STEP 2020 Solutions

Contents

Author's notes:

1. *At time of writing, I am not affiliated with Cambridge Assessment Admissions Testing. I did an undergraduate maths degree at Cambridge, so I sat the STEP II and III papers as an A-level student (in 2015), and I have also been one of a team of markers for the STEP exams (in 2019 and 2020). Any opinions given here are entirely my own, based on my own experiences of STEP.*

2. *These 'solutions' are not intended to be used as any sort of mark scheme. In terms of method, often there will be more than one correct way to answer a STEP question, and it is certainly not the case that the answers presented here are the only correct approaches to these questions. The worked solutions here were typed up after attempting the questions myself, and I have checked them against the official mark schemes published online. However, there is no guarantee that the solutions typed up here would achieve full marks. In particular, I have not provided diagrams for all questions due to the difficulties of typesetting them neatly. Many questions may ask the student to draw a diagram, and in these instances marks are usually awarded for this. Another point of consideration is explanation: sometimes marks are awarded for explicitly justifying an assumption used. I have tried to justify these as I think necessary, but there is no guarantee that these solutions justify all assumptions to the standards of the mark schemes.*

3. *If you are preparing to sit the STEP exams, I hope these can be of some help.*

STEP II

Section A: Pure Mathematics

Question 1

(i) Using the given substitution $x = \frac{1}{1-u}$, we have $u = 1 - \frac{1}{x} = \frac{x-1}{x}$:

$$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}} dx = \int \frac{(1-u)^{\frac{3}{2}}}{\left(\frac{1}{1-u} - 1\right)^{\frac{1}{2}}} \frac{d}{du}\left(\frac{1}{1-u}\right) du = \int \frac{(1-u)^{\frac{3}{2}}}{\left(\frac{1}{1-u} - 1\right)^{\frac{1}{2}}} \frac{1}{(1-u)^2} du$$

$$= \int \frac{1}{(1-(1-u))^{\frac{1}{2}}} du$$

$$= \int u^{-\frac{1}{2}} du$$

$$= 2u^{\frac{1}{2}} + c$$

$$= 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + c \quad ,$$

where c is some constant.

(ii) Let $x = t + 2$ where $t > 0$, then

$$\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}} dx = \int \frac{1}{t^{\frac{3}{2}}(t+3)^{\frac{1}{2}}} dt \quad .$$

Now let $t = \frac{3}{u-1}$ where $u > 1$, so $u = 1 + \frac{3}{t} = \frac{t+3}{t} = \frac{x+1}{x-2}$, then

$$\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}} dx = \int \frac{(u-1)^{\frac{3}{2}}}{3^{\frac{3}{2}}\left(\frac{3}{u-1} + 3\right)^{\frac{1}{2}}} \frac{d}{du}\left(\frac{3}{u-1}\right) du$$

$$= \int \frac{(u-1)^{\frac{3}{2}}}{3^{\frac{3}{2}}\left(\frac{3}{u-1} + 3\right)^{\frac{1}{2}}} \frac{-3}{(u-1)^2} du$$

$$= \frac{-3}{3^{\frac{3}{2}}} \int \frac{1}{(3+3(u-1))^{\frac{1}{2}}} du$$

$$= \frac{-1}{\sqrt{3}} \int (3u)^{-\frac{1}{2}} du$$

$$= -\int u^{-\frac{1}{2}} du$$

$$= -2u^{\frac{1}{2}} + c$$

$$= -2\left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + c \quad ,$$

where c is some constant.

(iii) Let $x = t + 1$, then

$$\int_2^\infty \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}dx = \int_1^\infty \frac{1}{t(t-1)^{\frac{1}{2}}(3t+1)^{\frac{1}{2}}}dt \quad .$$

Now let $t = \frac{1}{1-u}$, so $u = 1 - \frac{1}{t}$:

$$\begin{aligned}
\int_1^\infty \frac{1}{t(t-1)^{\frac{1}{2}}(3t+1)^{\frac{1}{2}}}dt &= \int_0^1 \frac{1-u}{\left(\frac{1}{1-u}-1\right)^{\frac{1}{2}}\left(\frac{3}{1-u}+1\right)^{\frac{1}{2}}}\frac{1}{(1-u)^2}du \\
&= \int_0^1 \frac{1}{(1-(1-u))^{\frac{1}{2}}(3+1-u)^{\frac{1}{2}}}du \\
&= \int_0^1 \frac{1}{u^{\frac{1}{2}}(4-u)^{\frac{1}{2}}}du \\
&= \int_0^1 (4u-u^2)^{-\frac{1}{2}}du \\
&= \int_0^1 (4-(4-4u+u^2))^{-\frac{1}{2}}du \\
&= \int_0^1 (4-(2-u)^2)^{-\frac{1}{2}}du \\
&= \frac{1}{2}\int_0^1 (1-(1-\tfrac{1}{2}u)^2)^{-\frac{1}{2}}du \quad .
\end{aligned}$$

Let $v = 1 - \frac{1}{2}u$, so $u = 2 - 2v$, then

$$\begin{aligned}
\frac{1}{2}\int_0^1 (1-(1-\tfrac{1}{2}u)^2)^{-\frac{1}{2}}du &= \frac{1}{2}\int_1^{\frac{1}{2}} (1-v^2)^{-\frac{1}{2}}(-2)dv \\
&= \int_{\frac{1}{2}}^1 (1-v^2)^{-\frac{1}{2}}dv \\
&= [\arcsin v]_{v=\frac{1}{2}}^{v=1} \\
&= \arcsin(1) - \arcsin\left(\tfrac{1}{2}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \quad ,
\end{aligned}$$

that is:

$$\int_2^\infty \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}dx = \frac{\pi}{3} \quad .$$

Question 2

(i) The differential equation here is separable, allowing us to solve it as follows:

$$
\begin{aligned}
\frac{dy}{dx} = \frac{kxy - y}{x - kxy} \quad \iff \quad \frac{1 - ky}{y}\frac{dy}{dx} &= \frac{kx - 1}{x} \\
(y^{-1} - k)\frac{dy}{dx} &= k - x^{-1} \\
\iff \quad \ln|y| - ky &= kx - \ln|x| + c \\
\ln|xy| &= k(x + y) + c \\
\implies \quad xy &= Ae^{k(x+y)} = A \cdot 2^{x+y} \\
\frac{1}{4}((x + y)^2 - (x - y)^2) &= A \cdot 2^{x+y} \\
(x - y)^2 &= (x + y)^2 - 4A \cdot 2^{x+y} \quad ,
\end{aligned}
$$

where c is some constant, and $A = e^c$ (since $|xy| = xy$).

For C_1, we substitute in $x = y = 1$ to find

$$0 = 4 - 4A_1 \cdot 4 \qquad \implies \qquad A_1 = \frac{1}{4} \quad ,$$

and so C_1 is given by the equation

$$(x - y)^2 = (x + y)^2 - 2^{x+y} \quad .$$

For C_2, we substitute in $x = y = -1$ to find

$$0 = 4 - 4A_2 \cdot \frac{1}{4} \qquad \implies \qquad A_2 = 4 \quad ,$$

and so C_2 is given by the equation

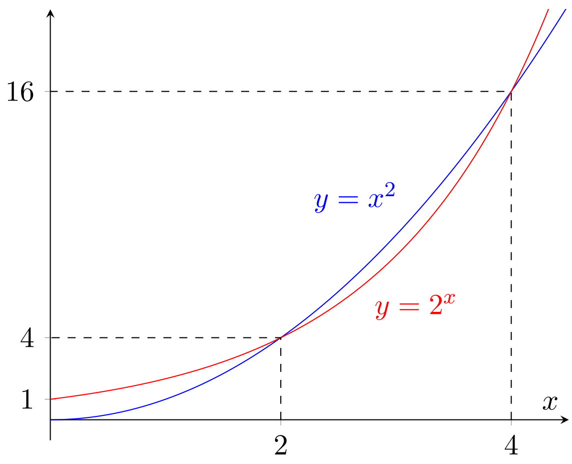
$$(x - y)^2 = (x + y)^2 - 16 \cdot 2^{x+y} \quad ,$$

or equivalently

$$(x - y)^2 = (x + y)^2 - 2^{x+y+4} \quad .$$

In both cases, the equations describing the curves are invariant under the transformation $(x, y) \mapsto (y, x)$, hence both curves are symmetrical about $y = x$.

