2}$	**M1**	For writing down the quadratic equation. Allow for line above also. OR for identifying values which can be used to show existence of a root via the IVT (if it is otherwise clear that this is the method they are using).
			If $x \geq -b$:		
			$x^2 + x + b - \frac{1}{2} = 0$	**A1**	For correctly analysing the absolute value above OR substituting in above values correctly.

Question	Answer	Marks	Guidance
	Roots are $\frac{-1\pm\sqrt{3-4b}}{2}$		
	$\frac{-1+\sqrt{3-4b}}{2} > -\frac{1}{2} > -b$ if $\frac{1}{2} < b < \frac{3}{4}$, so there must be at least one purely real root.	**E1**	For establishing that at least one real root exists, using any method. If the equation is incorrect, only award marks if analysis is practically unchanged compared to the correct equation. If they have not split into cases based on the absolute value, they must check that this is a solution to the original equation.
		[5]	

Question			Answer	Marks	Guidance
	ii		$\|z+2\|^2$ and 4 are both real, so z^3 must be real.		
			$z = x + iy$, $3x^2y - y^3 = 0$. OR: $z = k\omega^n$, k is real, ω is a cube root of unity and n is either 0, 1 or 2.	**B1**	Mark awarded for recognising that z^3 is real and the algebraic form of z or the relation between x and y that is implied. Can be awarded if one of the cases (e.g. n = 2) is missing.
			$n = 0$: $k^3 + k^2 + 4k = 0$ $k(k^2 + k + 4) = 0$	**M1**	Write down the cubic equation (in one variable) for k (or x or y) in the purely real (0) case. If working with z , then they need to give some indication that they are treating z as real.
			$k = 0$ is the only real solution as the discriminant of the quadratic factor is -15.	**A1**	For analysing the equation correctly in this case and finding 0 as the only real solution.
			$n \neq 0$ $k^3 + k^2 - 2k = 0$ $k(k-1)(k+2) = 0$	**M1**	Write down the cubic equation for k (or x or y) in the complex case. Can be a FT for similar method if the case is slightly wrong, or for an expansion with an algebraic error.
			$k = 0, 1, -2$	**A1**	For analysing the quadratic correctly in this case and finding all remaining solutions. Award for finding the correct x values in the case $y = 3x^2$. More needed if the case is just $y = \sqrt{3}x$ to demonstrate they have found all four solutions.

Question			Answer	Marks	Guidance
			Roots are $0,\ \omega,\ \omega^2, -2\omega, -2\omega^2$	**[5]**	

Question			Answer	Marks	Guidance
3	i		$\frac{dy}{dx} = \frac{(1-\ln x)}{x^2}$ Stationary point when $\frac{(1-\ln x)}{x^2} = 0 \Rightarrow x = \mathrm{e}$ Vertical asymptote $x = 0$, curve crosses the x-axis at $(1,0)$ and nothing to the left of the y-axis. Horizontal asymptote $y = 0$ and maximum at $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$.	**B1** **G1** **G1** **[3]**	G1 for (1,0), increasing on LHS and asymptote at $x = 0$ G1 for $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$, decreasing on RHS and asymptote
	ii	a	$\mathrm{e} < 3 < \pi$ From the graph: $\frac{\ln 3}{3} > \frac{\ln \pi}{\pi}$ $\pi \ln 3 > 3 \ln \pi$ $3^{\pi} > \pi^3$	**M1** **A1** **[2]**	Need to see some reference to $\mathrm{e} < 3 < \pi$, but this could be on their graph A1=CSO; algebra must be seen somewhere. Either a conclusion must be present, or they must have made clear that it was sufficient to show the M1 line.

Question			Answer	Marks	Guidance
		b	$\sqrt{5} < \frac{9}{4} < \mathrm{e}$ From the graph: $\frac{\ln\sqrt{5}}{\sqrt{5}} < \frac{\ln\frac{9}{4}}{\frac{9}{4}}$ $\frac{9}{4}\ln\sqrt{5} < \sqrt{5}\ln\frac{9}{4}$ $\sqrt{5}^{\frac{9}{4}} < \left(\frac{9}{4}\right)^{\sqrt{5}}$	**E1** **M1** **A1** **[3]**	Must acknowledge both inequalities somehow (can be on graph or reference to being in the increasing range); no formal justification required for either. A1=CSO; algebra can be skipped if they did it correctly in ii)a); conclusion of some form still needed.
	iii		$1 < x < 2 < \mathrm{e}$, so $\frac{\ln x}{x} < \frac{\ln 2}{2}$ $\mathrm{e} < 3 < x + 2 < 4$, so $\frac{\ln 4}{4} < \frac{\ln(x+2)}{x+2}$ $\frac{\ln 4}{4} < \frac{\ln(x+2)}{x+2}$	**M1**	Both inequalities found; must see some acknowledgement of these inequalities using decreasing/increasing ranges on graph ($x < 2$, $x + 2 > 3$ is enough, need not see e)

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	$\frac{\ln 4}{4} = \frac{2\ln 2}{4} = \frac{\ln 2}{2}$, so $\frac{\ln x}{x} < \frac{\ln(x+2)}{x+2}$	**M1**	Dependent on first M1
	$(x+2)\ln x < x\ln(x+2)$		
	$x^{x+2} < (x+2)^x$	**A1**	Algebra can be skipped if done correctly in earlier parts.
		[3]	
	Alternative (iii)		
	Correctly drawn graphs of $\frac{\ln x}{x}$ and $\frac{\ln(x+2)}{x+2}$	**M1**	
	Showing intersection of the two graphs is at $x = 2$	**M1**	
	Deducing the inequality $(x+2)^x > x^{(x+2)}$	**A1**	With fully correct working only.

Question			Answer	Marks	Guidance
`	**iv**		$9^{\sqrt{2}} = (3^2)^{\sqrt{2}} = 3^{2\sqrt{2}}$ $(\sqrt{2})^9 = \left((\sqrt{2})^3\right)^3 = (2\sqrt{2})^3$ So, the two inequalities are equivalent.	**E1**	Must either manipulate both sides or make reference to equivalence (i.e. cannot just have "⇒" all the way down). For the first line here a jump is ok. Must see working for the second.
			Since $e^2 < 8$, $e < 2\sqrt{2} < 3$ $e < 2\sqrt{2} < 3$	**B1**	Must see both inequalities (at least implicitly), must show $e < 2\sqrt{2}$ rigorously if trying to use decimal expansion; otherwise may simply be stated.
			$\frac{\ln 2\sqrt{2}}{2\sqrt{2}} > \frac{\ln 3}{3}$ $(2\sqrt{2})^3 > 3^{2\sqrt{2}}$ So $(\sqrt{2})^9 > 9^{\sqrt{2}}$	**M1** **A1**	First line of reasoning here can be implicit from a present $e < 2\sqrt{2}$ or by reference to/showing understanding in previous parts Algebra can be skipped if done correctly in an earlier part.

Question			Answer	Marks	Guidance
				[4]	
			Alternative (iv)		
			$9^{\sqrt{2}} = (3^2)^{\sqrt{2}} = 3^{2\sqrt{2}}$	**E1**	Must either manipulate both sides or make reference to equivalence (i.e. cannot just have "⇒" all the way down). For the first line here a jump is ok. Must see working for the second.
			$\left(\sqrt{2}\right)^9 = \left(\left(\sqrt{2}\right)^3\right)^3 = \left(2\sqrt{2}\right)^3$		
			So, the two inequalities are equivalent.		
			Consider $y = \frac{\ln x}{\sqrt{x}}$ and prove its maximum is at $x = e^2$	**M1**	
			Show by any valid method that the inequality at hand is equivalent to comparing $ln\,9\,/3$ and $ln\,8\,/2\sqrt{2}$	**M1**	
			Use $e^2 < 8 < 9$ to deduce that $ln\,9\,/3 < ln\,8\,/2\sqrt{2}$ and hence that $\sqrt{2}^9$ is the larger value	**A1**	
				[4]	

