Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2023

Examiners' Report

Mark Scheme

STEP MATHEMATICS 3

2023

Examiners' Report

STEP 3 Introduction

The total entry was a marginal increase on that of 2022 (by just over 1%). Two questions were attempted by more than 90% of candidates, another two by 80%, and another two by about two thirds. The least popular questions were attempted by more than a sixth of candidates. All the questions were perfectly answered by at least three candidates (but mostly more than this), with one being perfectly answered by eighty candidates. Very nearly 90% of candidates attempted no more than 7 questions.

One general comment regarding all the questions is that candidates need to make sure that they read the question carefully, paying particular attention to command words such as “hence” and “show that”.

Question 1

Although this was a very popular question, being attempted by nearly 93% of the candidature, it was very narrowly beaten into second place by question 5. It was the third most successfully attempted with a mean score of approximately 9.5/20.

Most candidates found the equation of the line, and of the circle and then solved simultaneously in part (i) to find common points, rather than using the perpendicular distance from a line formula; some using the distance formula misquoted it, with a common error being failure to include the modulus signs. Then they generally applied use of the discriminant, but with varying success.

In part (ii), most candidates successfully expressed the given expression as a quadratic in q, obtained the determinant and the two required expressions using Vieta's formulas, but failed to fully demonstrate the inequality.

Attempts at part (iii) were frequently inelegant and involved repeating work from previous parts of the question, rather than using the results of part (i) and (ii).

Question 2

This was the fifth most popular question on the paper, being attempted by about two thirds of the candidates, just a few more than question 8. However, it was the most successfully attempted with a mean score of over 12/20. Many candidates produced excellent responses to this question, and a number scored a perfect 20/20.

The two curves were generally well sketched in part (i), with the commonest fault being failure to obtain values for r at the points where $\theta = 0, \frac{1}{2}\pi, \pi$.

The derivation of the required result in (i) for the point of intersection as well as the result for the area of A in part (ii) were generally well done.

Similarly, in part (iii) the area of B was well attempted, although algebraic errors were more common here with the required result not being given in the question, unlike the area of A in part (ii).

Part iv) was found to be the most difficult part of the question, though marks for finding expressions for S and T were generally obtained. After this, the first challenge was to notice that $\alpha \to 0$, and it was important to justify this observation using the expression $\tan(\alpha) = \frac{1}{k}$ from part (i). A small number of candidates made heuristic arguments that $\alpha \to 0$, using their sketches - however, this was not acceptable for the mark without further justification. The next challenge was to compute the limits rigorously, and candidates found this to be the most challenging aspect of the question. Common mistakes included not noticing that α depended on k, and substituting $\alpha = 0$ prematurely, which, for example, led to the erroneous conclusion that $ksin(\alpha) \to 0$.

Question 3

This was the least popular of the Pure questions, being attempted by just under 45% of the candidates. Furthermore, it was not well answered yielding a mean score of 6/20.

Many incorrectly treated a and b as real numbers in part (i) which rendered the question very simple. On the other hand, there were some that correctly simplified their working by 'ignoring' a in each number and then translating the triangle by a after. The second part of (i) was often well-answered.

Most attempts at part (ii) were decent. Students that attempted it recognised that the roots should be written using a, b, s from part (i) and wrote down the sum/product of roots formulae for p and q. A few did not write down the equation for the coefficient of z^2, and without this it was not possible for them to earn further credit for simplifying p and q. Sign errors in q were not uncommon.

In part (iii), there was a large variety among the sketches seen. Only a few candidates specified the leading order behaviour at infinity. A fair number of candidates did not reflect the nature of the point of inflection in their drawing. Some did not specify intercepts. Pretty nearly all recognised the asymptote at -1/9.

In part (iv), the majority that had successfully drawn the sketch in part (iii) managed to successfully satisfy the logic, although some failed to obtain the reality of the expression even though this was explicitly required.

Question 4

This was only marginally more popular than question 3 and was the least successfully attempted question on the paper with a mean score of 5/20.

In the vast majority of cases, there was no substantially correct attempt except in part (i). Those that used de Moivre's theorem, expanded binomially, equated real parts and replaced $\sin^{2r}\theta$ by $(\cos^2\theta - 1)^r$, generally scored well in this part, but marks could not be credited where mathematical steps were glossed over when the question stated, 'Show that'. Some attempted use of proof by induction, but their conclusions were not supported by their mathematical argument.

Many attempting part (ii) wrote down the coefficient required but made no further progress, or made the expansion and went no further. There were some that did substitute +1 and -1, and solved this part.

Part (iii) was solved generally by those that had made the substitutions in (ii) and saw that differentiation might be useful.

Part (iv) could be answered using the given results from the previous parts of the question, however this part was almost exclusively only attempted by candidates that had had a reasonable level of success on the previous three parts. Those candidates that set out a careful and organised solution were more successful in part (iv).

Question 5

Whilst this was the most popular question, it was only the sixth most successful with a mean mark of a little under 9/20. Many of the candidates made substantial attempts at parts (i) and (ii) but found it more challenging to make progress with part (iii). Two common general errors were lack of precision when handling inequalities, and working backwards from a required result without demonstrating that the logic was reversible.

In part (i), a small number of candidates rearranged the first equation to remove denominators then wrote the required result without adequate intermediate steps of working. There were a very small number of arithmetic errors when finding the pairs of x and y.

In (ii) many candidates commented that as p and q were prime then the only possible factors of pq were $1, p, q$ and pq and went on to test each of these as possible values for $p + q + n$. Many candidates were able to form a relevant equation involving p and q and whilst most factorised it, similarly to part (i), a small number attempted alternative approaches. The most successful of these was to write p as a function of q and rewrite the improper fraction to see that $q - 2$ must divide 3. A small number of candidates spotted that p and q were the solutions to the quadratic equation $t^2 - (n + 1)t + (2n + 1) = 0$ but from here few were able to fully justify that the only solutions for (p, q) came from $n = 7$.

The first two results of part (iii) caused much confusion. Relatively few candidates realised at the start of their attempts that these were equivalent to $q < n$ and $p < n$. Those who did recognise this completed part (iii) with relative ease. For the second part, a pleasing number of candidates realised that $(p + q)^3$ would be a useful expression to consider and those who did usually managed to get to the difference of two cubes expression necessary to make progress. Some candidates were unsure where to go next but a good number realised the importance of the printed inequalities and correctly deduced that $p + q - n$ must be 1 or 3. From here candidates often managed to rule out one case but ruling out both successfully was relatively rare.

Question 6

The fourth most popular question being attempted by just over three quarters of the candidates, it was the second most successful with a mean of just over half marks. The best responses involved clear algebra and working, with the given results fully justified. Many candidates picked up marks by accurate differentiation, whilst the best candidates were able to sketch graphs showing all the main features and could carefully justify results. Many parts of the question asked candidates to show a given result, which meant that candidates needed to ensure they showed sufficient working before reaching the given result. In part (i) and part (ii)(b) candidates were required to use a specified method; candidates who did not use this method did not gain all the marks.

There were some good answers to part (ii), but many candidates failed to show a stationary point of inflection at the origin, possibly as they assumed that they had shown the graph was strictly increasing rather than increasing. Failing to show the asymptote limits was another common mistake.

Part (ii) (a) was found to be the hardest. Many candidates did not justify their results (such as the behaviour of g(x) or g'(x) as x tends to infinity). Some candidates drew graphs to help justify their result, but these generally did not explain why their graphs looked as they did. However, part (ii)(b) was generally done well, as was (c) by those that attempted it.

Many candidates failed to gain the marks in part (d), mainly through failing to consider the symmetry of $\frac{\cosh^2 x}{1+x^2}$.

Candidates found the graph in part (e) easier to sketch than the one in part (i). The most common mistake here was to have the graph reflected, with $\mathrm{g}(x)$ positive when x is positive, incorrectly.

Question 7

This was the third most popular question and was only marginally less successfully attempted than questions 1 and 11 with over 9.4/20 the mean mark. It was generally answered well by candidates, with many candidates earning more than half the marks for this question and many candidates earning full, or close to full, marks.

The majority of candidates successfully earned full credit in part (i).

They also did well on part (ii) though quite a number did not use $(\mathrm{g}(x) - x)^2 \geq 0$ when justifying why $\mathrm{g}(x) = x$, or incorrectly stated that $(\mathrm{g}(x) - x)^2 > 0$.

In part (iii), candidates who integrated $2\int_0^1 x\mathrm{h}'(x)\mathrm{d}x$ by parts generally went on to earn full, or close to full, marks for this part. A number of candidates began by writing down the equation from (ii) with $\mathrm{h}'(x)$ in place of $\mathrm{g}(x)$. In some cases, candidates successfully 'worked backwards', cancelling down each side to verify their initial equality, however less successful attempts simply assumed the initial equation, without justification.