(ii) Sketching $y = x^2$ and $y = 2^x$ for $x \geqslant 0$, we have the following.



We see that for real $t \geqslant 0$, we have $t^2 \geqslant 2^t$ if and only if $2 \leqslant t \leqslant 4$.

We know that C_1 lies entirely in the region $x > 0$, $y > 0$, thus for all points on C_1 we have $x + y \geqslant 0$, and from the equation above, we have

$$(x-y)^2 = (x+y)^2 - 2^{x+y} \quad \Longrightarrow (x+y)^2 - 2^{x+y} \geqslant 0$$
$$\Longrightarrow (x+y)^2 \geqslant 2^{x+y} \quad .$$

We thus conclude that for all points on C_1 we must have $2 \leqslant x + y \leqslant 4$.

If we set $x + y = 2$ in the equation for C_1, we get $(x - y)^2 = 0$; thus the curve C_1 intersects the line $x + y = 2$ at one point: $x = y = 1$. Similarly, if we set $x + y = 4$ in the equation for C_1, we again get $(x - y)^2 = 0$; thus the curve C_1 intersects the line $x + y = 4$ at one point: $x = y = 2$. Further, if we set $x = y = t$ in the original differential equation, we find that at these points where $y = x$ we have

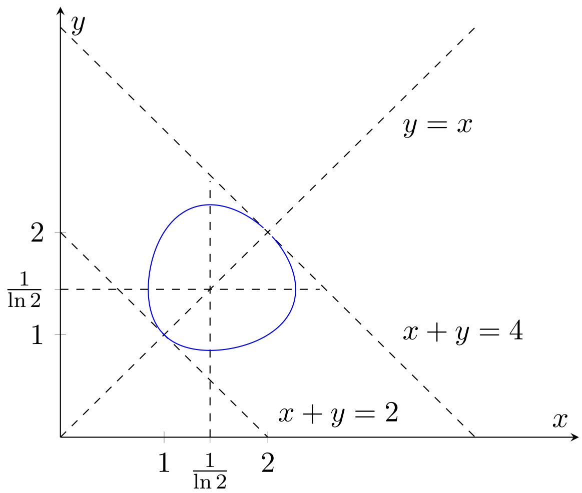
$$\frac{dy}{dx} = \frac{kt^2 - t}{t - kt^2} = -1 \quad ,$$

thus at $x = y = 1$, the curve C_1 is tangential to the line $x + y = 2$, and at $x = y = 2$, C_1 is tangential to the line $x + y = 4$.

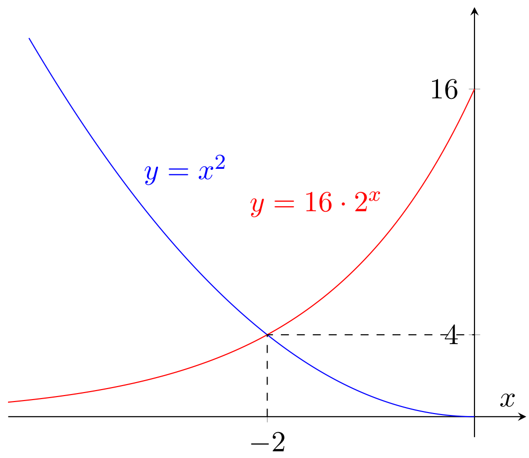
Looking for stationary points $\frac{dy}{dx} = 0$ or $\frac{dx}{dy} = 0$ in $x > 0$, $y > 0$, we find

$$\frac{dy}{dx} = 0 \iff \frac{y(kx-1)}{(1-ky)x} = 0 \iff x = k^{-1} = \frac{1}{\ln 2} \quad ,$$
$$\text{and} \quad \frac{dx}{dy} = 0 \iff \frac{(1-ky)x}{y(kx-1)} = 0 \iff y = k^{-1} = \frac{1}{\ln 2} \quad .$$

Sketching C_1 then, we have the following graph.



(iii) Sketching $y = x^2$ and $y = 16 \cdot 2^x$ for $x \leqslant 0$, we have the following.



We see that for real $t \leqslant 0$, we have $t^2 \geqslant 16 \cdot 2^t$ if and only if $t \leqslant -2$. Since C_2 lies entirely in the region $x < 0$, $y < 0$, we have that $x + y \leqslant 0$ for all points on C_2, and from the equation above we have

$$(x-y)^2 = (x+y)^2 - 16 \cdot 2^{x+y} \implies (x+y)^2 - 16 \cdot 2^{x+y} \geqslant 0$$
$$\implies (x+y)^2 \geqslant 16 \cdot 2^{x+y} \quad .$$

We thus conclude that for all points on C_2 we must have $x + y \leqslant -2$.

If we set $x+y=-2$ in the equation for C_2, we get $(x-y)^2=0$; thus the curve C_2 intersects the line $x+y=-2$ at one point: $x=y=-1$. Further, by the working in part (ii), we know that at $x=y=-1$, the gradient of the curve C_2 is -1, thus it is tangential to the line $x+y=-2$. Also by the working in part (ii) we know that there are no stationary points $\frac{dy}{dx}=0$ or $\frac{dx}{dy}=0$ in $x<0$, $y<0$.

Rewriting the equation for C_2 as

$$xy = 4\cdot 2^{x+y} \quad ,$$

and supposing we fix $x+y=t$, then we can solve (in terms of t):

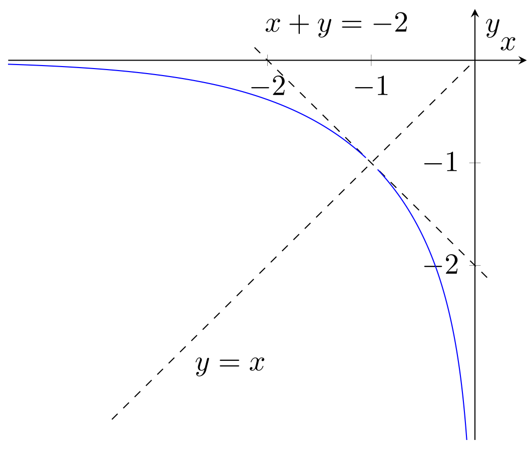
$$\begin{aligned} x(t-x) &= 4\cdot 2^t \\ x^2 - tx + 4\cdot 2^t &= 0 \\ \implies x = \frac{1}{2}\left(t \pm \sqrt{t^2-16\cdot 2^t}\right) \ , \ y = t - x &= \frac{1}{2}\left(t \mp \sqrt{t^2-16\cdot 2^t}\right) \quad . \end{aligned}$$

In particular, this shows that for $t<-2$ (such that the arguments in the square roots are positive), the curve C_2 intersects the line $x+y=t$ at exactly two points:

$$\begin{aligned} (x_1,y_1) &= \left(\frac{1}{2}\left(t+\sqrt{t^2-16\cdot 2^t}\right), \frac{1}{2}\left(t-\sqrt{t^2-16\cdot 2^t}\right)\right) \\ (x_2,y_2) &= \left(\frac{1}{2}\left(t-\sqrt{t^2-16\cdot 2^t}\right), \frac{1}{2}\left(t+\sqrt{t^2-16\cdot 2^t}\right)\right) \quad . \end{aligned}$$

Moreover if we let $t\to-\infty$, we see $x_1\to-\infty$, $y_1\to 0$, and $x_2\to 0$, $y_2\to-\infty$.

Sketching C_2 then, we have the following graph.



Question 3

(i) We will prove the contrapositive. Suppose that for a sequence of positive real numbers $u_1, u_2, \ldots, u_n$ there exists r with $2 \leqslant r \leqslant n-1$ such that

$$u_{r-1} \geqslant u_r \quad \text{and} \quad u_r < u_{r+1} \quad .$$

Since all terms here are positive, these together give

$$u_{r-1} \cdot u_{r+1} \geqslant u_r \cdot u_{r+1} > u_r \cdot u_r = u_r^2 \quad ,$$

which contradicts the property L. We conclude that if a sequence of positive real numbers does have property L and $u_{r-1} \geqslant u_r$ for some r, $2 \leqslant r \leqslant n-1$, then it must have $u_r \geqslant u_{r+1}$.