Question			Answer	Marks	Guidance
	v		$8^{\sqrt[3]{3}} = (2^3)^{\sqrt[3]{3}} = 2^{3\sqrt[3]{3}}$	B1	
			$\left(\sqrt[3]{3}\right)^8 = \left(\left(\sqrt[3]{3}\right)^4\right)^2 = \left(3\sqrt[3]{3}\right)^2$	B1	Enough to state both for the marks.
			$\left(3\sqrt[3]{3}\right)^3 = 81 > 64 = 4^3$, so $e < 4 < 3\sqrt[3]{3}$ $e < 4 < 3\sqrt[3]{3}$	M1	Don't need to see 4 > e at this point (clear enough and seen similar enough times). Must see some working.
			$\frac{\ln 4}{4} > \frac{\ln 3\sqrt[3]{3}}{3\sqrt[3]{3}}$	M1	
			$\frac{\ln 2}{2} > \frac{\ln 3\sqrt[3]{3}}{3\sqrt[3]{3}}$		
			$2^{3\sqrt[3]{3}} > \left(3\sqrt[3]{3}\right)^2$		
			$8^{\sqrt[3]{3}} > \left(\sqrt[3]{3}\right)^8$	A1	Must correctly and clearly compare the inequalities (not ok for logic to go in the wrong direction) for full marks. Again, algebra can be skipped if seen elsewhere.
				[5]	

Question	Answer	Marks	Guidance
	Alternative (v)		
	$8^{\sqrt[3]{3}} = (2^3)^{\sqrt[3]{3}} = 2^{3\sqrt[3]{3}}$	**B1**	
	$\left(\sqrt[3]{3}\right)^8 = \left(\left(\sqrt[3]{3}\right)^4\right)^2 = \left(3\sqrt[3]{3}\right)^2$	**B1**	
	$\left(3\sqrt[3]{3}\right)^3 = 81 > 64 = 4^3$, so $e < 4 < 3\sqrt[3]{3}$	**M1**	
	Consider $y = 2^t - t^2$, $\frac{dy}{dt} = (\ln 2)2^t - 2t > 0$ for $t \geq 4$ (by any valid method)	**M1**	Any valid proof that y increasing for $t \geq 4$
	$y(4) = 0$ and hence $y\left(3\sqrt[3]{3}\right) > 0$, whence $8^{\sqrt[3]{3}} > \left(\sqrt[3]{3}\right)^8$	**A1**	Any valid way to finish off
		[5]	

Question			Answer	Marks	Guidance
4	i		Let $\delta = x - \lfloor x \rfloor$ $x + n = \lfloor x \rfloor + n + \delta$ $\lfloor x \rfloor + n$ is an integer and $\delta < 1$, so $\lfloor x + n \rfloor = \lfloor x \rfloor + n$	 **M1** **E1** **[2]**	 Any attempt to show that $\lfloor x \rfloor + n$ satisfies the definition of $\lfloor x + n \rfloor$, either as given in the question or by using $\delta \in [0,1)$ (or any attempt to consider a 'decimal part' or similar, wrt $x + n$). Give BOD for $\delta > 0$. Correct explanation with $\delta < 1$ (can be lenient with stating that '$\lfloor x \rfloor + n$ is an integer'); or 'it is the largest possible integer $\leq x + n$'.
	ii	a	$f_n\left(x + \frac{1}{n}\right) - f_n(x)$ $= (\lfloor x + 1 \rfloor - \lfloor x \rfloor) - (\lfloor nx \rfloor - \lfloor nx + 1 \rfloor)$ $= 1 - 1 = 0$	**M1** **A1** **[2]**	Any attempt to compare $f_n\left(x + \frac{1}{n}\right)$ and $f_n(x)$, expanding $f_n\left(x + \frac{1}{n}\right)$ is enough. Must clearly show application of (i); it is enough to see the terms that cancel in brackets as shown.
		b	If $0 \leq t < \frac{1}{n}$, then $t + \frac{k}{n} < 1$ for all integers $0 \leq k \leq n - 1$, so $\left\lfloor t + \frac{k}{n} \right\rfloor = 0$ $nt < 1$, so $\lfloor nt \rfloor = 0$	**E1** **E1**	This can be implicit if $t + \frac{n-1}{n} < 1$ is stated. If the inequalities are stated without $\left\lfloor t + \frac{k}{n} \right\rfloor = 0$ or $\lfloor nt \rfloor = 0$, and the answer is not clear enough to give BOD (i.e. $f_n(t) = 0 + \cdots + 0 - 0$ is sufficient for BOD and

Question			Answer	Marks	Guidance
			Therefore $f_n(t) = 0$ for $0 \leq t < \frac{1}{n}$.	**E1** **[3]**	the final E mark) then give the corresponding marks for the inequalities and withhold the final E mark.

Question			Answer	Marks	Guidance
		c	By repeated application of (a): $f_n\left(y+\frac{k}{n}\right)=f_n(y)$ For any value of x, there is an integer k and a value $0\le\delta<\frac{1}{n}$ such that $x=\delta+\frac{k}{n}$ Therefore $f_n(x)=0$	**M1** **E1** **[2]**	Any attempt to 'iterate' (ii)(a) to some $k\in\mathbb{Z}$ other than 0,1. Any argument explicitly restricting to $x\ge 0$ gets E0. Claiming that $f_n\left(t+\frac{k}{n}\right)=f_n(t)$ for integer k, while only showing 'positive' iteration is sufficient for E1. Any response that does not specify $k\in\mathbb{Z}$, and it cannot be inferred from their argument gets E0.
	iii	**a**	Let $x=2k+\delta$, where k is an integer and $0\le\delta<2$ $\left\lfloor\frac{x}{2}\right\rfloor=k$ $\left\lfloor\frac{x+1}{2}\right\rfloor=\begin{matrix}k & 0\le\delta<1\\ k+1 & 1\le\delta<2\end{matrix}$ $\lfloor x\rfloor=\begin{matrix}2k & 0\le\delta<1\\ 2k+1 & 1\le\delta<2\end{matrix}$	**M1** **A1**	Identification of the two cases. Either case completed correctly.

Question	Answer	Marks	Guidance
	In both cases, $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$	**E1** **[3]**	Complete solution.
	Alternative (iii)(a)		
	Attempt to use part (ii)(c), $f_n(x) \equiv 0$.	**M1**	
	Used $n = 2$: $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor$ or equivalent.	**A1**	
	$x \longmapsto \frac{x}{2}$ or equivalent, correct solution.	**E1** **[3]**	

Question			Answer	Marks	Guidance
		b	Let $g_n(x) = \left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor$	**B1**	
			$g_n(x) = g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor$		
			$g_n(x) + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$	**M1**	
			$= g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$		
			Let $y = \frac{x+2^n}{2^n}$		
			$\frac{y}{2} = \frac{x+2^n}{2^{n+1}}$ and $\frac{y+1}{2} = \frac{x+2^{n+1}}{2^{n+1}}$		
			$g_n(x) + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$	**A1**	
			$= g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^n} \right\rfloor$		
			$g_0(x) + \left\lfloor \frac{x+2}{2} \right\rfloor = \lfloor x+1 \rfloor$	**M1**	
			Therefore $g_n(x) = \lfloor x+1 \rfloor - \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$	**A1**	

Question			Answer	Marks	Guidance
			For $x \geq 0$ and n sufficiently large, $\left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor = \left\lfloor \frac{x}{2^{n+1}} + 1 \right\rfloor = 1$		
			Therefore $g_n(x) = \lfloor x + 1 \rfloor - 1$ for large values of n	**E1**	

Question	Answer	Marks	Guidance
	For $x < 0$ and n sufficiently large, $\left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor = \left\lfloor \frac{x}{2^{n+1}} + 1 \right\rfloor = 0$ Therefore $g_n(x) = \lfloor x+1 \rfloor$ for large values of n	**E1**	
	$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor + \cdots$ $= \begin{matrix} \lfloor x+1 \rfloor - 1 & x \geq 0 \\ \lfloor x+1 \rfloor & x < 0 \end{matrix}$	**A1** **[8]**	
	Alternative (iii)(b)		
	Let $g_n(x) = \sum_{k=0}^{n} \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor$	**B1**	This mark can be implicit, i.e. $\sum_{k=0}^{\infty} \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor = \lim_{n\to\infty} \left(\lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor \right)$ gets the mark for considering the limit of the partial sum; any statement that $x/2^n$ goes to zero gets B1 for consideration of 'term at infinity' (even if incorrectly stated as $\left\lfloor \frac{x}{2^n} \right\rfloor$ goes to zero, since this loses the final three marks anyway).
	$\left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{x}{2^{k+1}} + \frac{1}{2} \right\rfloor =$	**M1**	Attempt to manipulate kth term using (iii)(a).
	$= \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor \frac{x}{2^{k+1}} \right\rfloor$, using (iii)(a).	**A1**	Correct simplification. (Instead of kth term, exhibiting multiple terms is enough.)