Many candidates observed that part (iv) could be solved by considering the integral $\int_0^1 \left(\mathrm{e}^{\frac{1}{2}ax}\,\mathrm{k}(x) - \mathrm{e}^{-\frac{1}{2}ax}\right)^2 \mathrm{d}x$. Sadly, many candidates failed to factorise the resulting quadratic in $\frac{1}{a}$, and another common error was simply to set the quadratic equal to zero without valid justification. Many, too, tried (unsuccessfully) to solve using integration by parts.

Question 8

This was only a little less popular than question 2 and was only answered with moderate success having a mean score of 7.3/20.

Part (i) was mostly well answered, although some candidates lost marks by not being thorough in demonstrating the inequality, or by using a characteristic equation and failing to verify their solution as required.

Part (ii) was found difficult, for although g_1 was almost always stated, many struggled to find g_2, often just flipping the sign of g_1. Even when a general solution was found, many candidates used the boundary conditions of (i) instead of appreciating the sign change of the derivative at $x = 1$.

Part (iii) was done extremely poorly, even by candidates who had the right functions and the algebraic relationship between them.

Candidates could often pick up marks on part (iv), even if they were less successful with the rest of the question, though notation of what derivatives were being taken was often ambiguous.

In attempting to answer part (v), many candidates knew they needed to use the result of part (iv) for part (a), although a significant number lost marks for giving no explanation or working. Part (b) proved much harder than part (a), since few candidates realized they had to match their function at $\frac{5}{4}\pi$, not at $-\frac{1}{4}\pi$ again. Some candidates fully solved both (a) and (b) directly via the characteristic equation which led to a very lengthy solution.

Question 9

This question only just beat question 11 to be the least popular question on the paper. Although its mean score was only 6.9/20, it was more successfully attempted than two of the Pure questions and the other Mechanics question.

Parts (i) and (ii) were well answered by a good number of candidates, if candidates once set up the problem and then got going. The diagrams were done well, and the derivatives didn't pose much of a problem for most, although there were some errors in the second derivatives and applying the chain rule, with candidates forgetting to multiply by $\dot{\theta}$ to produce $\dot{\theta}^2$.

Part (iii) was more mixed in terms of good responses. Those who did this by resolving forces horizontally and vertically set up the remainder of the question well, but some seemed to struggle with the first part and just assumed the equations were true. The biggest problem for a large number of candidates here was applying boundary conditions when integrating and justifying the choice of boundary conditions.

Parts (iv) and (v) were answered fairly well.

Part (vi) lead to a lot of marks lost, as they were required to justify the velocity being negative by finding a suitable time to ensure it happens, which most did not do.

Question 10

The most popular of the Applied questions, it was also the least successfully attempted, and was only slightly better attempted than question 4. If a candidate found this question difficult, it tended to be from the start, failing to draw a correct diagram of what was going on. If they did set up the problem correctly then finding different angles in terms of the given angle β proved problematic. This meant that often sin was found instead of cos and vice versa, within attempted solutions.

Many successfully took moments about A and resolved forces vertically and horizontally, but most were unable to use these to produce the inequality required in part (i).

Those few that did well in part (i) generally also did well in part (ii), with only one adjustment to the diagram needed, and the resulting algebra was worked through with little issue. The hint for the final part was used well by candidates, but only a few managed to turn the resulting square into a minimum value for μ.

Question 11

Although it was unpopular, those that attempted this scored better than on other Applied questions and, indeed, five of the Pure questions. Most candidates recognised that the expression on the left in the stem could be separated into two sums, one of which would produce $\mathrm{e}^x - 1$ while the other would produce $x\mathrm{e}^x$.

A small number of candidates falsely seemed to assume that since

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \mathrm{e}^x$$

it immediately follows that

$$\sum_{k=0}^{\infty} (k+1)\frac{x^k}{k!} = (x+1)\mathrm{e}^x$$

Some candidates chose to write out the sums without using summation notation, which made some parts of the justification more difficult to express clearly.

In part (i) most candidates were able to work out the value of $\mathrm{P}(D = 0)$ and a large number were able to give some justification of the first formula for $\mathrm{E}(D)$. Many however did not manage to justify fully how all parts of the required expression were deduced and in a question in which the answer to be reached is known, it is important that solutions clearly express the steps that are involved. In particular, many candidates simply stated that the sum over values of k ran from d to infinity without any comment that this is because there had to be at least d sides on the die. Similarly, many candidates failed to express the clear reasoning required to show the second form of $\mathrm{E}(D)$.

Candidates were generally successful in using the formula for $\mathrm{E}(D)$ to complete part (i)(c) and the majority recognised the significance of the result in the stem to the work here.

Many of the candidates who attempted part (ii) were able to make good progress, although there were some who failed to understand the sequence in which the events take place in this second situation. Most were able to find an initial expression for the value of $\mathrm{P}(Z = 0)$ and the majority recognised that this was a sum of a geometric series. However, several candidates calculated the sum to infinity instead of the required sum of n terms and some of those who correctly calculated the sum of n terms then made errors when dealing with the powers in the simplification.

Many of the candidates who attempted to calculate $\mathrm{E}(Z)$ were able to reach a correct form, but relatively few recognised that changing the order of summation (as in part (i)) would again help to simplify the expression. Those who found the correct expression were generally able to justify that $\mathrm{E}(Z) > \mathrm{E}(D)$, although some did not comment on the fact that the exponential term must be positive as part of their justification.

Question 12

This was only a little more popular than questions 9 and 11, and it was the seventh most successfully attempted with a mean score of 7.4/20 . While there were a number of attempts that did not manage to make any significant progress, those who were able to analyse the situation were able to make very good progress.

When finding the probability in part (i), the most common methods employed were to count the number of ways in which the outcome could be achieved or to create a product of individual probabilities by considering the socks being taken one at a time.

Answers to part (ii) were generally well done, with most candidates providing a good explanation of the role of each part of the required formula.

For the final part, most candidates chose to use algebra to show the given result, and this was generally done successfully, although a small number of candidates did not show fully the steps that were being taken. Almost all candidates who reached this point were able to apply the result shown to the formula for the expectation and most realised that several factors could be moved outside the sum. Only a small number did not realise that the summation that remained was the sum of the probabilities of all possible outcomes for that random variable.

STEP MATHEMATICS 3

2023

Mark Scheme

1. (i) The line through P and Q is

$$\frac{y-ap^2}{x-2ap}=\frac{y-aq^2}{x-2aq}$$

Alternatives

$$\frac{y-ap^2}{x-2ap}=\frac{ap^2-aq^2}{2ap-2aq}=\frac{p+q}{2}$$

$$\frac{y-aq^2}{x-2aq}=\frac{ap^2-aq^2}{2ap-2aq}=\frac{p+q}{2}$$

or multiplied to remove denominators.

M1

$$(y-ap^2)(x-2aq)=(y-aq^2)(x-2ap)$$

$$(2ap-2aq)y+2a^2(p^2q-pq^2)=(ap^2-aq^2)x$$

P and Q are distinct thus $p\neq q$ and so $p-q\neq 0$

Therefore $2y+2apq=(p+q)x$ that is $(p+q)x-2y-2apq=0$

A1

The perpendicular distance of $(0,3a)$ from the line PQ is $2a$ which requires

$$\left|\frac{-6a-2apq}{\sqrt{((p+q)^2+4)}}\right|=2a$$

M1 A1 A1 A1

that is $(pq+3)^2=(p+q)^2+4$ **M1 A1**

i.e. $(p+q)^2=p^2q^2+6pq+9-4=p^2q^2+6pq+5$ (*)

A1* (9)

Alternatives M1A1 as before

(I) $(p+q)x-2y-2apq=0$ meets $x^2+(y-3a)^2=4a^2$ when

$$4(y+apq)^2+(p+q)^2(y^2-6ay+5a^2)=0$$

M1 A1

$$(4+(p+q)^2)y^2-(6a(p+q)^2-8apq)y+(5a^2(p+q)^2+4a^2p^2q^2)=0$$

A1

Thus using $(p+q)^2=p^2q^2+6pq+5$, **M1**

$$(pq+3)^2y^2-2a(pq+3)(3pq+5)y+a^2(3pq+5)^2=0$$

which is a perfect square, **A1**

so $(pq+3)y-a(3pq+5)=0$ which only has a single root so the line is a tangent. **A1 A1***

(II) Foot of perpendicular from $(0,3a)$ to $(p+q)x-2y-2apq=0$ is at intersection with $(p+q)y+2x=3a(p+q)$ **M1 A1**

So solving $(p+q)^2y+4y+4apq=3a(p+q)^2$ **A1**

$y=\frac{3a(p+q)^2-4apq}{(p+q)^2+4}$ and $x=\frac{(p+q)}{2}\left(3a-\frac{3a(p+q)^2-4apq}{(p+q)^2+4}\right)=\frac{2a(p+q)(3+pq)}{(p+q)^2+4}$

A1

and so the square of the distance is **M1 A1**

$$\left[\frac{2a(p+q)(3+pq)}{(p+q)^2+4}\right]^2+\left[\frac{4a(3+pq)}{(p+q)^2+4}\right]^2=\left[\frac{2a(3+pq)}{(p+q)^2+4}\right]^2(4+(p+q)^2)$$
$$=\frac{(3+4a^2pq)^2}{(4+(p+q)^2)}=4a^2$$

using given condition. **A1***

(III) Method is possible by differentiation of circle equation. Partial or incorrect solution by this method zero marks; completely correct solution full marks; completely correct solution except minor inaccuracy, withhold one accuracy mark and final accuracy mark.