From this we see that any sequence of positive real numbers with property L is either: (1) strictly increasing, or (2) strictly increasing until a value u_m, such that $u_m \geqslant u_{m+1}$, after which the sequence is non-increasing. In case (1) the sequence is unimodal (as in the definition given) with $k = n$; in case (2) the sequence is unimodal (as in the definition given) with $k = m$. We conclude that any sequence of positive real numbers with property L is unimodal.

(ii) Trivially for $r = 2$, we have

$$u_2 - \alpha u_1 = \alpha^{2-2}(u_2 - \alpha u_1) \quad .$$

Now suppose that

$$u_k - \alpha u_{k-1} = \alpha^{k-2}(u_2 - \alpha u_1)$$

for some $k \geqslant 2$. This gives

$$\begin{aligned} u_{k+1} - \alpha u_k &= 2\alpha u_k - \alpha^2 u_{k-1} - \alpha u_k \\ &= \alpha(u_k - \alpha u_{k-1}) \\ &= \alpha \cdot \alpha^{k-2}(u_2 - \alpha u_1) \\ &= \alpha^{(k+1)-2}(u_2 - \alpha u_1) \quad , \end{aligned}$$

and so by induction we have that

$$u_r - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$$

for all $2 \leqslant r \leqslant n$. Using this, for any $2 \leqslant r \leqslant n-1$, we have

$$\begin{aligned} u_r^2 - u_{r-1}u_{r+1} &= u_r^2 - u_{r-1}(u_{r+1} - \alpha u_r) - \alpha u_r u_{r-1} \\ &= u_r(u_r - \alpha u_{r-1}) - u_{r-1}(\alpha^{r-1}(u_2 - \alpha u_1)) \\ &= u_r(u_r - \alpha u_{r-1}) - \alpha u_{r-1}(\alpha^{r-2}(u_2 - \alpha u_1)) \\ &= u_r(u_r - \alpha u_{r-1}) - \alpha u_{r-1}(u_r - \alpha u_{r-1}) \\ &= (u_r - \alpha u_{r-1})^2 \quad . \end{aligned}$$

Provided that $\alpha > 0$ and $0 < \alpha u_1 < u_2$, the first result gives that for any $r \geqslant 2$

$$\begin{aligned} u_{r+1} - \alpha u_r &= \alpha^{r-1}(u_2 - \alpha u_1) > 0 \\ \implies \quad u_{r+1} &> \alpha u_r \quad , \end{aligned}$$

and inducting this inequality gives

$$u_{r+1} > \alpha u_r > \alpha^2 u_{r-1} > \cdots > \alpha^{r-1} u_2 \quad \implies \quad u_{r+1} > 0 \quad ,$$

hence all entries in the sequence are positive. By the second result, we have for any $2 \leqslant r \leqslant n-1$

$$\begin{aligned} u_r^2 - u_{r-1}u_{r+1} &= (u_r - \alpha u_{r-1})^2 \geqslant 0 \\ \implies \quad u_{r-1}u_{r+1} &\leqslant u_r^2 \quad . \end{aligned}$$

thus this is a sequence of positive real numbers with property L, and so it is unimodal.

Now given $u_1 = 1$, $u_2 = 2$, we have

$$u_r - u_{r-1} = \alpha^{r-2}(2 - \alpha) \quad .$$

We have

$$(2-1)\alpha^{1-1} + 2(1-1)\alpha^{1-2} = \alpha^0 + 0 = 1 \quad ,$$

and

$$(2-2)\alpha^{2-1} + 2(2-1)\alpha^{2-2} = 0 + 2\alpha^0 = 2 \quad ,$$

so the induction hypothesis is true for $r = 1$ and $r = 2$. Now suppose it is true for $r = m-1$, then

$$u_{m-1} = (3-m)\alpha^{m-2} + 2(m-2)\alpha^{m-3} \quad ,$$

and we have

$$\begin{aligned} u_m - \alpha u_{m-1} &= \alpha^{m-2}(2-\alpha) \\ \implies \quad u_m &= \alpha u_{m-1} + 2\alpha^{m-2} - \alpha^{m-1} \\ &= (3-m)\alpha^{m-1} + 2(m-2)\alpha^{m-2} + 2\alpha^{m-2} - \alpha^{m-1} \\ &= (2-m)\alpha^{m-1} + 2(m-1)\alpha^{m-2} \quad , \end{aligned}$$

thus by induction the hypothesis is true for all r.

Now in the case $\alpha = 1 - \frac{1}{N}$ where N is an integer and $2 \leqslant N \leqslant n$, we compute

$$\begin{aligned} u_r - u_{r+1} &= (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2} - (1-r)\alpha^r - 2r\alpha^{r-1} \\ &= \alpha^{r-2}\big((2-r)\alpha + 2(r-1) - (1-r)\alpha^2 - 2r\alpha\big) \\ &= \alpha^{r-2}\big((2r-2) + (2-3r)\alpha - (1-r)\alpha^2\big) \\ &= \frac{\alpha^{r-2}}{N^2}\big((2r-2)N^2 + (2-3r)N(N-1) - (1-r)(N-1)^2\big) \\ &= \frac{\alpha^{r-2}}{N^2}\big(-N^2 + rN + r - 1\big) \quad . \end{aligned}$$

Substituting $r = N$, we have

$$u_N - u_{N+1} = \frac{\alpha^{N-2}}{N^2}(N-1) > 0 \quad ,$$

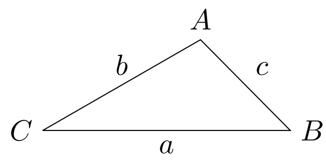
and substituting $r = N-1$, we have

$$u_{N-1} - u_N = \frac{\alpha^{N-3}}{N^2}(-2) < 0 \quad ,$$

hence $u_N > u_{N+1}$ and $u_N > u_{N-1}$. Since we know from above that this sequence is unimodal, this tell us that u_N is the unique maximum value in the sequence.

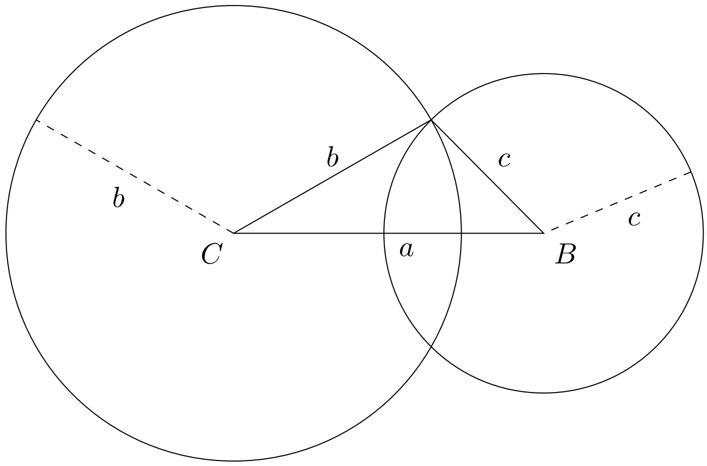
Question 4

(i) Given a triangle with side-lengths a, b, c, we label the vertices as shown.



We must have $c < a + b$, since the straight-line distance from A to B must be strictly less than the (piece-wise) straight line distance from A to C and then from C to B, where C is any point not on the line segment AB. By cyclic symmetry $(a, b, c) \mapsto (b, c, a) \mapsto (c, a, b)$, the other two inequalities follow.

(ii) Given three positive numbers a, b, c, such that $c < a + b$, $a < b + c$, and $b < c + a$; suppose without loss of generality that $a \geqslant b \geqslant c$, and consider the following diagram. Draw a straight line segment BC of length a, draw a circle centred at C with radius b, and draw a second circle centred at B with radius c.



Since $a \geqslant b$ and $a \geqslant c$, neither circle can contain the other; thus, given $a < b + c$, the circles must intersect at two points (not on the line segment BC). Labelling either intersection point as A then gives a triangle ABC with side lengths a, b, c.

(iii) We are given that a, b, c are positive and $c < a+b$, $a < b+c$, and $b < c+a$.

(A) Define $a' = a+1$, $b' = b+1$, $c' = c+1$. We have

$$\begin{aligned} & c < a+b \\ \Longrightarrow \quad & c+2 < a+1+b+1 \\ \Longrightarrow \quad & c'+1 < a'+b' \\ \Longrightarrow \quad & c' < a'+b' \quad , \end{aligned}$$

and by cyclic symmetry the other two inequalities follow. Thus a', b', c' can always form the sides of a triangle.