Question			Answer	Marks	Guidance
			so $g_n(x) = \sum_{k=0}^{n} \left\lfloor \frac{x}{2^k} \right\rfloor - \sum_{k=0}^{n} \left\lfloor \frac{x}{2^{k+1}} \right\rfloor$ $= \sum_{k=0}^{n} \left\lfloor \frac{x}{2^k} \right\rfloor - \sum_{k=1}^{n+1} \left\lfloor \frac{x}{2^k} \right\rfloor$ $\therefore g_n(x) = \lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor$ For $x \geq 0$, and n sufficiently large, $\left\lfloor \frac{x}{2^{n+1}} \right\rfloor = 0$, and therefore $g_n(x) = \lfloor x \rfloor$. For $x < 0$, and n sufficiently large, $\left\lfloor \frac{x}{2^{n+1}} \right\rfloor = -1$, and therefore $g_n(x) = \lfloor x \rfloor + 1$. The infinite sum is therefore: $\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor + \cdots =$ $= \begin{matrix} \lfloor x \rfloor & : x \geq 0 \\ \lfloor x \rfloor + 1 & : x < 0 \end{matrix}$	**M1** **A1** **E1** **E1** **A1** **[8]**	Attempt to re-index or use telescoping series Can be inside a limit. Cannot be inferred from statement as in second half of B1 guidance above.

Question		Answer	Marks	Guidance
5	i	$\frac{dx}{du} = -u^{-2}$	M1	Dx=-du/u^2 fine. Done correctly. Seeing –u^(-2) term in integrand is sufficient. Must include an attempt at substitution. Incorrect differentiation gets M0 May calculate du/dx= -x^(-2) and sub this in for M1, but need integrand in terms of u for A1
		As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$	B1	May be implicit in calculation. Must be earned in (i), not from correct attempts in (ii) onward. If not done explicitly, and in integrand there is no minus sign and limits are 0 to inf give B0 NGE. In this case, final two marks can be earned. Give BOD on limits when d(1/u) is used, until they are dealt with fully.
		$\int_0^\infty \frac{\sqrt{x}}{\sqrt{x(x^3+1)}}dx = \int_\infty^0 \frac{\sqrt{u^{-1}}}{\sqrt{u^{-1}(u^{-3}+1)}}(-u^{-2})du$ $= \int_0^\infty \frac{1}{u^2\sqrt{u}\sqrt{u^{-1}(u^{-3}+1)}}du$	A1	Substitution completed correctly. Allow incorrect limits. For correct form of the integrand, may just be the first/second term in the integrand dealt with correctly.
		$= \int_0^\infty \frac{1}{\sqrt{u(u^3+1)}}du$	A1	Fully simplified integrand. No errors in calculation up to this point. Needs to be some indication of legitimate simplification of their substitution.
		Therefore: $\int_0^\infty \frac{\sqrt{x}}{\sqrt{x(x^3+1)}}dx = \int_0^\infty \frac{1}{\sqrt{x(x^3+1)}}dx$, so $\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(x^3+1)}}dx = 0$	E1	Using their transformation, done accurately, to prove the given statement. A complete proof of the statement, including convincing manipulation of the integrals. A0E1 is not possible.

Question		Answer	Marks	Guidance
			[5]	
	ii	$\frac{dx}{du} = -2u^{-3}$ As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$ $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx = \int_\infty^0 \frac{1}{\sqrt{u^{-6}+1}} (-2u^{-3})\, du$ $= 2\int_0^\infty \frac{1}{\sqrt{u^6+1}} du$ $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx = 2\int_0^\infty \frac{1}{\sqrt{x^6+1}} dx$	**M1** **A1** **A1** **[3]**	Dx=-2du/u^3 fine. Differentiation done correctly. Seeing –2u^(-3) term in integrand is sufficient. Must include an attempt at substitution. If attempting to use (i) to solve (ii) will need 2 substitutions, M1 only after the second substitution is made. Substitution completed correctly. May have incorrect limits at this stage. For correct form of the integrand. Completely correct simplification, including limits dealt with correctly. Give BOD on limits when d(1/u^2) is used, until they are dealt with fully. Can give BOD for not changing dummy variable back to x. Candidates cannot gain this A mark if they leave out the differential element du while performing the change of variables, but after that step allow with BOD provided no other serious mistakes.

Question		Answer	Marks	Guidance
	iii	Consider $\int_0^\infty \frac{x^k}{\sqrt{x^p+1}}dx$, for fixed values of p and k for which the integral converges: Substitute $x = u^{-1}$: $\frac{dx}{du} = -u^{-2}$	**M1**	Dx=-du/u^2 fine. Done correctly. Seeing –u^(-2) term in integrand is sufficient. Must include an attempt at substitution. Need power of u in substitution to be -1
		As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$ $\int_0^\infty \frac{x^k}{\sqrt{x^p+1}}dx = \int_\infty^0 \frac{u^{-k}}{\sqrt{u^{-p}+1}}(-u^{-2})\,du$	**A1**	Substitution completed correctly. May have incorrect limits at this stage. For correct form of the integrand.
		$= \int_0^\infty \frac{1}{u^{k+2}\sqrt{u^{-p}+1}}du$ $= \int_0^\infty \frac{1}{\sqrt{u^{2k+4-p}+u^{2k+4}}}du$	**A1**	Correct form of integrand, with correct powers of *u* collected, including limits dealt with correctly. Numerator and denominator correct. Want to see numerator = 1 (or containing u^{r-1}), denominator containing $\sqrt{u^{p+1}}$. Can also be awarded for a form from which they have correctly extracted the necessary information.
		Therefore: $\int_0^\infty \frac{x^k}{\sqrt{x^p+1}} - \frac{1}{\sqrt{x^{2k+4-p}+x^{2k+4}}}dx = 0$		

Question		Answer	Marks	Guidance
		$\int_0^\infty \frac{x^{k+\frac{s}{2}}}{\sqrt{x^s(x^p+1)}} - \frac{1}{\sqrt{x^{2k+4-p}(x^p+1)}}\,dx = 0$ Therefore, if $s = 2k + 4 - p$, $\int_0^\infty \frac{x^{k+\frac{s}{2}-1}}{\sqrt{x^s(x^p+1)}}\,dx = 0$ $k = \frac{p+s-4}{2}$ $r = k + \frac{s}{2} = \frac{p}{2} + s - 2$	**M1** **A1** **A1** **[6]**	Writing the integrand in an appropriate form to compare powers of the dummy variable and attempting to obtain an equation for *r*. Incorrect justification must score M0. Correct equation making integral = 0. Correct final answer.

Question		Answer	Marks	Guidance
	iv	$\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx$		
		Use the substitution $x = u^{-k}$	M1	$dx = -k\frac{du}{u^{k+1}}$ fine. Seeing $\frac{-k}{u^{k+1}}$ term in integrand is sufficient. Must include an attempt at substitution. Power on u must be $-k$ May award M1 for error in power of u after substitution (e.g. $\frac{-k}{u^{k-1}}$) May use index other than k, and solve entire problem in terms of this arbitrary index. This is acceptable. Withold mark until clear this is the case. Other powers that "will work" are acceptable, e.g. $\frac{-q}{p}$, $\frac{-p}{q}$, $\frac{2}{2-p}$, $\frac{2-q}{2}$ $x = u^k$, $u^{\frac{p}{q}}$ is accepted, with the final answers being $p = 2 - \frac{2}{k}$, $q = 2 - 2k$ This is marked as normal
		$\frac{dx}{du} = -ku^{-k-1}$		
		For any positive value of k: As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$	E1	Must be clear that it applies for all positive values of k. . There must be a statement somewhere which can be read as being linked to the consideration of the limits, can be generous with how clearly this is stated.
		$\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx = \int_\infty^0 \frac{-ku^{-k-1}}{\sqrt{u^{-pk}+1}}du$	A1	Correct substitution, including limits dealt with correctly.