(ii) (*) can be re-written

$$q^2(p^2-1)+4pq+(5-p^2)=0$$

M1

Considering this as a quadratic equation for q , to be two distinct roots, $p^2-1\neq 0$ (it is given that $p^2\neq 1$) **E1** and the discriminant needs to be positive.

$$16p^2-4(p^2-1)(5-p^2)=4(p^4-2p^2+5)=4(p^2-1)^2+16>0$$

as required. **M1 A1**

$q_1+q_2=\frac{-4p}{(p^2-1)}$, $q_1q_2=\frac{(5-p^2)}{(p^2-1)}$ **A1 A1 (6)**

(iii) Given P, with $p^2\neq 1$, by (ii) points Q_1 and Q_2 can be defined with parameters q_1 and q_2 where q_1 and q_2 are the roots of (*). So by (i), PQ_1 and PQ_2 are tangents to the circle centre $(0,3a)$ radius $2a$. **E1**

The perpendicular distance of $(0,3a)$ from the line Q_1Q_2 is

$$\left|\frac{-6a-2aq_1q_2}{\sqrt{((q_1+q_2)^2+4)}}\right|=\left|\frac{-6a-2a\frac{(5-p^2)}{(p^2-1)}}{\sqrt{\left(\left(\frac{-4p}{(p^2-1)}\right)^2+4\right)}}\right|=2a\left|\frac{3(p^2-1)+(5-p^2)}{\sqrt{16p^2+4(p^2-1)^2}}\right|$$

M1 A1

$$= 2a\left|\frac{2p^2+2}{\sqrt{4p^4+16p^2+4}}\right| = 2a$$

A1

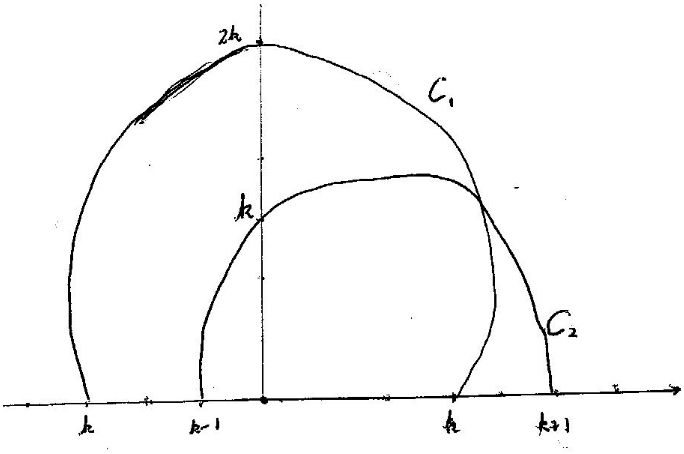
Alternative Q_1Q_2 is the third such line provided that $(q_1q_2+3)^2 = (q_1+q_2)^2+4$

$$(q_1q_2+3)^2-(q_1+q_2)^2-4 = \left[\frac{(5-p^2)}{(p^2-1)}+3\right]^2 - \left[\frac{-4p}{(p^2-1)}\right]^2 - 4$$
$$= \frac{4(p^2+1)^2-16p^2-4(p^2-1)^2}{(p^2-1)^2} = 0$$

M1 A1 A1

Thus PQ_1Q_2 is the triangle required. **E1 (5)**

2. (i)



G1 G1 G1 G1

At intersection, when $\theta = \alpha$, $k(1+\sin\theta) = k + \cos\theta$

Therefore, $k\sin\alpha = \cos\alpha$, that is, $\tan\alpha = \frac{1}{k}$ **B1* (5)**

(ii) Area A is

$$\frac{1}{2}\int_0^{\alpha}\left(k(1+\sin\theta)\right)^2 d\theta = \frac{k^2}{2}\int_0^{\alpha} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

M1

$$= \frac{k^2}{2}\int_0^{\alpha} 1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\, d\theta = \frac{k^2}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\alpha}$$

dM1 **A1**

$$= \frac{k^2}{2}\left\{\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{4}\sin 2\alpha + 2\right\} = \frac{k^2}{2}\left\{\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{2}\sin\alpha\cos\alpha + 2\right\}$$

$$= \frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1-\cos\alpha)$$

A1* (4)

(iii) Area B is

$$\frac{1}{2}\int_{\alpha}^{\pi}(k+\cos\theta)^2 d\theta = \frac{1}{2}\int_{\alpha}^{\pi} k^2 + 2k\cos\theta + \cos^2\theta\, d\theta = \frac{1}{2}\int_{\alpha}^{\pi} k^2 + 2k\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

M1 **M1**

$$= \frac{1}{2}\left[k^2\theta + 2k\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\alpha}^{\pi} = \frac{1}{2}\left\{k^2\pi + \frac{\pi}{2} - k^2\alpha - 2k\sin\alpha - \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right\}$$

A1

$$= \frac{1}{2}\left\{k^2\pi + \frac{\pi}{2} - k^2\alpha - 2k\sin\alpha - \frac{\alpha}{2} - \frac{1}{2}\sin\alpha\cos\alpha\right\}$$

$$= \frac{1}{4}\{2k^2\pi + \pi - 2k^2\alpha - 4k\sin\alpha - \alpha - \sin\alpha\cos\alpha\}$$

A1 (4)

(iv) As $k \to \infty$, α is small as $\tan\alpha = \frac{1}{k}$ so $\alpha \approx \sin\alpha \approx \tan\alpha = \frac{1}{k}$ and $\cos\alpha \approx 1 - \frac{1}{2k^2}$ **M1**

Area A is $\frac{k}{2} +$ terms of lower order in k **A1**

Area B is $\frac{k^2\pi}{2} +$ terms of lower order in k **A1**

So, area R is $\frac{k^2\pi}{2} +$ terms of lower order in k

Area T is

$$\frac{1}{2}\int_0^{\pi}(k + \cos\theta)^2 d\theta = \frac{1}{4}(2k^2\pi + \pi)$$

or alternatively, use of result from (iii) with $\alpha = 0$

which is $\frac{k^2\pi}{2} +$ terms of lower order in k **B1**

Thus, as required,

$$\frac{area\ of\ R}{area\ of\ T} = \frac{\frac{k^2\pi}{2} + \text{ terms of lower order in } k}{\frac{k^2\pi}{2} + \text{ terms of lower order in } k} \to 1$$

E1

Area S is

$$\frac{1}{2}\int_0^{\pi}\big(k(1 + \sin\theta)\big)^2 d\theta = \frac{k^2}{4} \times 3\pi + 2k^2 = k^2\left(\frac{3\pi}{4} + 2\right)$$

or alternatively, use of result from (ii) with $\alpha = \pi$

B1

Thus

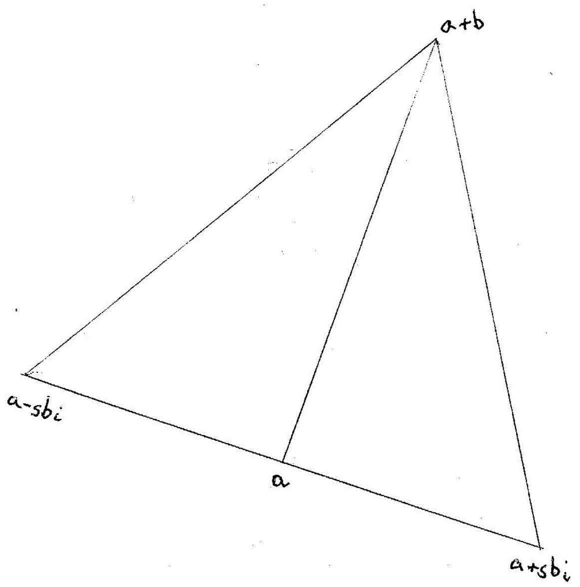
$$\frac{area\ of\ R}{area\ of\ S} \to \frac{\frac{\pi}{2}}{\left(\frac{3\pi}{4} + 2\right)} = \frac{2\pi}{3\pi + 8}$$

A1 (7)

3. (i) $sb\mathrm{i}$ represents a vector perpendicular to the vector represented by b. **E1** Thus, the two points represented by $a \pm sb\mathrm{i}$ are equidistant from the point represented by a **E1** and they are joined to it by vectors which are perpendicular to that joining it to C so they form a base of a triangle which has altitude from a to $a + b$ and has two equal length sides, by Pythagoras. **E1 (3)**