(B) Define $\hat{a} = \frac{a}{b}$, $\hat{b} = \frac{b}{c}$, $\hat{c} = \frac{c}{a}$. If $(a, b, c) = (1, 1, 1)$, then $(\hat{a}, \hat{b}, \hat{c}) = (1, 1, 1)$ – both of these are the sides of an equilateral triangle of side length 1. However if $(a, b, c) = (1, 1, \frac{1}{2})$ – which can form an isosceles triangle – then $(\hat{a}, \hat{b}, \hat{c}) = (1, 2, \frac{1}{2})$, which cannot form the sides of a triangle, since $\hat{b} > \hat{c} + \hat{a}$. Hence $\hat{a}$, $\hat{b}$, $\hat{c}$ can sometimes, but not always, form the sides of a triangle.

(C) Define $\check{a} = |a-b|$, $\check{b} = |b-c|$, $\check{c} = |c-a|$. Assume without loss of generality that $a \geqslant b \geqslant c$, then

$$\begin{aligned} \check{a} + \check{b} &= |a-b| + |b-c| \\ &= a-b+b-c \\ &= a-c \\ &= |c-a| = \check{c} \quad . \end{aligned}$$

Thus $\check{c} = \check{a} + \check{b}$ exactly, hence $\check{a}$, $\check{b}$, $\check{c}$ can never form the sides of a triangle.

(D) Define $\tilde{a} = a^2 + bc$, $\tilde{b} = b^2 + ca$, $\tilde{c} = c^2 + ab$. We have that

$$\begin{aligned} \tilde{a} + \tilde{b} - \tilde{c} &= a^2 + bc + b^2 + ca - c^2 - ab \\ &= a^2 - 2ab + b^2 + (a+b-c)c + ab \\ &= (a-b)^2 + (a+b-c)c + ab \\ \Longrightarrow \quad \tilde{a} + \tilde{b} - \tilde{c} &\geqslant (a+b-c)c + ab \qquad \text{(since } (a-b)^2 \geqslant 0) \\ \Longrightarrow \quad \tilde{a} + \tilde{b} - \tilde{c} &> (a+b-c)c \qquad \text{(since } a > 0,\ b > 0) \\ \Longrightarrow \quad \tilde{a} + \tilde{b} - \tilde{c} &> 0 \qquad \text{(since } a+b > c,\ c > 0) \\ \Longrightarrow \quad \tilde{a} + \tilde{b} &> \tilde{c} \quad , \end{aligned}$$

and by cyclic symmetry the other two inequalities follow. Thus $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ can always form the sides of a triangle.

(iv) Define $\alpha = f(a)$, $\beta = f(b)$, $\gamma = f(c)$. We have

$$\begin{aligned}
a+b > a > 0 \quad &\Longrightarrow \quad \frac{f(a+b)}{a+b} < \frac{\alpha}{a} \quad , \\
a+b > b > 0 \quad &\Longrightarrow \quad \frac{f(a+b)}{a+b} < \frac{\beta}{b} \quad , \\
\text{and} \quad a+b > c > 0 \quad &\Longrightarrow \quad f(a+b) > \gamma \quad .
\end{aligned}$$

Combining these inequalities, we find that

$$\begin{aligned}
& \alpha + \beta = a \cdot \frac{\alpha}{a} + b \cdot \frac{\beta}{b} \\
\Longrightarrow \quad & \alpha + \beta > a \cdot \frac{f(a+b)}{a+b} + b \cdot \frac{f(a+b)}{a+b} \\
\Longrightarrow \quad & \alpha + \beta > f(a+b) > \gamma \quad .
\end{aligned}$$

Hence $\alpha + \beta > \gamma$, and by cyclic symmetry the other two inequalities follow. Thus α, β, γ can always form the sides of a triangle.

Question 5

(i) Writing x in the given form, we have

$$x - d(x) = \sum_{r=0}^{n-1} a_r 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r(10^r - 1) \quad .$$

We have that $10^r \geqslant 1$ for all $r \geqslant 0$, and $10^r - 1$ is a multiple of 9 for all $r \geqslant 0$ (to see this; in base 10, $10^r - 1$ is a string of r 9s). Hence for all x, $x - d(x) \geqslant 0$ and $x - d(x)$ is a multiple of 9.

(ii) We have

$$x - 44d(x) = 44(x - d(x)) - 43x \quad ,$$

thus $x - 44d(x)$ is a multiple of 9 if and only if $43x$ is a multiple of 9. Since $\text{hcf}(43, 9) = 1$, this occurs if and only if x is a multiple of 9. Hence $x - 44d(x)$ is a multiple of 9 if and only if x is a multiple of 9.

Given that $x = 44d(x)$ and x has n digits, we know that $x = \sum_{r=0}^{n-1} a_r 10^r$, where $a_{n-1} \geqslant 1$, thus $x \geqslant 10^{n-1}$. We also have

$$x = 44d(x) = 44\sum_{r=0}^{n-1} a_r \leqslant 44\sum_{r=0}^{n-1} 9 = 9n \cdot 44 = 396n \quad ,$$

that is: $x \leqslant 396n$. Together these require that $396n \geqslant 10^{n-1}$. As n increases, 10^{n-1} grows much faster than $396n$, and $396 \cdot 5 = 1980 < 10000 = 10^{5-1}$; hence we must have $n \leqslant 4$.

Given that $x = 44d(x)$, we know that x must be a multiple of 44 and as proved above it must be a multiple of 9. Since $\text{hcf}(43, 9) = 1$ we therefore know that x must be a multiple of 396 with at most 4 digits. Thus we check all positive multiples of 396 less than or equal to $396 \cdot 4 = 1584$:

k	$x = 396k$	$\frac{1}{44}x = 9k$	$d(x)$
1	396	9	18
2	792	18	18
3	1188	27	18
4	1584	36	18

By inspection $x = 792$ satisfies $\frac{1}{44}x = d(x)$, hence it is a solution and is the only solution to $x = 44d(x)$.

(iii) Similar to above, we write

$$\begin{aligned} x - 107d(d(x)) &= x - 107d(x) + 107(d(x) - d(d(x))) \\ &= 107(x - d(x)) + 107(d(x) - d(d(x))) - 106x \quad . \end{aligned}$$

We know that $x - 107d(d(x)) = 0$ (a multiple of 9) and by (i) both $x - d(x)$ and $d(x) - d(d(x))$ are multiples of 9, thus $106x$ is a multiple of 9. Since we have $\text{hcf}(106, 9) = 1$, this implies that x is a multiple of 9. Similarly, $x - 107d(d(x)) = 0$ (a multiple of 107) and both $107(x - d(x))$ and $107(d(x) - d(d(x)))$ are multiples of 107, thus $106x$ is a multiple of 107. Since $\text{hcf}(106, 107) = 1$ this implies that x is a multiple of 107. Further, since $\text{hcf}(107, 9) = 1$, these together imply that x is a multiple of $107 \cdot 9 = 963$.

To get an upper bound on the possible solutions, suppose x has n digits (so $x \geqslant 10^{n-1}$). We know $d(x) \leqslant 9n$ and, by (i), $d(x) - d(d(x)) \geqslant 0$ (so $d(d(x)) \leqslant d(x)$). Thus we know

$$\begin{aligned} & d(d(x)) \leqslant d(x) \leqslant 9n \\ \implies & 107d(d(x)) \leqslant 963n \\ \implies & x \leqslant 963n \quad . \end{aligned}$$

Again as n increases, 10^{n-1} grows much faster than $963n$ and we have $963 \cdot 5 = 4815 < 10000 = 10^{5-1}$, hence $n \leqslant 4$. Thus we check all positive multiples of 963 less than or equal to $963 \cdot 4 = 3852$:

k	$x = 963k$	$\frac{1}{107}x = 9k$	$d(x)$	$d(d(x))$
1	963	9	18	9
2	1926	18	18	9
3	2889	27	27	9
4	3852	36	18	9

By inspection $x = 963$ satisfies $\frac{1}{107}x = d(d(x))$, hence it is a solution and is the only solution to $x = 107d(d(x))$.