Question	Answer	Marks	Guidance
	$= k\int_0^\infty \frac{1}{\sqrt{u^{2k+2-pk}+u^{2k+2}}}du$	**M1**	Writing the integrand in an appropriate form to compare powers of the dummy variable and attempting to obtain (or obtaining) an equation for p or q in terms of k by comparing appropriate powers of u. Want to see numerator of +-1, limits correct, denominator simplified as given, or factorised. The equation $q = pk$ alone is not sufficient.
	If $p = \frac{2(k+1)}{k}$, $u^{2k+2-pk} = 1$ and $\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx = k\int_0^\infty \frac{1}{\sqrt{x^q+1}}dx$	**A1**	At least one correct explicit equation for p or q in terms of k, unless from obviously incorrect working. If equations are implicit for $p(k)$, $q(k)$ there must be two correct equations including $q/p = k$ or equivalent.
	Where q$= 2(k+1)$	**A1**	Both p and q correctly identified. Might not both be explicitly given in terms of k (e.g. $p(k)$ given, $q = kp$). Should be some specification of p and q in terms of k, not just $k(p)$ etc (e.g. "choose k = ..." is A0 NGE). If neither p or q given explicitly in terms of k, then candidate must **convincingly** argue that given equations for k, p, q, can be solved for p, q for all $k > 0$.
		[6]	

Question			Answer	Marks	Guidance
6	i		Let point of contact be (x_1, y_1):		
			The centre of the circle is $(0, a)$		
			Gradient of radius of circle to contact point $= \frac{y_1 - a}{x_1}$		
			$2k\frac{dy}{dx} = 2x$	**M1** **A1**	M1: differentiating both equations (or considering two relevant gradients). Need to get $\frac{dy}{dx}$ by itself (i.e. need to divide by x) - done implicitly or explicitly. A1: correct gradients (can be gradient of a perpendicular).
			Gradient of tangent at contact point $= \frac{x_1}{k}$		
			$\left(\frac{y_1 - a}{x_1}\right)\left(\frac{x_1}{k}\right) = -1$		
			$y_1 - a = -k$	**M1**	Having an equation that equates two gradients.
			$x_1 = \pm\sqrt{2ka - 2k^2}$ $y_1 = a - k$		
			$2ka - 2k^2 + k^2 = r^2$		

Question			Answer	Marks	Guidance
			$r^2 = k(2a-k)$	**A1**	CSO
			Points of intersection between circle and parabola:		
			$2ky + (y-a)^2 = r^2$	**M1**	Substitution (r^2 and then x or y).
			$y^2 + 2(k-a)y + a^2 - k(2a-k) = 0$		
			$y^2 + 2(k-a)y + a^2 - 2ak + k^2 = 0$		
			$y^2 + 2(k-a)y + (k-a)^2 = 0$		
			$(y + k - a)^2 = 0$		
			$y = a - k$		
			$x^2 = 2k(a-k)$, which is positive since $a > k$ so there are two points of intersection.	**E1**	Showing 2 intersections ($a > k$ stated or equivalent).
			Points of intersection are $\left(\pm\sqrt{2k(a-k)}, a-k\right)$		

Question			Answer	Marks	Guidance
			Gradient of tangent to $2ky = x^2$ at $\left(\pm\sqrt{2k(a-k)}, a-k\right)$ is $\pm\sqrt{\frac{2(a-k)}{k}}$ Gradient of radius of circle to $\left(\pm\sqrt{2k(a-k)}, a-k\right)$ is $\frac{-k}{\pm\sqrt{2k(a-k)}} = \mp\frac{k}{\sqrt{2k(a-k)}}$ $\pm\sqrt{\frac{2(a-k)}{k}} \times \mp\frac{k}{\sqrt{2k(a-k)}} = -1$, so the circle touches the parabola tangentially at both points.	**E1** **[7]**	Showing tangential.

Question			Answer	Marks	Answer
	ii		Gradients of tangents are ± 1 so the points of intersection are the lower intersections between the circle and the lines $y = a \pm x$		
			$2(y-a)^2 = r^2$	M1	May be seen graphically. Going towards one coordinate. Need something like equation only in one coordinate in the context of a valid method.
			Lower intersection is at $y = a - \frac{r\sqrt{2}}{2}$		
			Points of intersection are $\left(\pm\frac{r\sqrt{2}}{2}, a - \frac{r\sqrt{2}}{2}\right)$		
			$y - \left(a - \frac{r\sqrt{2}}{2}\right) = \pm\left(x \mp \frac{r\sqrt{2}}{2}\right)$	M1	Substituting back in for other coordinate (quite easy to get). Say something reasonable about there being one *y* (okay if discriminant not stated to be zero yet).
			$y = \pm x - \frac{r\sqrt{2}}{2} + a - \frac{r\sqrt{2}}{2}$		
			$y = a - r\sqrt{2} \pm x$		
			Therefore, $c = a - r\sqrt{2}$		
			$r = \pm\frac{\sqrt{2}}{2}(a-c)$		
			Points of contact: $\left(\pm\frac{a-c}{2}, \frac{a+c}{2}\right)$	A1	CSO (need to give r^2 or correct value of r).
				[3]	

Question			Answer	Marks	Answer
	iii	a	$r_1^2 = k(2a_1 - k)$ and $r_1 = \frac{\sqrt{2}}{2}(a_1 - c)$ $a_1^2 - 2a_1c + c^2 = 4a_1k - 2k^2$ Similarly. $a_2^2 - 2a_2c + c^2 = 4a_2k - 2k^2$ $a_1^2 - a_2^2 - 2c(a_1 - a_2) = 4(a_1 - a_2)k$ $(a_1 - a_2)(a_1 + a_2 - 2c - 4k) = 0$ $a_1 \neq a_2$, so $a_1 + a_2 = 2c + 4k$ $a_1^2 + a_2^2 - 2c(a_1 + a_2) + 2c^2 = 4(a_1 + a_2)k - 4k^2$ $a_1^2 + a_2^2 = (2c + 4k)^2 - 2c^2 - 4k^2$	**M1** **A1**	Write down a single expression or equality with part i) result and whatever their part ii) result was. Correct expression for $a_1 + a_2$, need to say $a_1 \neq a_2$ or equivalent. (Can be solved via quadratic/Vieta, need to also say $a_1 \neq a_2$ in that case. Take "sum of roots" etc. to imply that roots are distinct.).

Question			Answer	Marks	Answer
			$a_1^2 + a_2^2 = 2c^2 + 16kc + 12k^2$	A1 [3]	Correct expression for $a_1^2 + a_2^2$ (if they get M1, they can use the given expression for $a_1 + a_2$ to get this).
		b	The points $\left(\pm\frac{a_1-c}{2}, \frac{a_1+c}{2}\right)$ and $\left(\pm\frac{a_2-c}{2}, \frac{a_2+c}{2}\right)$ are on the circle. $\left(\frac{a_1-c}{2}\right)^2 + \left(\frac{a_1+c}{2} - d\right)^2 = p^2$ $\left(\frac{a_1-c}{2}\right)^2 + \left(\frac{a_1+c}{2} - d\right)^2 = p^2$ $\left(\frac{a_1-c}{2}\right)^2 - \left(\frac{a_2-c}{2}\right)^2 = \left(\frac{a_1+a_2-2c}{2}\right)\left(\frac{a_1-a_2}{2}\right)$ $= k(a_1 - a_2)$ $\left(\frac{a_1+c}{2} - d\right)^2 - \left(\frac{a_2+c}{2} - d\right)^2$ $= \left(\frac{a_1+a_2+2c}{2} - 2d\right)\left(\frac{a_1-a_2}{2}\right)$ $= (c + k - d)(a_1 - a_2)$	M1	Correctly process difference between the two equations for one corresponding pair of terms. (Substitute in coords to circle equation then get rid of p by subtracting).