Alternative Distance $a + b$ to $a + sb\mathrm{i}$ is $|(a + sb\mathrm{i}) - (a + b)| = |b||s\mathrm{i} - 1| = |b|\sqrt{s^2 + 1}$ as s is real, **E1** and distance $a + b$ to $a - sb\mathrm{i}$ is $|(a - sb\mathrm{i}) - (a + b)| = |b||-s\mathrm{i} - 1| = |b|\sqrt{s^2 + 1}$, **E1** so two equal length sides. **E1**

a is represented by the midpoint of the base. **B1** b is represented by the vector joining the midpoint of the base to the other vertex. **B1** s is the scale factor that the magnitude of the altitude is multiplied by to obtain half the base. **B1**



(3)

(ii) We require complex a and b and real s such that

$$(a + sb\mathrm{i}) + (a - sb\mathrm{i}) + (a + b) = 0 \Rightarrow b = -3a$$

M1 **A1**

and

$$(a + sb\mathrm{i})(a - sb\mathrm{i}) + (a - sb\mathrm{i})(a + b) + (a + b)(a + sb\mathrm{i}) = p$$

so

$$a^2 + s^2b^2 + 2a(a + b) = p \Rightarrow 3a^2(3s^2 - 1) = p$$

A1

and

$$(a + sb\mathrm{i})(a - sb\mathrm{i})(a + b) = -q \Rightarrow -2a^3(9s^2 + 1) = -q$$

A1

Therefore

$$\frac{p^3}{q^2} = \frac{[3a^2(3s^2-1)]^3}{[2a^3(9s^2+1)]^2} = \frac{27(3s^2-1)^3}{4(9s^2+1)^2}$$

A1* (5)

(iii)

$$y = \frac{(3x-1)^3}{(9x+1)^2}$$

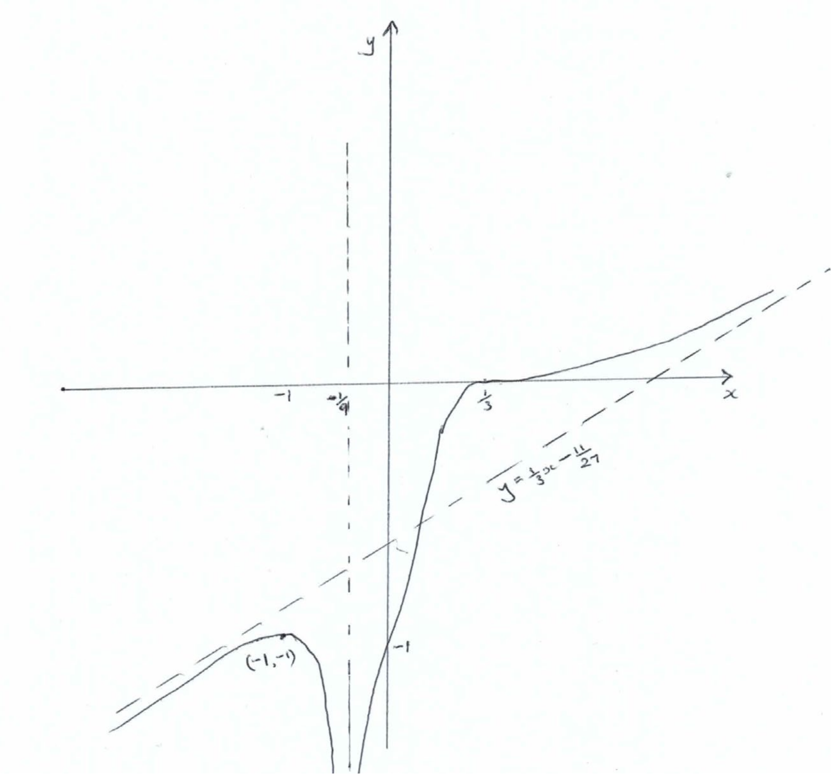
has x intercept at $\left(\frac{1}{3}, 0\right)$, y intercept at $(0, -1)$ **G1** a vertical asymptote at $x = -\frac{1}{9}$ and an asymptote $y = \frac{1}{3}x - \frac{11}{27}$ as $x \to \pm\infty$. **G1**

$$\frac{dy}{dx} = \frac{(9x+1)^2 9(3x-1)^2 - (3x-1)^3 18(9x+1)}{(9x+1)^4}$$

$$= \frac{9(3x-1)^2(9x+1-6x+2)}{(9x+1)^3} = \frac{27(3x-1)^2(x+1)}{(9x+1)^3}$$

M1 A1

Thus, the stationary points are a maximum at $(-1, -1)$ and a point of inflection at $\left(\frac{1}{3}, 0\right)$. **G1 G1**



(6)

(iv) If the roots of $z^3 + pz + q = 0$ represent the vertices of an isosceles triangle, then by (ii), $\frac{p^3}{q^2}$ must be real **E1** and as $s^2 > 0$, from (iii) $\frac{p^3}{q^2} > \frac{27}{4} \times -1 = \frac{-27}{4}$ **E1** as $y = \frac{(3x-1)^3}{(9x+1)^2}$ is increasing for $x > 0$. **E1 (3)**

4. (i) By de Moivre,

$$\cos\big((2n+1)\theta\big) + \mathrm{i}\sin\big((2n+1)\theta\big) = (\cos\theta + \mathrm{i}\sin\theta)^{2n+1}$$

Expanding by the binomial theorem and equating real parts

$$\cos\big((2n+1)\theta\big) = \cos^{2n+1}\theta - \binom{2n+1}{2}\cos^{2n-1}\theta\,\sin^2\theta + \cdots + (-1)^n\binom{2n+1}{2n}\cos\theta\ \sin^{2n}\theta$$

M1 A1

$$= \cos^{2n+1}\theta + \binom{2n+1}{2}\cos^{2n-1}\theta\,(\cos^2\theta - 1) + \cdots + \binom{2n+1}{2n}\cos\theta\ (\cos^2\theta - 1)^n$$

M1

$$= \sum_{r=0}^{n}\binom{2n+1}{2r}\cos^{2n+1-2r}\theta\,(\cos^2\theta - 1)^r$$

A1* (4)

Notice, for (iv), that this expression only contains odd powers of $\cos\theta$.

(ii) The coefficient of x^{2n+1} in $p(x)$ is

$$\sum_{r=0}^{n}\binom{2n+1}{2r}$$

B1

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \sum_{r=0}^{n}\binom{2n+1}{2r}x^{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}x^{2r+1}$$

Substituting $x = 1$,

$$2^{2n+1} = \sum_{r=0}^{n}\binom{2n+1}{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

and substituting $x = -1$, **M1**

$$0 = \sum_{r=0}^{n}\binom{2n+1}{2r} - \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

A1

Adding these two results,

$$2^{2n+1} = 2\sum_{r=0}^{n}\binom{2n+1}{2r}$$

and so the required coefficient is 2^{2n} as required. **A1* (4)**

(iii) The coefficient of x^{2n-1} in $\mathrm{p}(x)$ is

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r}$$

B1

As

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \sum_{r=0}^{n}\binom{2n+1}{2r}x^{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}x^{2r+1}$$

differentiating with respect to x

$$(2n+1)(1+x)^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r}x^{2r-1} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}x^{2r}$$

M1

Substituting $x = 1$,

$$(2n+1)2^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}$$

M1

and substituting $x = -1$,

$$0 = -\sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}$$

A1

Subtracting these two results

$$(2n+1)2^{2n} = 2\sum_{r=0}^{n} 2r\binom{2n+1}{2r} = 4\sum_{r=0}^{n} r\binom{2n+1}{2r}$$

and so the required coefficient is

$$-(2n+1)2^{2n} \div 4 = -(2n+1)2^{2n-2}$$

A1* (5)

Alternative

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\sum_{r=0}^{n} r\,\frac{(2n+1)!}{(2n-2r+1)!\,(2r)!} = -\frac{2n+1}{2}\sum_{r=1}^{n}\frac{(2n)!}{(2n-2r+1)!\,(2r-1)!}$$

M1

$$= -\frac{2n+1}{2}\sum_{r=1}^{n}\binom{2n}{2r-1} = -\frac{2n+1}{2}\sum_{r=0}^{n-1}\binom{2n}{2r+1}$$

M1

As in (ii),

$$\sum_{r=0}^{n-1}\binom{2n}{2r+1} = \frac{1}{2}2^{2n}$$

A1

so

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\frac{2n+1}{2}\frac{1}{2}2^{2n} = -(2n+1)2^{2n-2}$$

A1* (5)

(iv) Suppose

$$\mathrm{q}(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$$

then

$$\mathrm{p}(x) = (x+1)[ax^n + bx^{n-1} + cx^{n-2} + \cdots]^2$$

$$= (x+1)(a^2x^{2n} + 2abx^{2n-1} + (b^2+2ac)x^{2n-2} + \cdots)$$

$$= a^2x^{2n+1} + (a^2+2ab)x^{2n} + (b^2+2ac+2ab)x^{2n-1} + \cdots$$

M1 A1

Thus $a^2 = 2^{2n}$, $a^2 + 2ab = 0$, and $b^2 + 2ac + 2ab = -(2n+1)2^{2n-2}$ **dM1 A1**