Question 6

(i) We evaluate $\boldsymbol{M}^2$ directly:

$$\boldsymbol{M}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \quad ,$$

giving

$$\begin{aligned} \operatorname{tr}(\boldsymbol{M}^2) &= a^2+bc+bc+d^2 \\ &= a^2+2ad+d^2-2ad+2bc \\ &= (a+d)^2-2(ad-bc) \\ &= (\operatorname{tr}(\boldsymbol{M}))^2-2\det(\boldsymbol{M}) \quad . \end{aligned}$$

(ii) From above, we know

$$\begin{aligned} \boldsymbol{M}^2 &= \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} = \begin{pmatrix} a(a+d)-(ad-bc) & b(a+d) \\ c(a+d) & d(a+d)-(ad-bc) \end{pmatrix} \\ &= \begin{pmatrix} a\tau-\delta & b\tau \\ c\tau & d\tau-\delta \end{pmatrix} \quad , \end{aligned}$$

where $\tau = a+d = \operatorname{tr}(\boldsymbol{M})$, $\delta = ad-bc = \det(\boldsymbol{M})$. Thus $\boldsymbol{M}^2 = \pm\boldsymbol{I}$ if and only if

$$a\tau-\delta=\pm1 \ , \quad b\tau=0 \ , \quad c\tau=0 \ , \quad d\tau-\delta=\pm1 \quad ,$$

which, taking the fourth equation away from the first, is equivalent to

$$(a-d)\tau=0 \ , \quad b\tau=0 \ , \quad c\tau=0 \ , \quad d\tau-\delta=\pm1 \quad .$$

Hence $\boldsymbol{M}^2 = \pm\boldsymbol{I}$ if and only if

$$\begin{aligned} &\tau=0 \ , \quad \delta=\mp1 \quad , \\ \text{or} \quad &a=d \ , \quad b=c=0 \ , \quad bc+d^2=\pm1 \quad (\implies d^2=\pm1) \quad . \end{aligned}$$

For the case $\boldsymbol{M}^2 = \boldsymbol{I}$, this second set of conditions gives $a=d=\pm1,\ b=c=0$ which is exactly $\boldsymbol{M}=\pm\boldsymbol{I}$; while the first set of conditions is $\operatorname{tr}(\boldsymbol{M})=0,\ \det(\boldsymbol{M})=-1$ (which does not include $\boldsymbol{M}=\pm\boldsymbol{I}$ since, for example $\det(\pm\boldsymbol{I})=1$). We conclude that $\boldsymbol{M}^2=\boldsymbol{I}$ but $\boldsymbol{M}\neq\pm\boldsymbol{I}$ if and only if $\operatorname{tr}(\boldsymbol{M})=0$ and $\det(\boldsymbol{M})=-1$.

For the case $\boldsymbol{M}^2=-\boldsymbol{I}$, the second set of conditions requires $d^2=-1$, hence this gives no real solutions; while the first set of conditions is $\operatorname{tr}(\boldsymbol{M})=0,\ \det(\boldsymbol{M})=1$. We conclude that $\boldsymbol{M}^2=-\boldsymbol{I}$ if and only if $\operatorname{tr}(\boldsymbol{M})=0$ and $\det(\boldsymbol{M})=1$.

(iii) By part (ii), if $\boldsymbol{M}^2 \neq \pm \boldsymbol{I}$ then we have

$$(\boldsymbol{M}^2)^2 = \boldsymbol{I} \quad \Longrightarrow \quad \mathrm{tr}(\boldsymbol{M}^2) = 0 \text{ and } \det(\boldsymbol{M}^2) = -1 \quad .$$

However, by the properties of the determinant $\det(\boldsymbol{M}^2) = \det(\boldsymbol{M})^2$, and since $\boldsymbol{M}$ is real, we must have that $\det(\boldsymbol{M})$ is real. Thus $\det(\boldsymbol{M})^2 = -1$ is not possible. Trivially, if $\boldsymbol{M}^2 = \pm\boldsymbol{I}$, then $\boldsymbol{M}^4 = (\boldsymbol{M}^2)^2 = (\pm 1)^2 \boldsymbol{I}^2 = \boldsymbol{I}$. We conclude that $\boldsymbol{M}^4 = \boldsymbol{I}$ if and only if $\boldsymbol{M}^2 = \pm\boldsymbol{I}$.

Again by part (ii) we have

$$(\boldsymbol{M}^2)^2 = -\boldsymbol{I} \quad \Longleftrightarrow \quad \mathrm{tr}(\boldsymbol{M}^2) = 0 \text{ and } \det(\boldsymbol{M}^2) = 1 \quad .$$

By the result of part (i), and using the properties of the determinant again, this is equivalent to

$$\mathrm{tr}(\boldsymbol{M})^2 - 2\det(\boldsymbol{M}) = 0 \text{ and } \det(\boldsymbol{M})^2 = 1 \quad ,$$

This is equivalent to the two conditions:

$$\begin{aligned} &\mathrm{tr}(\boldsymbol{M})^2 = 2 \;\text{ and }\; \det(\boldsymbol{M}) = 1 \\ \text{or} \quad &\mathrm{tr}(\boldsymbol{M})^2 = -2 \;\text{ and }\; \det(\boldsymbol{M}) = -1 \quad . \end{aligned}$$

Since $\boldsymbol{M}$ is real, $\mathrm{tr}(\boldsymbol{M})$ is real, and thus the second condition allows no solutions. We conclude that

$$\begin{aligned} \boldsymbol{M}^4 = -\boldsymbol{I} \quad &\Longleftrightarrow \quad \mathrm{tr}(\boldsymbol{M})^2 = 2 \text{ and } \det(\boldsymbol{M}) = 1 \\ &\Longleftrightarrow \quad \mathrm{tr}(\boldsymbol{M}) = \pm\sqrt{2} \text{ and } \det(\boldsymbol{M}) = 1 \quad . \end{aligned}$$

(iv) By the above results we have

$$\begin{aligned} (\boldsymbol{M}^2)^4 = \boldsymbol{I} \quad &\Longleftrightarrow \quad (\boldsymbol{M}^2)^2 = \pm\boldsymbol{I} \\ &\Longleftrightarrow \quad \boldsymbol{M} = \pm\boldsymbol{I} \text{ or } (\mathrm{tr}(\boldsymbol{M}) = 0, \det(\boldsymbol{M}) = -1) \\ &\qquad\quad \text{or } (\mathrm{tr}(\boldsymbol{M}) = 0, \det(\boldsymbol{M}) = 1) \\ &\qquad\quad \text{or } (\mathrm{tr}(\boldsymbol{M}) = \pm\sqrt{2}, \det(\boldsymbol{M}) = 1) \quad . \end{aligned}$$

The general forms for reflection matrices and rotation matrices are

$$\text{rotation: } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \text{reflection: } \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \quad ,$$

hence a simple sufficient condition to ensure our matrix $\boldsymbol{M}$ is neither a rotation nor a reflection is to specify $a^2 + c^2 \neq 1$. For example, we could choose the conditions $\mathrm{tr}(\boldsymbol{M}) = 0, \det(\boldsymbol{M}) = -1$ and give the matrix

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad .$$

Question 7

(i) If $z = 3 + ti$ where $t \in \mathbb{R}$, then

$$\begin{aligned}|w-1| &= \left|\frac{2}{z-2} - 1\right| = \left|\frac{4-z}{z-2}\right| = \left|\frac{1-ti}{1+ti}\right| = \frac{|1-ti|}{|1+ti|}\\ &= \frac{(1+t^2)^{\frac{1}{2}}}{(1+t^2)^{\frac{1}{2}}} = 1 \quad ,\end{aligned}$$

independent of t. Hence if z lies on $\mathrm{Re}(z) = 3$, then w lies on a circle with radius 1 centred at $1 + 0i$.