Question	Answer	Marks	Answer
	$k(a_1 - a_2) + (c + k - d)(a_1 - a_2) = 0$		
	$d = c + 2k$	**A1**	CSO.
	$\left(\frac{a_1 - c}{2}\right)^2 + \left(\frac{a_1 + c}{2} - d\right)^2 = p^2$ $\left(\frac{a_1 - c}{2}\right)^2 + \left(\frac{a_1 + c}{2}\right)^2 - d(a_1 + c) + d^2 = p^2$ $\frac{a_1^2 + c^2}{2} - d(a_1 + c) + d^2 = p^2$		
	So $\frac{a_1^2 + a_2^2 + 2c^2}{2} - d(a_1 + a_2 + 2c) + 2d^2 = 2p^2$	**M1**	Expanding equation and substituting in d.
	$\frac{2c^2 + 16kc + 12k^2 + 2c^2}{2} - (c + 2k)(4c + 4k) + 2(c + 2k)^2 = 2p^2$	**M1**	Sum completely correct with substitutions made (i.e. expression with p^2,c , k correct but may be long) from their (reasonable) value of d
	$p^2 = 2kc + 3k^2$	**A1**	CSO.
		[5]	

Question			Answer	Marks	Answer
		c	$k(2d-k)=k(2c+4k-k)$		
			$=2kc+3k^2=p^2$	M1	Reasonable substitution.
			$d>k>0$, so by part (i) the circle also touches the parabola tangentially at two points.	E1	Using part i) of question and simply stating $d>k>0$ without need to show this inequality.
				[2]	

Question		Answer	Marks	Guidance
7	i	$\frac{d}{dt}(x^2) = 2x\frac{dx}{dt}$	**M1**	Reasonable attempt to turn this into 1st order Differential equation
		$\frac{dx}{dt} = x^2 + c$	**A1**	
		At $t = 0$:	**B1**	
		$a^2 = a^2 + c \Rightarrow c = 0$		
		$\int \frac{1}{x^2} dx = \int dt$	**M1**	Separate the variables
		$-\frac{1}{x} = t + c_2$	**A1**	
		At $t = 0$:		
		$\frac{-1}{a} = c_2$		
		$\frac{-1}{x} = t - \frac{1}{a}$		
		$x = \frac{a}{1-at}$	**A1**	Find c_2 and write a correct expression for x. FT allowed but dependent on M1.

Question		Answer	Marks	Guidance
		As t increases from 0, $1-at$ decreases from 1, so the particle moves away from the origin.	**E1** **[7]**	Must have some reference to the equation otherwise NGE Must mention behaviour near t=0 not just at the limit $t \to \infty$. FT allowed. In all explanations, if derivatives are used they must be directly applied to the position, x.

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$ $\int \frac{1}{x^2+p^2} dx = \int dt$ $\frac{1}{p}\arctan\left(\frac{x}{p}\right) = t + c_2$ At $t = 0$: $\frac{1}{p}\arctan\left(\frac{a}{p}\right) = c_2$ $x = p\tan\left(pt + \arctan\left(\frac{a}{p}\right)\right)$ At t increases from 0, x moves away from the origin.	**B1** **M1** **A1** **A1** **E1** **[5]**	Reasonable attempt to solve the integral with arctan. Correct expression up to constant (x need not be the subject) Correct expression in terms of x. FT allowed but dependent on M1 and must be a genuinely different type of expression to (i). FT allowed, but must be a genuinely different type of expression to (i). Must mention behaviour near t=0 not just at the limit $t \to \infty$. Must have some reference to the equation or a graph otherwise NGE

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	**B1**	
	iii	At $t = 0$: $a^2 - q^2 = a^2 + c \Rightarrow c = -q^2$ $\int \frac{2}{x^2-q^2} dx = \int 2dt$ $\frac{2q}{x^2-q^2} = \frac{1}{x-q} - \frac{1}{x+q}$ $\ln\left\|A\frac{(x-q)}{(x+q)}\right\| = 2tq$ $A\frac{(x-q)}{(x+q)} = e^{2tq}$	**B1** **M1** **M1**	An attempt at manipulating the fraction Attempt at integrating to ln|... |, must include the absolute value signs

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	**B1**	
		At $t = 0$: $A\frac{(a-q)}{(a+q)} = 1 \Rightarrow A = \frac{a+q}{a-q}$	**M1**	Reasonable attempt to find the constant.
		$(a+q)(x-q) = (a-q)e^{2tq}(x+q)$ $x = \frac{q(a+q) + qe^{2tq}(a-q)}{(a+q) - e^{2tq}(a-q)}$ $= -q + \frac{2q(a+q)}{(a+q) - e^{2tq}(a-q)}$	**A1**	Just for correct expression with x as the subject. Lose this mark if the absolute value signs disappear without explanation or without being absorbed into a constant.
		If $q = a$, then $x = a$, so the particle does not move.	**E1**	

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	**B1**	
		If $q < a$, then $(a+q) - e^{2tq}(a-q)$ is decreasing as t increases from 0. Therefore x is increasing and so the particle moves away from the origin.	**E1**	Must mention behaviour near $t = 0$, not just at the limit $t \to \infty$. FT allowed, but must be a genuinely different type of expression to (i)/(ii).
		If $q > a$, then $(a+q) - e^{2tq}(a-q)$ is increasing as t increases from 0. Therefore x is decreasing and so the particle moves towards the origin.	**E1** **[8]**	Must mention behaviour near $t = 0$, not just at the limit $t \to \infty$. FT allowed, but must be a genuinely different type of expression to (i)/(ii).

Question		Answer	Marks	Guidance
8	**i**	The sum of the elements in each subset must be 18.	**B1**	Implied by a correct pair of subsets.
		$\sum_{r=1}^{8} r^2 = \frac{8}{6}(8+1)(2\times 8+1) = 204$	**M1**	Implied by a correct pair of subsets.
		The sum of the squares of the elements in each subset must be 102.	**A1**	Implied by a correct pair of subsets.
		For the subset containing 8, the other three numbers must sum, to 10 and the squares must sum to 38.	**M1**	Choice of one element for a particular set.
		$\{2,3,5,8\}$ and $\{1,4,6,7\}$	**A1**	
			[5]	
	ii	$\sum_{k=1}^{m}(c+b_k)^3 = \sum_{k=1}^{m} c^3 + 3c^2 b_k + 3cb_k^2 + b_k^3$	**M1**	Correct binomial expansion.
		$mc^3 + 3c^2\sum_{k=1}^{m} b_k + 3c\sum_{k=1}^{m} b_k^2 + \sum_{k=1}^{m} b_k^3$	**M1**	Split into separate sums with constants extracted.
		$mc^3 + 3c^2\sum_{k=1}^{m} a_k + 3c\sum_{k=1}^{m} a_k^2 + \sum_{k=1}^{m} b_k^3$	**A1**	Changed terms to refer to the other sequence. Make clear that they are using the properties of balanced subsets.

Question		Answer	Marks	Guidance
		$\sum_{k=1}^{m}(c+a_k)^3$ $mc^3+3c^2\sum_{k=1}^{m}b_k+3c\sum_{k=1}^{m}b_k^2+\sum_{k=1}^{m}a_k^3$ Therefore $\sum_{k=1}^{m}a_k^3+\sum_{k=1}^{m}(c+b_k)^3=\sum_{k=1}^{m}(c+a_k)^3+\sum_{k=1}^{m}b_k^3$	**E1** **A1** **[5]**	Justification, e.g. by "by given result." Fully clear justification.
	iii	Suppose $\sum_{k=1}^{m}a_k=\sum_{k=1}^{m}b_k$ $\sum_{k=1}^{m}a_k^2=\sum_{k=1}^{m}b_k^2$ $\sum_{k=1}^{m}a_k+c=\sum_{k=1}^{m}b_k+c$	 **B1**	 Statement that the sums remain equal if a constant is added to all terms.