Therefore $a = 2^n$ (as $a > 0$), **B1**

$$b = \frac{-a}{2} = -2^{n-1}$$

A1

and

$$2^{2n-2} + 2^{n+1}c - 2^{2n} = -(2n+1)2^{2n-2}$$

so

$$2^{n-3} + c - 2^{n-1} = -(2n+1)2^{n-3}$$

$$c = 2^{n-3}(4-1-2n-1) = 2^{n-2}(1-n)$$

as required. **A1*(7)**

5. (i)

$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7}$$

$$7y+14x=2xy$$

$$2xy-7y-14x+49=49$$

$$(2x-7)(y-7)=49$$

B1*

Thus $2x-7=1$, $y-7=49$, or $2x-7=7$, $y-7=7$, or $2x-7=49$, $y-7=1$

M1

and so $(x,y)=(4,56)$, $(7,14)$, or $(28,8)$ **A1 (3)**

(ii)

$$p^2+pq+q^2=n^2$$

$$p^2+2pq+q^2-n^2=pq$$

$$(p+q)^2-n^2=pq$$

$$(p+q+n)(p+q-n)=pq$$

B1*

$p+q+n\neq p$ and $p+q+n\neq q$ as p , q , and n are all positive. $p+q+n>p+q-n$ so $p+q+n\neq 1$ as that would require $p+q-n=pq>1$. **M1**

Thus $p+q+n=pq$ and $p+q-n=1$ as required. **A1***

Therefore $p+q+p+q-1=pq$ **M1**

$$pq-2p-2q+4=3$$

$$(p-2)(q-2)=3$$

dM1

Thus $p-2=1$, $q-2=3$, or $p-2=3$, $q-2=1$

Alternative (I)

$$pq-2p-2q+4=3$$

$$pq-2p-2q+1=0$$

$$p(q-2)=2q-1$$

$$p=\frac{2q-1}{q-2}=2+\frac{3}{q-2}$$

as $q\neq 2$ ($q=2$ would yield -4+4-3=0) so $q-2=1\ or\ 3$ **E1**

and so $(p,q)=(3,5)$, or $(5,3)$ **A1 (6)**

Alternative (II) $p+q+n=pq$ and $p+q-n=1$ yield $p+q=n+1$ and $pq=2n+1$

Therefore, p and q are solutions of $t^2-(n+1)t+(2n+1)=0$

Hence $t=\frac{(n+1)\pm\sqrt{(n+1)^2-4(2n+1)}}{2}=\frac{(n+1)\pm\sqrt{(n-3)^2+12}}{2}$

For integer t we require that $(n-3)^2+12$ is a perfect square (in fact an even perfect square).

Thus the difference of squares between $(n-3)^2+12$ and $(n-3)^2$ is 12. Successive squares, z^2 and $(z+1)^2$ differ by $2z+1$, which for $z\geq 6$ is ≥ 13. Thus $(n-3)\leq 5$. Then, either by listing potential solutions exhaustively, or justifying that $(n-3)^2+12$ and $(n-3)^2$ have to be squares differing by $7+5$ and hence $(n-3)^2=2^2$ giving $(p,q)=(3,5)$, or $(5,3)$. **E1 A1 (6)**

(iii) If $p^3+q^3+3pq^2=n^3$, and as p, q, and hence n are all positive, then $p^3<n^3$ and $q^3<n^3$ so $p<n$ and $q<n$, **E1** and hence $p+q-n<p$ and $p+q-n<q$. **A1***

If

$$p^3+q^3+3pq^2+3p^2q=n^3+3p^2q$$

M1

$$(p+q)^3-n^3=3p^2q$$

dM1

$$(p+q-n)((p+q)^2+(p+q)n+n^2)=3p^2q$$

A1

As $p+q-n<p$ and $p+q-n<q$, so $p+q-n=1$ or 3 **A1**

If $p+q-n=1$ then $(n+1)^3-n^3=3p^2q$ and hence $3n^2+3n+1=3p^2q$ **M1** which is not possible as LHS is not a multiple of 3 and RHS is. **E1**

If $p+q-n=3$, then $(n+3)^3-n^3=3p^2q$ and hence $9n^2+27n+27=3p^2q$, that is $3(n^2+3n+3)=p^2q$. **M1** So p or q must divide 3 and hence must be 3 as p and q are prime. **E1**

If $p=3$, then $q-n=0$ but $q<n$ and vice versa if $q=3$ **E1* (11)**

6. (i)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = 1 + \frac{x^2}{2!} + \cdots$$

$$\cosh^2 x = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 = 1 + x^2 + \frac{x^4}{3} + \cdots \geq 1 + x^2$$

B1*

as all terms are of even degree with positive coefficients.

Alternative

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \cdots \geq 1 + \frac{x^2}{2!}$$

$$\cosh^2 x \geq \left(1 + \frac{x^2}{2!}\right)^2 = 1 + x^2 + \frac{x^4}{4} \geq 1 + x^2$$

B1*

$$\mathrm{f}(x) = \tan^{-1} x - \tanh x$$

$$\mathrm{f}'(x) = \frac{1}{1 + x^2} - \mathrm{sech}^2\, x = \frac{\cosh^2 x - (1 + x^2)}{(1 + x^2)\cosh^2 x}$$

M1

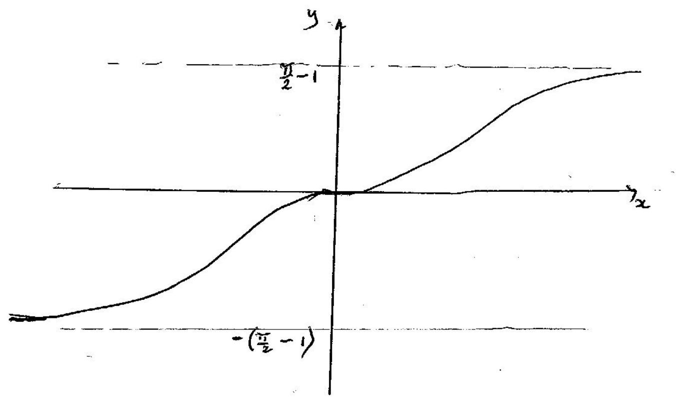
We have shown that the numerator $\cosh^2 x - (1 + x^2) \geq 0$ and the denominator is positive so $\mathrm{f}'(x) \geq 0$ and hence the function f is increasing. **E1***

When $x = 0$, $\mathrm{f}(x) = \mathrm{f}'(x) = 0$ and for all other x, $\mathrm{f}'(x) > 0$

$$\mathrm{f}(-x) = -\mathrm{f}(x)$$

G1

As $x \to \pm\infty$, $\mathrm{f}(x) \to \pm\left(\frac{\pi}{2} - 1\right)$ respectively. **G1 (5)**



(ii) (a)

$$g(x) = \tan^{-1} x - \frac{1}{2}\pi \tanh x$$

$$g'(x) = \frac{1}{1+x^2} - \frac{1}{2}\pi \operatorname{sech}^2 x = \frac{2\cosh^2 x - \pi\,(1+x^2)}{2\,(1+x^2)\cosh^2 x}$$

M1

As in (i), the denominator is positive. When $x = 0$, the numerator $= 2 - \pi < 0$. **A1**

The numerator$= (2-\pi)(1+x^2) + 2\left(\frac{x^4}{3} + \cdots\right) \to \infty$ as $x \to \infty$. **M1** Thus, there is a value of $x \neq 0$ for which $g'(x) = 0$ and as $g'(x)$ is an even function, there is also the value $-x$. **E1** Hence, there are at least two stationary points for g. **(4)**

Alternative

$g(0) = 0$ **E1** and $g(x) \to 0$ as $x \to \infty$ **E1** and $g(x)$ is not identically zero **E1** so there must be a stationary point for positive x, and similarly for negative. **E1**

(b)

$$\frac{d}{dx}[(1+x^2)\sinh x - x\cosh x] = (1+x^2)\cosh x + 2x\sinh x - x\sinh x - \cosh x$$

M1

$$= x^2\cosh x + x\sinh x \geq 0$$

for $x \geq 0$ **A1**

as $x^2 \geq 0$ and $\cosh x \geq 1$ for all x and $\sinh x \geq 0$ for $x \geq 0$ **E1**

When $x = 0$, $(1+x^2)\sinh x - x\cosh x = 0$ and we have shown $(1+x^2)\sinh x - x\cosh x$ is increasing for $x \geq 0$, thus $(1+x^2)\sinh x - x\cosh x$ is non-negative for $x \geq 0$. **E1 (4)**

(c)

$$\frac{d}{dx}\left[\frac{\cosh^2 x}{1+x^2}\right] = \frac{(1+x^2)2\cosh x\sinh x - 2x\cosh^2 x}{(1+x^2)^2} = \frac{2\cosh x\left((1+x^2)\sinh x - x\cosh x\right)}{(1+x^2)^2}$$

M1 A1

$\frac{2\cosh x}{(1+x^2)^2} > 0$ for all x and by (b) $(1+x^2)\sinh x - x\cosh x \geq 0$ for $x \geq 0$

so $\frac{\cosh^2 x}{1+x^2}$ is increasing for $x \geq 0$. **E1 (3)**

(d)