Now suppose z lies on V, so $z = p + ti$ where $p \in \mathbb{R}$, $p \neq 2$, $t \in \mathbb{R}$. The distance of w from a given point $x + iy$ is

$$\begin{aligned}|w-(x+iy)| &= \left|\frac{2}{(p-2)+ti} - (x+iy)\right| = \left|\frac{2-((p-2)+ti)(x+iy)}{(p-2)+ti}\right|\\ &= \left|\frac{(2-(p-2)x+ty)+(-tx-(p-2)y)i}{(p-2)+ti}\right|\\ &= \frac{|(2-(p-2)x+ty)+(-tx-(p-2)y)i|}{|(p-2)+ti|}\\ &= \left(\frac{(2-(p-2)x+ty)^2+(tx+(p-2)y)^2}{(p-2)^2+t^2}\right)^{\frac{1}{2}}\\ &= \left(\frac{(2-(p-2)x)^2+(p-2)^2y^2+4yt+(x^2+y^2)t^2}{(p-2)^2+t^2}\right)^{\frac{1}{2}} \quad .\end{aligned}$$

This is independent of t if and only if

$$\begin{aligned} & 4y = 0 \text{ and } (2-(p-2)x)^2 + (p-2)^2y^2 = (x^2+y^2)(p-2)^2\\ \iff & y = 0 \text{ and } 4 - 4(p-2)x + (p-2)^2x^2 = x^2(p-2)^2\\ \iff & y = 0 \text{ and } 4 - 4(p-2)x = 0\\ \iff & y = 0 \text{ and } x = \frac{1}{p-2} \quad .\end{aligned}$$

In this case $\left|w - \frac{1}{p-2}\right| = (x^2+y^2)^{\frac{1}{2}} = \frac{1}{|p-2|}$. Hence if z lies on V, then w lies on a circle of radius $\frac{1}{|p-2|}$, centred at $\frac{1}{p-2} + 0i$. We have

$$\begin{aligned}\mathrm{Im}(w) > 0 &\iff \mathrm{Im}\left(\frac{2}{z-2}\right) > 0 \iff \mathrm{Im}\left(\frac{2(\bar{z}-2)}{|z-2|^2}\right) > 0\\ &\iff \mathrm{Im}(\bar{z}-2) > 0\\ &\iff \mathrm{Im}(\bar{z}) > 0\\ &\iff \mathrm{Im}(z) < 0 \quad .\end{aligned}$$

Hence for z on V with $t < 0$ we have $\mathrm{Im}(w) > 0$.

(ii) Now suppose z lies on H, so $z = t + qi$ where $q \in \mathbb{R}$, $q \neq 0$, $t \in \mathbb{R}$. From above, substituting $(p,t) \mapsto (t,q)$, the distance of w from a given point $x + iy$ is

$$\begin{aligned} |w - (x+iy)| &= \left(\frac{(2-(t-2)x)^2 + (t-2)^2 y^2 + 4yq + (x^2+y^2)q^2}{(t-2)^2 + q^2} \right)^{\frac{1}{2}} \\ &= \left(\frac{(x^2+y^2)(t-2)^2 - 4x(t-2) + 4 + 4yq + (x^2+y^2)q^2}{(t-2)^2 + q^2} \right)^{\frac{1}{2}} \end{aligned}$$

This is independent of t if and only if

$$\begin{aligned} &-4x = 0 \ \text{ and } \ 4 + 4yq + (x^2+y^2)q^2 = (x^2+y^2)q^2 \\ \iff\ & x = 0 \ \text{ and } \ 4 + 4yq = 0 \\ \iff\ & x = 0 \ \text{ and } \ y = -\frac{1}{q} \quad . \end{aligned}$$

In this case $\left| w - \left(-\frac{1}{q}i\right) \right| = (x^2+y^2)^{\frac{1}{2}} = \frac{1}{|q|}$. Hence if z lies on H, then w lies on a circle of radius $\frac{1}{|q|}$, centred at $0 - \frac{1}{q}i$. We have

$$\begin{aligned} \operatorname{Re}(w) > 0 \iff \operatorname{Re}\left(\frac{2}{z-2}\right) > 0 &\iff \operatorname{Re}\left(\frac{2(\bar{z}-2)}{|z-2|^2}\right) > 0 \\ &\iff \operatorname{Re}(\bar{z}-2) > 0 \\ &\iff \operatorname{Re}(\bar{z}) > 2 \\ &\iff \operatorname{Re}(z) > 2 \quad . \end{aligned}$$

Hence for z on H with $t > 2$ we have $\operatorname{Re}(w) > 0$.

Question 8

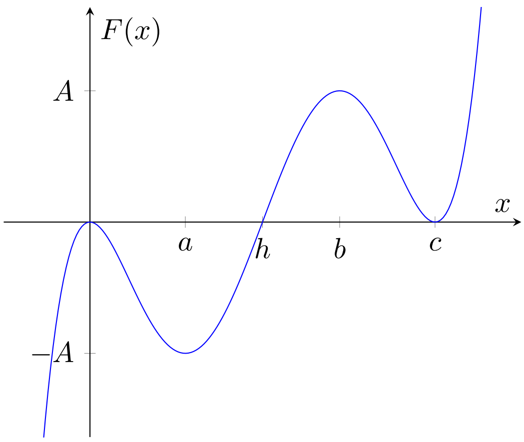
(i) Since $f(x)$ is a quartic polynomial in x with coefficient of x^4 equal to 1, we know that $F(x)$ must be a quintic polynomial in x with coefficient of x^5 equal to $\frac{1}{5}$. Immediately from the definition of $F(x)$, we must have $F(0) = 0$. Since we have $f(x) = F'(x)$ and $f(0) = 0$, we know that $x = 0$ must be a double root of $F(x)$. From the form of $f(x)$, we know that the area enclosed between the curve $y = f(x)$ and the x-axis lies below the axis between $x = 0$ and $x = a$, above the x-axis between $x = a$ and $x = b$, and then below the x-axis again between $x = b$ and $x = c$. Thus, if we let A denote the (positive) area enclosed between the curve $y = f(x)$ and the x-axis between $x = 0$ and $x = a$, we have that

$$F(a) = -A \quad , \quad F(b) = -A + 2A = A \quad , \quad F(c) = -A + 2A - A = 0 \quad .$$

Thus $x = c$ is a root of $F(x)$ and further, since $F'(c) = f(c) = 0$, it is a double root of $F(x)$. Since $F(a) < 0 < F(b)$, we conclude that the fifth root of $F(x)$ lies between $x = a$ and $x = b$. That is, $F(x)$ must take the form

$$F(x) = \frac{1}{5}x^2(x-h)(x-c)^2 \quad ,$$

where $a < h < b$ (and so $0 < h < c$). We note that the turning points $F'(x) = 0$ are at $f(x) = 0$, that is at $x = 0, a, b, c$. Our sketch is as follows,



and we have

$$\begin{aligned} F(x) + F(c-x) &= \frac{1}{5}x^2(x-h)(x-c)^2 + \frac{1}{5}(c-x)^2(c-x-h)(c-x-c)^2 \\ &= \frac{1}{5}x^2(x-h)(x-c)^2 + \frac{1}{5}(x-c)^2(c-x-h)x^2 \\ &= \frac{1}{5}x^2(x-h+c-x-h)(x-c)^2 \\ &= \frac{1}{5}(c-2h)x^2(x-c)^2 \quad . \end{aligned}$$

(ii) If we restrict only to $0 \leqslant x \leqslant c$, then $F(x)$ has a unique minimum at $x = a$, $F(a) = -A = -F(b)$, hence for all x in $0 \leqslant x \leqslant c$ we have $F(x) + F(b) \geqslant 0$, with equality if and only if $x = a$. Setting $x = c - b$ (which is necessarily greater than zero, since $b < c$, and necessarily less than c, since $b > 0$) then we have that

$$F(b) + F(c-b) \geqslant 0 \quad , \quad \text{with equality if and only if } c - b = a \quad .$$

Using the above expression for $F(x) + F(c-x)$, this gives

$$\frac{1}{5}(c-2h)b^2(c-b)^2 \geqslant 0 \quad , \quad \text{with equality if and only if } c - b = a \quad ,$$

and since $b > 0$ and $c - b > 0$ this simplifies to

$$c \geqslant 2h \quad , \quad \text{with } c = 2h \text{ if and only if } c = a + b \quad .$$

Thus we deduce that $c > 2h$ or $c = 2h = a + b$.

Similarly, $F(x)$ has a unique maximum at $x = b$, $F(b) = A = -F(a)$, hence for all x in $0 \leqslant x \leqslant c$ we have $F(x) + F(a) \leqslant 0$, with equality if and only if $x = b$. Setting $x = c - a$ (which is necessarily greater than zero, since $a < c$, and necessarily less than c, since $a > 0$) then we have that

$$F(a) + F(c-a) \leqslant 0 \quad , \quad \text{with equality if and only if } c - a = b \quad .$$

Again using the expression for $F(x) + F(c-x)$, this gives

$$\frac{1}{5}(c-2h)a^2(c-a)^2 \leqslant 0 \quad , \quad \text{with equality if and only if } c - a = b \quad ,$$

and since $a > 0$ and $c - a > 0$ this simplifies to

$$c \leqslant 2h \quad , \quad \text{with } c = 2h \text{ if and only if } c = a + b \quad .$$

Since we cannot have both $c > 2h$ and $c < 2h$, we deduce that $c = 2h = a + b$.