Question		Answer	Marks	Guidance
		$\sum_{k=1}^{m}(b_k + c)^2$	**M1**	Show that the sums of squares remain equal if a constant is added to all terms.
		$\sum_{k=1}^{m} c^2 + 2c\sum_{k=1}^{m} b_k + \sum_{k=1}^{m} {b_k}^2$		
		$\sum_{k=1}^{m} c^2 + 2c\sum_{k=1}^{m} a_k + \sum_{k=1}^{m} {a_k}^2$		
		$\sum_{k=1}^{m}(a_k + c)^2$	**A1**	
		Let the sequence a_k be 1, 4, 6, 7 and the sequence b_k be 2, 3, 5, 8.		
		Choosing c=8.	**M1**	
		The sets:	**E1**	Explanation that sums and sums of squares are equal.
		$\{1,4,6,7,2+8,3+8,5+8,8+8\}$ and $\{2,3,5,8,1+8,4+8,6+8,7+8\}$ the required properties.	**E1**	Explanation that the sums of cubes are equal.
		$\{1,4,6,7,10,11,13,16\},\{2,3,5,8,9,12,14,15\}$	**A1**	
			[7]	

Question		Answer	Marks	Guidance
	iv	$\{1,3,4,5,9,11\}$ and $\{0,2,6,7,8,10\}$ is a balanced partition of the set $\{0,1,2,\ldots,11\}$ and each subset contains 6 elements. Therefore, $\{n+1, n+3, n+4, n+5, n+9, n+11\}$ and $\{n, n+2, n+6, n+7, n+n+8, n+10\}$ is a balanced partition of the set $\{n, n+1, n+2, \ldots, n+11\}$ $\{(n+1)^2, (n+3)^2, (n+4)^2, (n+5)^2, (n+9)^2, (n+11)^2\}$ and $\{n^2, (n+2)^2, (n+6)^2, (n+7)^2, (n+8)^2, (n+10)^2\}$ is a pair of sets with the required property.	**B1** **E1** **E1** **[3]**	If 0 not included, then none of the three marks are available

Question		Answer	Marks	Guidance
9	i	GPE = 0 at distance r below A and B. At the point of collision: Let the angle that the string makes with the vertical be θ. $\sin\theta = \frac{\sqrt{2}}{2}$, so $\theta = \frac{\pi}{4}$ Gain in GPE for particle = $mgr\left(1 - \frac{\sqrt{2}}{2}\right)$	**M1**	Or Considering conservation of energy
		For each particle: Initial $KE = \frac{1}{2}mu^2$ Let final speed be v. $mgr\left(1 - \frac{\sqrt{2}}{2}\right) + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$ $v^2 = u^2 - gr\left(2 - \sqrt{2}\right)$	**A1**	Show how the given relationship for v arises.
		Horizontal component of velocity before collision has magnitude $\frac{v\sqrt{2}}{2}$. For collision: Horizontal speed of approach = $v\sqrt{2}$		

Question		Answer	Marks	Guidance
		Horizontal speed of separation = $ev\sqrt{2}$	**M1**	
		For each particle, horizontal component of speed = $\frac{1}{2}ev\sqrt{2}$	**A1**	
		Vertical component is unchanged, so = $\frac{1}{2}v\sqrt{2}$	**B1**	
			[5]	
	ii	Horizontally: $$x = \frac{1}{2}ev\sqrt{2}t$$		
		Vertically: $$y = \frac{1}{2}v\sqrt{2}t - \frac{1}{2}gt^2$$	**M1**	Set up equations for projectile motion.
		Coordinates of point where particle is located taking the point of suspension as the origin: $$\left(\frac{r\sqrt{2}}{2} - \frac{1}{2}ev\sqrt{2}t, -\frac{r\sqrt{2}}{2} + \frac{1}{2}v\sqrt{2}t - \frac{1}{2}gt^2\right)$$		

Question		Answer	Marks	Guidance
		String becomes taut when: $\left(\frac{r\sqrt{2}}{2}-\frac{1}{2}ev\sqrt{2}t\right)^2+\left(-\frac{r\sqrt{2}}{2}+\frac{1}{2}v\sqrt{2}t-\frac{1}{2}gt^2\right)^2 = r^2$ $(r-evt)^2+\left(-r+vt-\frac{\sqrt{2}}{2}gt^2\right)^2=2r^2$	**M1** **A1** **[3]**	Answer given.

Question		Answer	Marks	Guidance
	iii	$\left(1-\frac{evt}{r}\right)^2+\left(-1+\frac{vt}{r}-\frac{\sqrt{2}}{2r}gt^2\right)^2=2$		
		$(1-ez)^2+\left(-1+z-\frac{v^2t^2}{r^2c}\right)^2=2$		
		$(1-ez)^2+\left(-1+z-\frac{z^2}{c}\right)^2=2$	M1	Substitute the given variables to remove all instances of g and t.
		$1-2ez+e^2z^2+1+z^2+\frac{z^4}{c^2}-2z+\frac{2z^2}{c}-\frac{2z^3}{c}=2$		
		$\frac{z^4}{c^3}-\frac{2z^3}{c^2}+\frac{z^2}{c}(e^2+1)+\frac{2z^2}{c^2}-\frac{2ez}{c}-\frac{2z}{c}=0$	M1	Obviously quartic
		$z\left(\left(\frac{z}{c}\right)^3-2\left(\frac{z}{c}\right)^2+\left(\frac{z}{c}\right)(e^2+1)+\left(\frac{2}{c}\right)\left(\frac{z}{c}\right)-\frac{2}{c}(e+1)\right)=0$		

Question		Answer	Marks	Guidance
		$z = 0$ corresponds to the time of the collision, so is not one of the required solutions.	**E1**	Justify division by z $t \neq 0$ is enough
		$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1+e) = 0$	**A1** **[4]**	Answer given

Question		Answer	Marks	Guidance
	iv	If there are three equal roots then each root must be equal to $\frac{2}{3}$ (since the sum of the roots is the coefficient of z^2).	**B1**	Can also write $(z-R)^3=0$ and then expand and find $R=\frac{2}{3}$ for this mark.
		$\frac{2}{c}+1+e^2=3\left(\frac{2}{3}\right)^2$		
		$\frac{2}{c}(1+e)=\left(\frac{2}{3}\right)^3$		
		$e=\frac{4c}{27}-1$		
		$\frac{2}{c}+1+\left(\frac{4c}{27}-1\right)^2=\frac{4}{3}$		
		$(1+e)\left(\frac{4}{3}-1-e^2\right)=\frac{8}{27}$		
		$9(1+e)(1-3e^2)=8$		
		$27e^3+27e^2-9e-1=0$	**M1**	For arranging into a cubic = 0
		$27\left(\frac{1}{3}\right)^3+27\left(\frac{1}{3}\right)^2-9\left(\frac{1}{3}\right)-1=1+3-3-1=0$	**M1**	For showing $e=\frac{1}{3}$ solves the cubic. Can also show $e=\frac{1}{3}$ and $c=9$ solves the *c*-*e* equations, but in this case M0M1A0
		Therefore $e=\frac{1}{3}$ is a solution.	**A1**	For correct *e* and v^2

Question		Answer	Marks	Guidance
		Sum of roots $= -1$ and product of roots $= -1$, so the other two roots are both negative and therefore not possible values for e.	**E1**	(or find the other roots and make the same observations)
		Since the root is repeated, the string must become taut at a point where the direction of motion is tangential to the circle.	**E1**	Ignore subsequent discussion
		.	**[6]**	

Question		Answer	Marks	Guidance
	v	KE immediately after collision: $\frac{1}{2}m\left(\left(\frac{1}{6}v\sqrt{2}\right)^2+\left(\frac{1}{2}v\sqrt{2}\right)^2\right)=\frac{5}{18}mv^2$ Since no other energy is lost: $\frac{1}{2}mU^2=\frac{5}{18}mv^2+mgr\left(1-\frac{\sqrt{2}}{2}\right)$ $U^2=\frac{5}{9}v^2+2gr\left(1-\frac{\sqrt{2}}{2}\right)$ $U^2=\frac{5}{2}\sqrt{2}rg+2gr\left(1-\frac{\sqrt{2}}{2}\right)$ $=\left(2+\frac{3\sqrt{2}}{2}\right)rg$	**M1** **A1** **[2]**	Must substitute $e=\frac{1}{3}$

Question		Answer	Marks	Guidance
10	**i**	$AM = BM = PM = a$, so A, B and P lie on a circle with centre M.	**E1**	triangles are isosceles, stated or clear from diagram
		Since AB is a diameter of the circle, the angle at P, which is angle APB must be a right angle.	**E1**	
		$BP = 2a\sin\theta$	**B1**	Must be in the final simplified form (which can be given if it is seen later within the calculations)
		$AP = \sqrt{(2a)^2 - (2a\sin\theta)^2} = 2a\cos\theta$	**B1**	
		$\lvert T_A\rvert = \frac{\lambda(2a\cos\theta - l)}{l}$	**B1**	
		$\lvert T_B\rvert = \frac{\lambda(2a\sin\theta - l)}{l}$	**B1**	
		Let $\hat{\boldsymbol{a}}$ be a unit vector in the direction PA. Let $\hat{\boldsymbol{b}}$ be a unit vector in the direction PB.	**M1**	Setting up the pair of vectors.