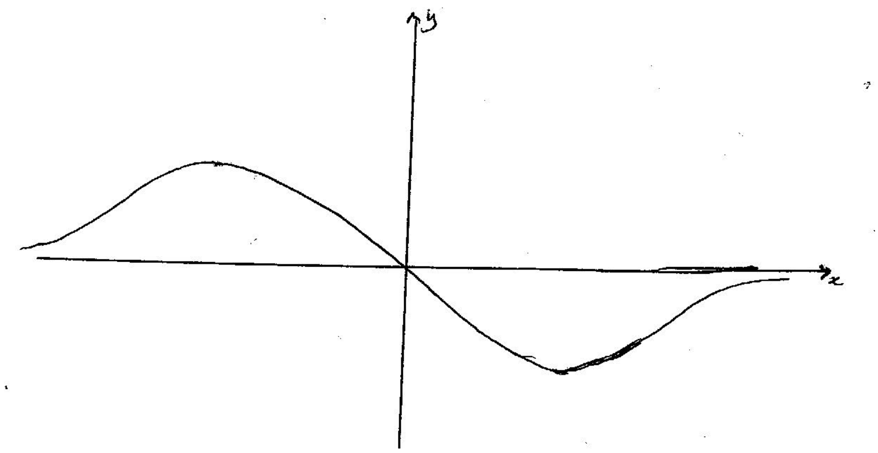
$$g'(x) = \frac{1}{1+x^2} - \frac{1}{2}\pi \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}\left[\frac{\cosh^2 x}{1+x^2} - \frac{1}{2}\pi\right]$$

By (c) , g' is increasing for $x \geq 0$, and thus there is exactly one value of x for $x > 0$ that $g'(x) = 0$

Similarly, as g' is an even function, there is exactly one value of x for $x < 0$ that $g'(x) = 0$

Thus there are exactly two stationary points. **E1 (1)**

(e)



G3 (3)

7. (i)

Let $x = u^2$, $\frac{dx}{du} = 2u$, $\sqrt{x} = u$ **M1**

$$\int_0^1 \mathrm{f}(\sqrt{x})\, dx = \int_0^1 \mathrm{f}(u)\ 2u\, du = 2\int_0^1 x\mathrm{f}(x)\ dx$$

as required. **A1* (2)**

(ii)

$$\int_0^1 (\mathrm{g}(x) - x)^2\, dx = \int_0^1 \left(\mathrm{g}(x)\right)^2 dx - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \int_0^1 x^2\, dx$$

M1

$$= \int_0^1 \mathrm{g}(\sqrt{x})\, dx - \frac{1}{3} - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \int_0^1 x^2\, dx$$

$$= 2\int_0^1 x\,\mathrm{g}(x)\, dx - \frac{1}{3} - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \left[\frac{x^3}{3}\right]_0^1$$

M1

$$= 0 - \frac{1}{3} + \frac{1}{3} = 0$$

A1*

$(\mathrm{g}(x) - x)^2 \geq 0$

So, the area under the graph of $y = (\mathrm{g}(x) - x)^2 \geq 0$, and the area can only equal zero if $(\mathrm{g}(x) - x)^2 = 0$ for $0 \leq x \leq 1$, that is $\mathrm{g}(x) = x$. **E1 (4)**

(iii)

$$\int_0^1 (\mathrm{h}'(x) - x)^2\, dx = \int_0^1 \left(\mathrm{h}'(x)\right)^2 - 2x\mathrm{h}'(x) + x^2\, dx$$

M1

We are given that

$$\int_0^1 \left(\mathrm{h}'(x)\right)^2 = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx - \frac{1}{3}$$

Integrating by parts

$$\int_0^1 2x\mathrm{h}'(x)\, dx = \left[2x\mathrm{h}(x)\right]_0^1 - 2\int_0^1 \mathrm{h}(x)\ dx = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx$$

M1 A1

and

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

So,

$$\int_0^1 (\mathrm{h}'(x) - x)^2\,dx = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx - \frac{1}{3} - \left(2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx\right) + \frac{1}{3} = 0$$

A1

As in (ii) with g, $\mathrm{h}'(x) = x$. Thus $\mathrm{h}(x) = \frac{1}{2}x^2 + c$ but $\mathrm{h}(0) = 0$ so $c = 0$ and thus $\mathrm{h}(x) = \frac{1}{2}x^2$

E1 **M1 A1** **A1 (8)**

(iv)

$$\int_0^1 \left(e^{\frac{1}{2}ax}\, k(x) - e^{-\frac{1}{2}ax}\right)^2 dx = \int_0^1 e^{ax}\big(\mathrm{k}(x)\big)^2 - 2\mathrm{k}(x) + e^{-ax}\,dx$$

M1 **dM1**

$$= 2\int_0^1 \mathrm{k}(x)\,dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - 2\int_0^1 \mathrm{k}(x)\,dx - \left[\frac{e^{-a}}{a}\right]_0^1$$

$$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{e^{-a}}{a} + \frac{1}{a} = -\frac{1}{a^2} + \frac{1}{a} - \frac{1}{4} = -\frac{4 - 4a + a^2}{4a^2} = -\frac{(2-a)^2}{4a^2}$$

A1 **A1**

As before, $\int_0^1 \left(e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax}\right)^2 dx \geq 0$ but $-\frac{(2-a)^2}{4a^2} \leq 0$

Therefore, $\int_0^1 \left(e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax}\right)^2 dx = 0$ and $\frac{(2-a)^2}{4a^2} = 0$ **E1**

Thus $e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax} = 0$ and $2 - a = 0$

So $a = 2$ and $\mathrm{k}(x) = e^{-ax} = e^{-2x}$ **A1 (6)**

8. (i)

$$y = xe^{-x}$$

$$\frac{dy}{dx} = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$\frac{d^2y}{dx^2} = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (x-2)e^{-x} + 2(1-x)e^{-x} + xe^{-x} = 0$$

M1 A1

$$x = 0, \qquad y = xe^{-x} = 0, \qquad \frac{dy}{dx} = (1-x)e^{-x} = 1$$

B1*

For $x \leq 1$, $(1-x) \geq 0$, $e^{-x} > 0$, so $\frac{dy}{dx} = (1-x)e^{-x} \geq 0$ **E1 (4)**

(ii) From (i),

$$\mathrm{g}_1(x) = xe^{-x}$$

B1

Consider

$$y = \mathrm{g}_2(x) = (a + bx)e^{x}$$

for $x \geq 1$ **B1**

Then g_2 must be a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, $\mathrm{g}_1(1) = \mathrm{g}_2(1)$, and $\mathrm{g}'_1(1) = \mathrm{g}'_2(1)$

$$\frac{dy}{dx} = be^{x} + (a+bx)e^{x} = \big((a+b) + bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} = be^{x} + \big((a+b) + bx\big)e^{x} = \big((a+2b) + bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \big((a+2b) + bx\big)e^{x} - 2\big((a+b) + bx\big)e^{x} + (a+bx)e^{x} = 0$$

as required.

$$\mathrm{g}_1(1) = \mathrm{g}_2(1) \Rightarrow e^{-1} = (a+b)e$$

$$\mathrm{g}'_1(1) = \mathrm{g}'_2(1) \Rightarrow 0 = (a+2b)e$$

M1 A1

So $a = -2b$ and thus $b = -e^{-2}$

Hence, $\mathrm{g}_2(x) = (2e^{-2} - e^{-2}x)e^{x} = (2-x)e^{x-2}$ **A1ft (5)**

(iii) $y = \mathrm{g}_2(x)$ is a reflection of $y = \mathrm{g}_1(x)$ in $x = 1$, **B1** which can be justified by substituting for x using $x' = 2 - x$ in $y = \mathrm{g}_1(x)$, $xe^{-x} = (2-x')e^{x'-2}$ as expected. **E1 (2)**

(iv) If $y = \mathrm{k}(c-x)$, then $\frac{dy}{dx} = -\mathrm{k}'(c-x)$, and $\frac{d^2y}{dx^2} = \mathrm{k}''(c-x)$ **M1**

So $\frac{d^2y}{dx^2} - p\frac{dy}{dx} + qy = \mathrm{k}''(c-x) + p\mathrm{k}'(c-x) + q\mathrm{k}(c-x) = 0$ **A1** provided that $r \le c - x \le s$

i.e. $c - s \le x \le c - r$ **B1 (3)**

(v) If $\mathrm{h}(x) = e^{-x}\sin x$, then $\mathrm{h}'(x) = -e^{-x}\sin x + e^{-x}\cos x$,

so $\mathrm{h}'\left(\frac{\pi}{4}\right) = -e^{-\frac{\pi}{4}}\sin\frac{\pi}{4} + e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} = 0$ as required. **B1***