(iii) Substituting $h = \frac{1}{2}c$ into the expression for $F(x)$ we have

$$\begin{aligned} F(x) &= \frac{1}{5}x^2\left(x - \frac{1}{2}c\right)(x-c)^2 \\ &= \frac{1}{10}(2x^3 - cx^2)(x^2 - 2xc + c^2) \\ &= \frac{1}{10}(2x^5 - 5cx^4 + 4c^2x^3 - c^3x^2) \quad , \end{aligned}$$

and so differentiating with respect to x we have

$$\begin{aligned} f(x) &= x^4 - 2cx^3 + \frac{6}{5}c^2x^2 - \frac{1}{5}c^3x \\ &= x(x-c)\left(x^2 - cx - \frac{1}{5}c^2\right) \quad . \end{aligned}$$

Further differentiation gives

$$f'(x) = 4x^3 - 6cx^2 + \frac{12}{5}c^2x \quad ,$$

and

$$\begin{aligned} f''(x) &= 12x^2 - 12cx + \frac{12}{5}c^2 \\ &= 12\left(x^2 - cx - \frac{1}{5}c^2\right) \quad . \end{aligned}$$

In particular, comparing this to the factorised expression for $f(x)$ in terms of x and c only, this gives that

$$f(x) = \frac{1}{12}x(x-c)f''(x) \quad ,$$

and so

$$f''(x) = 0 \quad \Longrightarrow \quad f(x) = 0 \quad .$$

This says that the points of inflection on the curve $y = f(x)$ (which are the points $f''(x) = 0$) occur only at points where $f(x) = 0$, and so lie on the x-axis.

Section B: Mechanics

Question 9

Set point N to be the origin of our coordinate system, and (without loss of generality) set B to be in the positive x-direction – so B is the point $(d, 0)$. Suppose that the particles collide at time t_c. Implicitly, the particles collide at a height y_c where $y_c \geqslant 0$. Note also that $\tan(\beta) = \frac{h}{d}$.

The trajectory of the particle projected from A is given by

$$\begin{aligned} x_A(t) &= Vt \quad , \\ y_A(t) &= h - \tfrac{1}{2}gt^2 \quad . \end{aligned}$$

The trajectory of the particle projected from B is given by

$$\begin{aligned} x_B(t) &= d - U\cos(\theta)t \quad , \\ y_B(t) &= U\sin(\theta)t - \tfrac{1}{2}gt^2 \quad . \end{aligned}$$

Assuming collision at time $t = t_c$, equating the coordinates gives

$$\begin{aligned} & Vt_c = d - U\cos(\theta)t_c \quad , \quad h - \tfrac{1}{2}gt_c^2 = U\sin(\theta)t_c - \tfrac{1}{2}gt_c^2 \\ \implies \quad & t_c = \frac{d}{V + U\cos(\theta)} \quad , \quad t_c = \frac{h}{U\sin(\theta)} \quad . \end{aligned}$$

Now comparing these, we must have

$$\begin{aligned} \frac{d}{V + U\cos(\theta)} &= \frac{h}{U\sin(\theta)} \\ Ud\sin(\theta) &= Vh + Uh\cos(\theta) \\ d\sin(\theta) - h\cos(\theta) &= \frac{Vh}{U} \quad . \end{aligned}$$

(i) Manipulating the above result:

$$\begin{aligned} \sin(\theta) - \tan(\beta)\cos(\theta) &= \frac{Vh}{Ud} \\ \sin(\theta)\cos(\beta) - \sin(\beta)\cos(\theta) &= \frac{Vh}{Ud}\cos(\beta) \quad , \end{aligned}$$

and we know that the right-hand side here is positive, since β is acute. Thus

$$\begin{aligned} \sin(\theta)\cos(\beta) - \sin(\beta)\cos(\theta) &> 0 \\ \sin(\theta - \beta) &> 0 \\ \implies \quad \theta - \beta &> 0 \quad , \end{aligned}$$

since $\theta - \beta$ must lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Hence we conclude that $\theta > \beta$.

(ii) The particles collide at height $y_c = y_A(t_c) = y_B(t_c)$, which (as implicitly assumed) is at least above the ground level, thus:

$$\begin{aligned}
y_B(t_c) = U\sin(\theta)t_c - \tfrac{1}{2}gt_c^2 &\geqslant 0 \\
\implies \quad U\sin(\theta)t_c &\geqslant \tfrac{1}{2}gt_c^2 \\
U\sin(\theta)\frac{h}{U\sin(\theta)} &\geqslant \frac{gh^2}{2U^2\sin^2(\theta)} \\
\implies U^2\sin^2(\theta) &\geqslant \frac{gh}{2} \quad .
\end{aligned}$$

Now since θ is strictly greater than β, we must have $\sin(\theta) > \sin(\beta) > 0$, and so we may square root both sides taking the positive root on each side. This gives

$$U\sin(\theta) \geqslant \sqrt{\frac{gh}{2}} \quad .$$

(iii) Returning again to our initial result, we have

$$\begin{aligned}
\sin(\theta) - \tan(\beta)\cos(\theta) &= \frac{Vh}{Ud} = \frac{V}{U}\tan(\beta) \\
\implies \quad \sin(\theta)\cos(\beta) - \sin(\beta)\cos(\theta) &= \frac{V}{U}\sin(\beta) \\
\sin(\theta - \beta) &= \frac{V}{U}\sin(\beta) \quad .
\end{aligned}$$

Since $\beta > 0$, we must have $\sin(\theta - \beta) < 1$, and thus

$$\begin{aligned}
1 &> \frac{V}{U}\sin(\beta) \\
\implies \quad \sin(\beta) &< \frac{U}{V} \quad .
\end{aligned}$$

Finally, we have that the particles collide at a height greater than $\frac{1}{2}h$ if and only if

$$\begin{aligned}
y_A(t_c) &> \tfrac{1}{2}h \\
h - \tfrac{1}{2}gt_c^2 &> \tfrac{1}{2}h \\
\tfrac{1}{2}h &> \tfrac{1}{2}gt_c^2 \\
h &> \frac{gh^2}{U^2\sin^2(\theta)} \\
\iff \quad U^2\sin^2(\theta) &> gh \quad .
\end{aligned}$$

The particle projected from B is moving upwards at the time of collision if and only if

$$\begin{aligned} & \left.\frac{dy_B}{dt}\right|_{t=t_c} > 0 \\ \iff \quad & U\sin(\theta) - gt_c > 0 \\ U\sin(\theta) - & \frac{gh}{U\sin(\theta)} > 0 \\ \iff & U^2\sin^2(\theta) > gh \quad . \end{aligned}$$

Thus the particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

Question 10

A diagram can be very useful here.

(i) Note we are given $l < 2a$. The (extended) length of the spring PH is $2a\cos\theta$, thus the tension in the spring is given by

$$T = \lambda\frac{2a\cos\theta - l}{l} \quad .$$

If there is an equilibrium at $\theta = \alpha$ then, resolving tangentially to the circle, we must have

$$\begin{aligned} T\sin\alpha &= mg\sin(2\alpha) \\ \frac{\lambda(2a\cos\alpha - l)}{l} &= 2mg\cos\alpha \\ (2\lambda a - 2mgl)\cos\alpha &= \lambda l \\ \cos\alpha &= \frac{\lambda l}{2(\lambda a - mgl)} \quad . \end{aligned}$$

In order for this equilibrium to be feasible, we need $\cos\alpha < 1$, thus

$$\begin{aligned} \frac{\lambda l}{2(\lambda a - mgl)} &< 1 \\ \lambda l &< 2\lambda a - 2mgl \\ (2a - l)\lambda &> 2mgl \\ \lambda &> \frac{2mgl}{2a - l} \quad , \end{aligned}$$

since $2a - l > 0$.