Question		Answer	Marks	Guidance
		$T_A + T_B = \frac{\lambda(2a\cos\theta - l)}{l}\widehat{\boldsymbol{a}} + \frac{\lambda(2a\sin\theta - l)}{l}\widehat{\boldsymbol{b}}$	**M1**	Write the sum of the tensions in vector form (can FT if they previously got incorrect tension expressions) Award equivalent marks if resolving in any two perpendicular directions.
		$= \frac{\lambda}{l}\left(2a\cos\theta\,\widehat{\boldsymbol{a}} + \boldsymbol{2}a\sin\theta\,\widehat{\boldsymbol{b}}\right) - \lambda\left(\widehat{\boldsymbol{a}} + \widehat{\boldsymbol{b}}\right)$	**A1**	Must be correct and split into the components that include l and those that don't
		$2a\cos\theta\,\widehat{\boldsymbol{a}} + \boldsymbol{2}a\sin\theta\,\widehat{\boldsymbol{b}} = 2\overrightarrow{\boldsymbol{PM}}$	**E1**	
		$\widehat{\boldsymbol{a}} + \widehat{\boldsymbol{b}}$ is a vector of length $\sqrt{2}$ in the direction of the bisector of angle APB (since APB is a right angle).	**E1**	

Question		Answer	Marks	Guidance
		Therefore: $T_A + T_B = \frac{2\lambda}{l}\overrightarrow{\boldsymbol{PM}} - \lambda\sqrt{2}\boldsymbol{v}$ where $\boldsymbol{v}$ is a unit vector in the direction of the bisector of angle APB (in the direction towards the ground). Since $PM = a$, the magnitude of $\frac{2\lambda}{l}\overrightarrow{\boldsymbol{PM}}$ is $\frac{2a\lambda}{l}$	**A1** **A1** **[13]**	Clear statement that the second bullet point is shown. Clear statement that the first bullet point is shown.
	ii	Taking moments about M: $a\cos\theta\, T_B + mg\left(\frac{1}{2}a\cos 2\theta\right) = a\sin\theta\, T_A$ $\cos\theta\left(\frac{\lambda(2a\sin\theta - l)}{l}\right) + \frac{1}{2}mg\cos 2\theta = \sin\theta\left(\frac{\lambda(2a\cos\theta - l)}{l}\right)$ $-2\lambda\cos\theta + mg\cos 2\theta = -2\lambda\sin\theta$ $mg\cos 2\theta = 2\lambda(\cos\theta - \sin\theta)$	**M1** **A1**	If applying (i): $mg\left(\frac{1}{2}a\cos 2\theta\right) = \lambda\sqrt{2}(a\sin(45° - \theta))$ Require all in terms of a single angle for this mark (should be θ but they may have relabelled). Must have the mass term correct for this method mark but can be errors with other terms

Question		Answer	Marks	Guidance
		Since $\theta < 45°$,	**E1**	
		$\cos\theta + \sin\theta = \frac{2\lambda}{mg}$		
		$\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45°)$	**M1**	Quick alt method here: $(\cos\theta + \sin\theta)^2 = 1 + sin\,2\theta$. (M1 earned here)
		Therefore, $1 < \frac{2\lambda}{mg} < \sqrt{2}$		Since $0 < \theta < 45°$ (not essential to state here), then $0 < sin\,2\theta < 1$. $1 < (\cos\theta + \sin\theta)^2 < 2$ Again since $0 < \theta < 45°$, $(\cos\theta + \sin\theta)$ is positive and $1 < (\cos\theta + \sin\theta) < \sqrt{2}$ So that $1 < \frac{2\lambda}{mg} < \sqrt{2}$
		$\frac{1}{2}mg < \lambda < \frac{1}{2}\sqrt{2}mg$	**A1** **[5]**	A1 for fully correct solution either way.

Question		Answer	Marks	Guidance
	iii	Taking moments about P: $mg\left(\frac{1}{2}a\cos 2\theta\right) + Fa\sin 2\theta = Na\cos 2\theta$ $\frac{1}{2}mg + F\tan 2\theta = N$ $N - F\tan 2\theta = \frac{1}{2}mg$	**M1** **A1** **[2]**	CSO

Question			Answer	Marks	Guidance
11	i		$\sum_{r=1}^{\infty} rx^{r-1} = \sum_{s=0}^{\infty}\left(\sum_{r=0}^{\infty} x^{s+r}\right)$ $= \sum_{s=0}^{\infty} \frac{x^s}{1-x}$ $= \frac{1}{1-x}\sum_{s=0}^{\infty} x^s$ $= \frac{1}{(1-x)^2}$	**M1** **M1** **A1** **[3]**	Double sum Use of formula for geometric sum AG
	i **(Alt 1)**		$\sum_{r=1}^{\infty} x^r = \frac{x}{1-x}$ Differentiate both sides wrt x: $\sum_{r=1}^{\infty} rx^{r-1} = \frac{1}{1-x} + \frac{x}{(1-x)^2}$ $= \frac{1}{(1-x)^2}$	**M1** **M1** **A1** **[3]**	Use of formula for geometric sum Differentiating both sides wrt x AG

Question			Answer	Marks	Guidance
	i **(Alt 2)**		Using the Binomial Theorem: $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}(n+1)!\,x^n}{n!}$ $= \sum_{n=0}^{\infty} (n+1)x^n$	**M1** **M1** **A1** **[3]**	Binomial Theorem Simplifying the terms AG
	ii	**a**	$P(\text{Even, but not 2k}) = \frac{k-1}{2k}$ $P(\text{Ali wins}) = \sum_{r=0}^{\infty} \left(\frac{k-1}{2k}\right)^r \left(\frac{1}{2k}\right)$ $\left\|\frac{k-1}{2k}\right\| < 1$, so $P(\text{Ali wins}) = \frac{\left(\frac{1}{2k}\right)}{1-\left(\frac{k-1}{2k}\right)} = \frac{1}{k+1}$	**M1** **A1** **A1** **[3]**	Can be inferred from next line/correct working Can be awarded for P(Ali loses) Don't require sight of $\left\|\frac{k-1}{2k}\right\| < 1$

Question			Answer	Marks	Guidance
	ii	b	$E(Win) = \sum_{r=1}^{\infty} r\left(\frac{k-1}{2k}\right)^{r-1}\left(\frac{1}{2k}\right)$ $= \frac{1}{2k} \times \frac{1}{\left(1-\frac{k-1}{2k}\right)^2} = \frac{2k}{(k+1)^2}$	**M1** **A1** **[2]**	
	iii		$\int \sum_{r=0}^{n} (r+1)\binom{n}{r} x^r \, dx = \sum_{r=0}^{n} \binom{n}{r} x^{r+1} + C$ $= x\sum_{r=0}^{n} \binom{n}{r} x^r \; + \; C$ $= x(1+x)^n \; + \; C$ $\sum_{r=0}^{n} (r+1)\binom{n}{r} x^r = \frac{d}{dx}(x(1+x)^n)$ $= (1+x)^n + nx(1+x)^{n-1}$ $(1+(n+1)x)(1+x)^{n-1}$	**M1** **M1** **A1** **M1** **A1**	Termwise integration (sums can be written out in full) Condone missing constant of integration Factorisation Condone missing constant of integration Condone missing constant of integration Differentiation of $x(1+x)^n$ wrt x