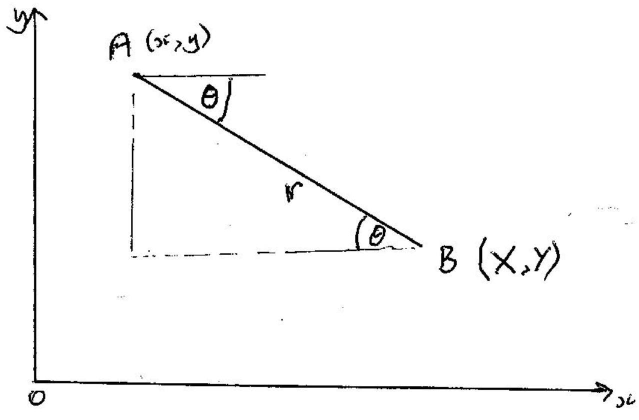
(a) Using (iv), the solution for $\frac{1}{4}\pi \le x \le \frac{5}{4}\pi$ must be $y = e^{-(c-x)}\sin(c-x)$ **M1** where

$-\frac{3}{4}\pi \le (c-x) \le \frac{1}{4}\pi$. **M1** That is $c = \frac{1}{2}\pi$. So $y = e^{x-\frac{1}{2}\pi}\cos x$ **A1**

(b) Similarly, the solution for $\frac{5}{4}\pi \le x \le \frac{9}{4}\pi$ must be $y = e^{c-x-\frac{1}{2}\pi}\cos(c-x)$ where

$\frac{1}{4}\pi \le (c-x) \le \frac{5}{4}\pi$. That is $c = \frac{5}{2}\pi$. **B1** So $y = e^{2\pi - x}\sin x$ **B1 (6)**

9. (i)



G1 (1)

(ii)

$$X = x + r\cos\theta$$

$$\dot{X} = \dot{x} - r\sin\theta\,\dot{\theta}$$

$$\ddot{X} = \ddot{x} - r\cos\theta\ \dot{\theta}^2 - r\sin\theta\ \ddot{\theta}$$

$$Y = y - r\sin\theta$$

$$\dot{Y} = \dot{y} - r\cos\theta\ \dot{\theta}$$

$$\ddot{Y} = \ddot{y} + r\sin\theta\ \dot{\theta}^2 - r\cos\theta\ \ddot{\theta}$$

B1 B1 (2)

(iii) The acceleration of A perpendicular to the string is $\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta$, **E1** and likewise for B is $\ddot{X}\ \sin\theta + \ddot{Y}\ \cos\theta$ so resolving for each in that direction, $\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta = 0$ and $\ddot{X}\ \sin\theta + \ddot{Y}\ \cos\theta = 0$ as only force is parallel to the string. **E1**

(alternatively, resolving forces in x,y for both particles, and adding necessary equations gives both results – need to show each equation for each E mark)

Substituting for $\ddot{X}$ and $\ddot{Y}$ using the results in (i),

$$\left(\ddot{x} - r\cos\theta\ \dot{\theta}^2 - r\sin\theta\ \ddot{\theta}\right)\sin\theta + \left(\ddot{y} + r\sin\theta\ \dot{\theta}^2 - r\cos\theta\ \ddot{\theta}\ \right)\cos\theta = 0$$

M1

(also may just notice $\boldsymbol{\sin\theta\left(\ddot{X} - \ddot{x}\right) + \cos\theta\left(\ddot{Y} - \ddot{y}\right) = 0}$)

Thus

$$(\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta) - r\ddot{\theta} = 0$$

A1ft

and so $r\ddot{\theta} = 0$, i.e. $\ddot{\theta} = 0$ **A1**

Integrating, $\dot{\theta} = k$ Initially, $\dot{y} = u$ and $\dot{Y} = 0$ when $\theta = 0$ so using $\dot{Y} = \dot{y} - r\cos\theta\ \dot{\theta}$, initially $0 = u - r\ \dot{\theta}$ and so $k = \frac{u}{r}$. **M1 A1**

$$\dot{\theta} = \frac{u}{r}$$

and so, integrating, $\theta = \frac{u}{r}\, t + c$, and using the initial conditions, $c = 0$

Hence,

$$\theta = \frac{ut}{r}$$

as required. **M1 A1* (9)**

(iv) Resolving in the x direction for m, $m\ddot{x} = T\cos\theta$, and for M, $M\ddot{X} = -T\cos\theta$, so adding, $m\ddot{x} + M\ddot{X} = 0$. Likewise in the y direction, $m\ddot{y} = -T\sin\theta$, $M\ddot{Y} = T\sin\theta$, giving $m\ddot{y} + M\ddot{Y} = 0$.
E1

Integrating this, $m\dot{y} + M\dot{Y} = mu$, using initial conditions. Integrating again and applying initial conditions, $my + MY = mut$. **M1 A1 (3)**

(v) As $Y = y - r\sin\theta$, $my + M(y - r\sin\theta) = mut$, so $my + My - Mr\sin\left(\frac{ut}{r}\right) = mut$ and thus,

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right)$$

E1 A1* (2)

(vi) Differentiating,

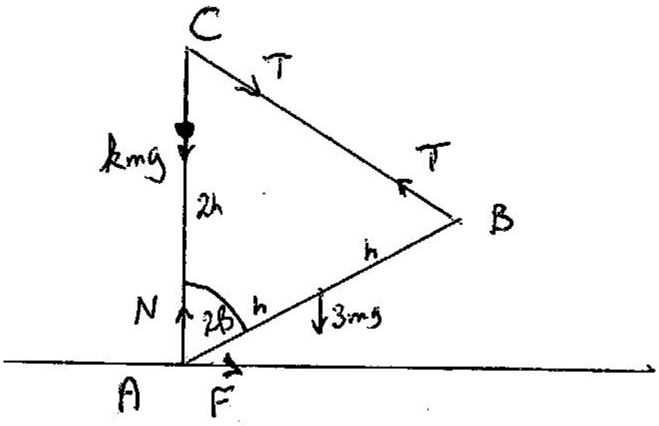
$$\dot{y} = \frac{1}{m+M}\left(mu + Mr\,\frac{u}{r}\cos\left(\frac{ut}{r}\right)\right) = \frac{u}{m+M}\left(m + M\cos\left(\frac{ut}{r}\right)\right)$$

M1 A1

When $\left(\frac{ut}{r}\right) = \pi$, $\dot{y} = \frac{u}{m+M}\,(m - M) < 0$ if $M > m$ as required.

E1 (3)

10.



G3

(i) Resolving vertically for the particle $T = kmg$ **B1**

Taking moments about A for the beam $3mgh\sin 2\beta = T2h\cos\beta$ **M1A1**

Thus $k = 3\sin\beta$ **A1 (7)**

Resolving horizontally for the beam $F = T\cos\beta = kmg\cos\beta$ **M1**

Resolving vertically for the beam $N + T\sin\beta = 3mg$ **B1**

Thus $N = 3mg - kmg\sin\beta = 3mg - 3mg\sin^2\beta = 3mg\cos^2\beta$ **A1**

As $F \leq \mu N$, $kmg\cos\beta \leq \mu 3mg\cos^2\beta$ so $k \leq 3\mu\cos\beta$ **M1**

Thus $k^2 \leq 9\mu^2\cos^2\beta = \mu^2(9 - 9\sin^2\beta) = \mu^2(9 - k^2) = 9\mu^2 - \mu^2k^2$

So $k^2 + \mu^2k^2 \leq 9\mu^2$ and so $k^2 \leq \frac{9\mu^2}{\mu^2+1}$ as required. **M1 A1* (6)**

Alternative

Considering, total force at A as R, there are three forces acting on the beam which must be concurrent, **M1** and so line of action of R passes through midpoint of BC, **A1** and thus the angle of friction must be at least β . **A1** $\mu \geq \tan\beta = \frac{k/3}{\sqrt{1-(k/3)^2}}$ **M1** so $\mu^2 \geq \frac{k^2}{9-k^2} \Rightarrow k^2 \leq \frac{9\mu^2}{\mu^2+1}$ **M1A1 (6)**

(ii) From (i) $F = T\cos\beta = kmg\cos\beta = 2mg\cos\beta$ **B1**

Moments about A for the beam $3mgh\sin 2\beta + mgxh\sin 2\beta = T2h\cos\beta = 4mgh\cos\beta$

Hence $3\sin\beta + x\sin\beta = 2$ and thus $\sin\beta = \frac{2}{3+x}$ **M1A1**

Resolving vertically $N + T\sin\beta = 3mg + mg$ and so $N = 4mg - 2mg\frac{2}{3+x} = 4mg\frac{2+x}{3+x}$

$$\frac{F^2}{N^2} = \frac{4m^2g^2\cos^2\beta}{16m^2g^2}\frac{(3+x)^2}{(2+x)^2} = \frac{(3+x)^2}{4(2+x)^2}\left(1 - \left(\frac{2}{3+x}\right)^2\right)$$

$$= \frac{1}{4(2+x)^2}\left((3+x)^2 - 4\right) = \frac{x^2+6x+5}{4(2+x)^2}$$

as required. **M1 A1***

$$\frac{1}{3}-\frac{F^2}{N^2}=\frac{4(2+x)^2-3(x^2+6x+5)}{12(2+x)^2}=\frac{x^2-2x+1}{12(2+x)^2}=\frac{(x-1)^2}{12(2+x)^2}\geq 0$$

Thus

$$\frac{F^2}{N^2}\leq\frac{1}{3}$$

and so to be in equilibrium whatever the value of x, we require $\mu\geq\frac{1}{\sqrt{3}}$ and hence $\frac{1}{\sqrt{3}}$ is the minimum value of μ. **M1A1(7)**

11.

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = \sum_{k=1}^{\infty} \frac{x^k}{k!} + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}$$

M1

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 + x \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

M1

$$= e^x - 1 + x \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x - 1 + xe^x = (x+1)e^x - 1$$

as required. **A1* (3)**

Alternative

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = \frac{d}{dx}\left(\sum_{k=1}^{\infty} \frac{x^{k+1}}{k!}\right)$$

M1

$$= \frac{d}{dx}\left(x \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1\right) = \frac{d}{dx}\left(x(e^x - 1)\right)$$

M1

$$= (x+1)e^x - 1$$

as required. **A1* (3)**

(i) (a) $\mathrm{P}(D=0) = \mathrm{P}(N=0) = e^{-n}$ **B1**

(b)

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} d\ \mathrm{P}(D=d)$$

M1

$$\mathrm{P}(D=d) = \sum_{k=d}^{\infty} \mathrm{P}(D=d|Y=k)\ \mathrm{P}(Y=k) = \sum_{k=d}^{\infty} \frac{1}{k} \frac{n^k e^{-n}}{k!}$$

M1

So

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} d \sum_{k=d}^{\infty} \frac{1}{k} \frac{n^k e^{-n}}{k!}$$

as required. **A1*(4)**

$$\sum_{d=1}^{\infty} \sum_{k=d}^{\infty} = (1,1) + (1,2) + \cdots + (2,2) + (2,3) + \cdots + (3,3) + (3,4) + \cdots$$

$$= (1,1) + (1,2) + (2,2) + (1,3) + (2,3) + (3,3) + \cdots$$

$$=\sum_{k=1}^{\infty}\sum_{d=1}^{k}$$

E1 A1

So

$$\mathrm{E}(D)=\sum_{d=1}^{\infty} d\sum_{k=d}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}=\sum_{k=1}^{\infty}\sum_{d=1}^{k} d\,\frac{1}{k}\frac{n^k e^{-n}}{k!}=\sum_{k=1}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}\sum_{d=1}^{k} d$$

A1*(3)

(c)

Thus

$$\mathrm{E}(D)=\sum_{k=1}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}\frac{k(k+1)}{2}=\frac{e^{-n}}{2}\sum_{k=1}^{\infty}\frac{k+1}{k!}n^k=\frac{e^{-n}}{2}\big((n+1)e^n-1\big)$$

B1 **M1 A1**

by using the result of the stem

$$=\frac{1}{2}\,(n+1-e^{-n})$$

A1*(4)

(ii) (a)

$$\mathrm{P}(Z=0)=\sum_{k=1}^{n}\mathrm{P}(Z=0|X_n=X_k)\ \mathrm{P}(X_n=X_k)=\sum_{k=1}^{n}\frac{1}{n}\,e^{-k}$$

M1

$$=\frac{1}{n}\,e^{-1}\frac{1-e^{-n}}{1-e^{-1}}=\frac{1-e^{-n}}{n(e-1)}$$

A1 (2)

(b)

$$\mathrm{E}(Z)=\sum_{s=1}^{\infty}s\,\mathrm{P}(Z=s)=\sum_{s=1}^{\infty}s\sum_{k=1}^{n}\frac{1}{n}\,\mathrm{P}(X_k=s)$$

M1

$$=\frac{1}{n}\sum_{k=1}^{n}\sum_{s=1}^{\infty}s\mathrm{P}(X_k=s)$$

M1

$$=\frac{1}{n}\sum_{k=1}^{n}k\ =\frac{1}{n}\frac{n(n+1)}{2}=\frac{1}{2}\,(n+1)>\frac{1}{2}\,(n+1-e^{-n})=\mathrm{E}(D)$$

A1 **A1(4)**

12. (i) There are $\binom{2n}{2k}$ ways of choosing $2k$ socks from $2n$. **E1** If there is no pair of socks, then the $2k$ socks must be of different colours; the colours can be chosen $\binom{n}{2k}$ ways and for each colour there are 2 ways of choosing a sock of that colour. **E1** Hence the probability of no pairs is

$$\frac{\binom{n}{2k}2^{2k}}{\binom{2n}{2k}}$$

B1 (3)

Alternative

The probability that all socks chosen do not include any pairs is

$$\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-2}\times\cdots\times\frac{2n-2(2k-1)}{2n-2k+1}$$

as having removed r different socks leaves only $2n-2r$ possibilities from the remaining $2n-r$.

E1

$$\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-2}\times\cdots\times\frac{2n-2(2k-1)}{2n-2k+1}=\frac{2^{2k}n(n-1)\cdots(n-2k+1)}{(2n)!\big/(2n-2k)!}$$

$$=\frac{2^{2k}n!}{(n-2k)!}\div\frac{(2n)!}{(2n-2k)!}=2^{2k}\times\frac{n!}{(n-2k)!\,(2k)!}\div\frac{(2n)!}{(2n-2k)!\,(2k)!}$$

E1

$$=\frac{\binom{n}{2k}2^{2k}}{\binom{2n}{2k}}$$

B1 (3)

(ii) For $X_{n,k}=r$, there must be $2r$ socks that are pairs and $2k-2r$ that are of different colours. **E1**

The colours of the pairs can be chosen $\binom{n}{r}$ ways and the colours of the remaining $2k-2r$ individual socks can be chosen from the remaining $n-r$ colours $\binom{n-r}{2k-2r}$ ways **E1**: for each colour chosen for an individual sock there are two choices of which sock of the pair is chosen. **E1**

Hence,

$$\mathrm{P}(X_{n,k}=r)=\frac{\binom{n}{r}\binom{n-r}{2k-2r}2^{2k-2r}}{\binom{2n}{2k}}=\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

A1*(4)

(iii)

$$\frac{k(2k-1)}{2n-1}\mathrm{P}\big(X_{n-1,k-1}=r-1\big)=\frac{k(2k-1)}{2n-1}\frac{\binom{n-1}{r-1}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n-2}{2k-2}}$$

M1A1

$$=\frac{k(2k-1)(n-1)!\,(2k-2)!\,(2n-2k)!}{(2n-1)(n-r)!\,(r-1)!\,\ (2n-2)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=\frac{1}{(2n-1)(2n-2)!}\frac{k(2k-1)(2k-2)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1A1

$$=\frac{1}{(2n-1)!}\frac{k(2k-1)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1

$$=\frac{2n}{(2n)!}\frac{(2k)!}{2}\frac{(n-1)!\,(2n-2k)!}{(r-1)!\,(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$=\frac{2n(n-1)!}{2(2n)!}\frac{r(2n-2k)!}{r!\,(n-r)!}\frac{(2k)!}{1}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$=r\frac{n!}{r!\,(n-r)!}\frac{(2n-2k)!\,(2k)!}{(2n)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=r\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}=r\mathrm{P}\big(X_{n,k}=r\big)$$

A1*(8)

Or alternatively, in the OPPOSITE DIRECTION

$$r\mathrm{P}\big(X_{n,k}=r\big)=r\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

M1A1

$$=r\frac{n!}{r!\,(n-r)!}\frac{(2n-2k)!\,(2k)!}{(2n)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=\frac{2n(n-1)!}{2(2n)!}\frac{r(2n-2k)!}{r!\,(n-r)!}\frac{(2k)!}{1}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1A1

$$= \frac{2n}{(2n)!}\frac{(2k)!}{2}\frac{(n-1)!\,(2n-2k)!}{(r-1)!\,(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1

$$= \frac{1}{(2n-1)!}\frac{k(2k-1)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{1}{(2n-1)(2n-2)!}\frac{k(2k-1)(2k-2)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{k(2k-1)(n-1)!\,(2k-2)!\,(2n-2k)!}{(2n-1)(n-r)!\,(r-1)!\,\,(2n-2)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{k(2k-1)}{2n-1}\frac{\binom{n-1}{r-1}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n-2}{2k-2}}$$

A1

$$= \frac{k(2k-1)}{2n-1}\mathrm{P}\left(X_{n-1,k-1} = r-1\right)$$

A1*(8)

$$\mathrm{E}\left(X_{n,k}\right) = \sum_{r=0}^{k} r\mathrm{P}\left(X_{n,k} = r\right) = \sum_{r=1}^{k} r\mathrm{P}\left(X_{n,k} = r\right) = \sum_{r=1}^{k} \frac{k(2k-1)}{2n-1}\mathrm{P}\left(X_{n-1,k-1} = r-1\right)$$

M1 **M1A1**

$$= \frac{k(2k-1)}{2n-1}\sum_{r=0}^{k-1}\mathrm{P}\left(X_{n-1,k-1} = r\right) = \frac{k(2k-1)}{2n-1}$$

M1 **A1(5)**