Question			Answer	Marks	Guidance
				[5]	
	iii (Alt 3)		$\sum_{r=0}^{n}\binom{n}{r}x^{r+1} = x\sum_{r=0}^{n}\binom{n}{r}x^{r}$	**M1**	Factorisation (sums can be written out in full)
			$= x(1+x)^n$	**A1**	
			Differentiating wrt x gives: $\sum_{r=0}^{n}(r+1)\binom{n}{r}x^{r} = \frac{d}{dx}[x(1+x)^n]$	**M1**	Termwise differentiation of $\sum_{r=0}^{n}\binom{n}{r}x^{r+1}$ wrt x
			$= (1+x)^n + nx(1+x)^{n-1}$	**M1**	Differentiation of $x(1+x)^n$ wrt x
			$(1+(n+1)x)(1+x)^{n-1}$	**A1**	
				[5]	

Question			Answer	Marks	Guidance
	iv		The number of winning sequences of $(r+1)$ rolls is equal to the number of different combinations of r integers between 1 and $2k-1$ (which are rolled in ascending order, followed by $2k$).	**E1**	
			$P(\text{Zen wins £(r + 1)}) = \frac{1}{(2k)^{r+1}}\binom{2k-1}{r}$	**M1**	
			$$E(\text{Winnings}) = \sum_{r=0}^{2k-1} \frac{r+1}{(2k)^{r+1}}\binom{2k-1}{r}$$		
			Using the result from (iii) with $n = 2k-1$ and $x = \frac{1}{2k}$:		
			$E(\text{Win}) = \frac{1}{2k}\left(1+\frac{2k}{2k}\right)\left(1+\frac{1}{2k}\right)^{2k-2}$	**M1**	For using result from (iii) (condone incorrect values of n or x)
			$= \frac{1}{k}\left(1+\frac{1}{2k}\right)^{2k-2}$	**A1**	
				[4]	

Question			Answer	Marks	Guidance
	v		For large values of k, $E(Z\ win) \approx \dfrac{e}{k\left(1+\frac{1}{2k}\right)^2}$		
			$\dfrac{E(Z\ win)}{E(A\ win)} \approx \dfrac{e}{k\left(1+\frac{1}{2k}\right)^2 \frac{2k}{(k+1)^2}}$	M1	Attempting to compute $\frac{E(Zen's\ Winnings)}{E(Ali's\ Winnnings)}$ and substituting for e
			$= \dfrac{(k+1)^2 e}{2\left(k+\frac{1}{2}\right)^2} \approx \dfrac{e}{2}$	A1	
			Since $\frac{e}{2}$ is slightly more than 1.35, Zen's expected earnings are a little over 35% more than Ali's expected earnings.	E1	Explaining why a factor of $\frac{e}{2}$ corresponds to a percentage increase of a little over 35%
				[3]	

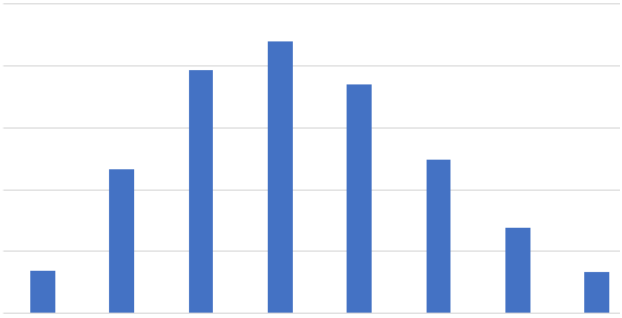
Question		Answer	Marks	Guidance
12	i	$d_r = \frac{e^{-\lambda}\lambda^r}{r!} - \frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}$	M1	Must be seen for general r
		$= \frac{e^{-\lambda}\lambda^{r-1}}{r!}(\lambda - r)$	A1	Accept correct simplified equivalent forms, e.g. $\frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}\left(\frac{\lambda}{r} - 1\right)$
		$\frac{e^{-\lambda}\lambda^{r-1}}{r!} > 0$, so If $r < \lambda$, $d_r = p_r - p_{r-1} > 0 \Rightarrow p_r > p_{r-1}$ If $r > \lambda$, $d_r = p_r - p_{r-1} < 0 \Rightarrow p_r < p_{r-1}$	E1	Sign of d_r for both cases stated (accept weak inequalities) Must see $\frac{e^{-\lambda}\lambda^{r-1}}{r!} > 0$ OE (unless candidate has written in terms of a probability, e.g. $d_r = P(X = r-1)\left(\frac{\lambda}{r} - 1\right)$, when there is no need to state explicitly that $P(X = r-1) > 0$)

Question		Answer	Marks	Guidance
		Therefore, the maximum value of p_r occurs for the largest integer value of r that is less than λ. Since λ is not an integer, this is equivalent to $\lambda - 1 < r < \lambda$	**E1** **[4]**	AG Must include: • maximum value of p_r occurs for the largest integer value of $r < \lambda$ • λ is not an integer • Conclusion Just using $d_r > 0$ and $d_{r+1} < 0$ to reach the conclusion earns E0 (this only shows that the given answer is necessary, not sufficient)

Question		Answer	Marks	Guidance
	ii	$\frac{d_r}{d_{r-1}} = \frac{\frac{e^{-\lambda}\lambda^{r-1}}{r!}(\lambda - r)}{\frac{e^{-\lambda}\lambda^{r-2}}{(r-1)!}(\lambda - r + 1)}$	M1	
		$= \frac{\lambda(\lambda - r)}{r(\lambda - r + 1)}$	A1	
		d_r will be positive for $r \leq m$ and negative for $r > m$, so the minimum value of d_r occurs for a value of $r > \lambda$.	B1	Statement that arg min is $> \lambda$
		For $r - 1$ in this range, the value of d_r decreases while $\frac{d_r}{d_{r-1}} = \frac{\lambda(\lambda - r)}{r(\lambda - r + 1)} > 1$	M1	Can also award for setting equal to 1
		Since $\lambda - r + 1 < 0$: $\lambda(\lambda - r) < r(\lambda - r + 1)$	E1	
		$\Rightarrow r^2 - r(1 + 2\lambda) + \lambda^2 < 0$		

Question	Answer	Marks	Guidance
	Critical values are $\lambda+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}}, \lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$	**A1**	OE. E.g. if candidate uses $r+1$ in place of r, accept $\lambda-\frac{1}{2}\pm\sqrt{\lambda+\frac{1}{4}}$
	Therefore, the minimum value of d_r occurs at k, the largest integer value of r that is less than $\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$	**E1**	
	Since $\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$ is not an integer, this is equivalent to $k<\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}<k+1$	**E1**	AG Must include $\lambda+\frac{1}{2}\sqrt{\lambda+\frac{1}{4}}$ is not an integer
		[8]	

Question		Answer	Marks	Guidance
	iii	$d_r \to 0$ as $r \to \infty$, so the maximum value of d_r can only occur at d_1 if $d_1 \geq 0$ $\Leftrightarrow \lambda \geq 1$	**B1**	Award for strict inequality (although such solutions will lose the final E1)
		The value of d_r increases for $r < \frac{2\lambda + 1 - \sqrt{4\lambda + 1}}{2}$ and then decreases for $\frac{2\lambda + 1 - \sqrt{4\lambda + 1}}{2} < r < \lambda$ after which d_r becomes negative.	**E1**	
		Therefore the maximum value of d_r occurs at $r = 1$ if $d_2 < d_1$	**M1**	OE (such as $d_2 - d_1 < 0, \frac{d_1}{d_2} > 1,\ 2 > \lambda + \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}$ etc.)
		$\frac{\lambda}{2}(\lambda - 2) < \lambda - 1$ $\lambda^2 - 4\lambda + 2 < 0$		

Question		Answer	Marks	Guidance
		$2-\sqrt{2}<r<2+\sqrt{2}$	**A1**	Must include both roots
		But λ is not an integer, so the maximum value of d_r occurs at $r=1$ if: $1<\lambda<2+\sqrt{2}$	**E1**	Must include explanation of why $\lambda=1$ is excluded.
			[5]	Candidates who instead try to find the condition that the *minimum* value of d_r occurs at $r=1$ by plugging $r=1$ into the result of (ii) earn no credit
	iv	$\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}=3.36+0.5+\sqrt{3.61}=5.76$	**B1**	Can be awarded for bounding above and below by suitable bounds (to facilitate computing the argmin of d_r)
		The minimum value of d_r occurs at $r=5$		
			**G1**	Maximum value is for $x=3$.
			G1	Largest drop in value occurs from $x=4$ to $x=5$.
				Continuous graphs earn G0 G0
			[3]	

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