(ii) The total energy of the particle-spring system (kinetic, gravitational potential, and elastic potential) when angle PHL is θ and the particle has speed v is

$$E = \frac{1}{2}mv^2 - mga\cos(2\theta) + \frac{\lambda}{2l}(2a\cos\theta - l)^2 \quad .$$

Equating this energy for the particle at rest at $\theta = \beta$ and for the particle with speed u at L, where $\theta = 0$ gives

$$\begin{aligned} -mga\cos(2\beta) + \frac{\lambda}{2l}(2a\cos\beta - l)^2 &= \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2 \\ \left(\frac{2a^2\lambda}{l} - 2mga\right)\cos^2\beta - 2a\lambda\cos\beta + mga + \frac{\lambda l}{2} &= \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2 \\ \frac{2(a\lambda - mgl)}{\lambda l}\cos^2\beta - 2\cos\beta + \frac{l}{2a} &= \frac{mu^2}{2a\lambda} - \frac{2mg}{\lambda} + \frac{1}{2al}(2a - l)^2 \quad , \end{aligned}$$

and using the expression for $\cos\alpha$ from above:

$$\begin{aligned}
\frac{1}{\cos\alpha}\cos^2\beta - 2\cos\beta + \frac{l}{2a} &= \frac{mu^2}{2a\lambda} - \frac{2mg}{\lambda} + \frac{1}{2al}(2a-l)^2 \\
\frac{1}{\cos\alpha}\cos^2\beta - 2\cos\beta + \frac{l}{2a} &= \frac{mu^2}{2a\lambda} - \frac{2mg}{\lambda} + \frac{1}{2al}(4a^2-4al+l^2) \\
\frac{1}{\cos\alpha}\cos^2\beta - 2\cos\beta &= \frac{mu^2}{2a\lambda} - \frac{2mg}{\lambda} + \frac{2a}{l} - 2 \\
\frac{1}{\cos\alpha}\cos^2\beta - 2\cos\beta &= \frac{mu^2}{2a\lambda} + \frac{2(a\lambda - mgl)}{\lambda l} - 2 \\
\frac{1}{\cos\alpha}\cos^2\beta - 2\cos\beta &= \frac{mu^2}{2a\lambda} + \frac{1}{\cos\alpha} - 2 \\
\cos^2\beta - 2\cos\alpha\cos\beta &= \frac{mu^2}{2a\lambda}\cos\alpha + 1 - 2\cos\alpha \\
\cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha &= \frac{mu^2}{2a\lambda}\cos\alpha + 1 - 2\cos\alpha + \cos^2\alpha \\
(\cos\alpha - \cos\beta)^2 &= \frac{mu^2}{2a\lambda}\cos\alpha + (1-\cos\alpha)^2 \quad .
\end{aligned}$$

Solving for $\cos\beta$, we have

$$\cos\beta = \cos\alpha - \left(\frac{mu^2}{2a\lambda}\cos\alpha + (1-\cos\alpha)^2\right)^{\frac{1}{2}} \quad ,$$

where we have the minus sign on the right-hand side, since the point at which the particle comes to rest necessarily lies higher than the equilibrium (and so $\cos\beta < \cos\alpha$). In order for the particle to come to rest below H, we must have $\beta < \frac{1}{2}\pi$, and so $\cos\beta > 0$. We thus have

$$\begin{aligned}
\cos\alpha &> \left(\frac{mu^2}{2a\lambda}\cos\alpha + (1-\cos\alpha)^2\right)^{\frac{1}{2}} \\
\cos^2\alpha &> \frac{mu^2}{2a\lambda}\cos\alpha + (1-\cos\alpha)^2 \\
2\cos\alpha - 1 &> \frac{mu^2}{2a\lambda}\cos\alpha \\
u^2 &< \frac{2a\lambda(2\cos\alpha - 1)}{m\cos\alpha} \\
u^2 &< \frac{2a\lambda}{m}(2 - \sec\alpha) \quad ,
\end{aligned}$$

where the squaring of both sides is permissible since both $\cos\alpha$, and the expression under the squre root are positive.

Section C: Probability and Statistics

Question 11

(i) The probability that the game has not ended after $2n$ turns is the probability that the coin turns up HT n times, or TH n times, which is

$$\mathbb{P}\{\text{no winner in } n \text{ turns}\} = (pq)^n + (qp)^n = 2 \cdot (pq)^n \quad .$$

The probability that the game never ends is thus $\lim_{n\to\infty} (2 \cdot (pq)^n) = 0$, since $0 \leqslant pq < 1$.

Given that the first toss is H. A will win if and only if the second toss is H, or if the subsequent coin tosses are TH some integer multiple of times followed by a final H. Thus

$$\begin{aligned}\mathbb{P}\{\text{A wins} \mid \text{first toss H}\} &= \sum_{n=0}^{\infty} (pq)^n \cdot p \\ &= \frac{p}{1-pq} \quad ,\end{aligned}$$

by summing the geometric series (since $|pq| < 1$). If the first toss is T. A will win if and only if the second toss is H and the subsequent coin tosses are TH some integer multiple of times followed by a final H. Thus

$$\begin{aligned}\mathbb{P}\{\text{A wins} \mid \text{first toss T}\} &= \sum_{n=0}^{\infty} p \cdot (pq)^n \cdot p \\ &= \frac{p^2}{1-pq} \quad ,\end{aligned}$$

by summing the geometric series (since $|pq| < 1$). Thus the total probability that A wins is

$$\begin{aligned}\mathbb{P}\{\text{A wins}\} &= \frac{p}{1-pq}\mathbb{P}\{\text{first toss H}\} + \frac{p^2}{1-pq}\mathbb{P}\{\text{first toss T}\} \\ &= \frac{p^2}{1-pq} + \frac{p^2 q}{1-pq} \\ &= \frac{p^2(1+q)}{1-pq} \quad .\end{aligned}$$

(ii) We note that the probability of A winning after the first T occurs is equal to the probability of A winning given that the first toss is T. Given the first toss is H, A wins if and only if (1) the next two tosses are HH, (2) the next tosses are HT and then some sequence with no more than two Ts before three Hs, (3) the next toss is a T followed by some sequence with no more than two Ts before three Hs. If we let $\mathbb{P}_{A,H}$ denote the probability that A wins given the first toss is H, and let $\mathbb{P}_{A,T}$ denote the probability that A wins given the first toss is T, then

$$\begin{aligned}\mathbb{P}_{A,H} &= \mathbb{P}\{H,H\} + \mathbb{P}\{H,T\}\mathbb{P}_{A,T} + \mathbb{P}\{T\}\mathbb{P}_{A,T} \\ &= p^2 + (pq+q)\mathbb{P}_{A,T} \quad .\end{aligned}$$

Similarly, if the first toss is a T, A wins if and only if there are fewer than two more consecutive Ts followed by some H and then a sequence with which A wins given the first toss was a H:

$$\begin{aligned}\mathbb{P}_{A,T} &= \mathbb{P}\{H\}\mathbb{P}_{A,H} + \mathbb{P}\{T,H\}\mathbb{P}_{A,H} \\ &= (p+pq)\mathbb{P}_{A,H} \quad .\end{aligned}$$

From these two equations, we have

$$\begin{aligned}\mathbb{P}_{A,H} &= p^2 + q(p+1)\mathbb{P}_{A,T} \\ &= p^2 + q(p+1)\cdot p(1+q)\mathbb{P}_{A,H}\end{aligned}$$

hence

$$\begin{aligned}(1 - pq(1+p)(1+q))\mathbb{P}_{A,H} &= p^2 \\ (1-(1-q)(1-p)(1+p)(1+q))\mathbb{P}_{A,H} &= p^2 \\ (1-(1-p^2)(1-q^2))\mathbb{P}_{A,H} &= p^2 \\ \mathbb{P}_{A,H} &= \frac{p^2}{1-(1-p^2)(1-q^2)} \quad .\end{aligned}$$

This in turn gives

$$\begin{aligned}\mathbb{P}_{A,T} = p(1+q)\mathbb{P}_{A,H} &= \frac{p^2\cdot(1+q)}{1-(1-p^2)(1-q^2)} \\ &= \frac{p^2(1-q^2)}{1-(1-p^2)(1-q^2)} \quad .\end{aligned}$$

Letting $\mathbb{P}_A$ denote the probability that A wins then, we have

$$\begin{aligned}\mathbb{P}_A = p\mathbb{P}_{A,H} + q\mathbb{P}_{A,T} &= \frac{p^2(p+q(1-q^2))}{1-(1-p^2)(1-q^2)} \\ &= \frac{p^2(p+q-q^3)}{1-(1-p^2)(1-q^2)} \\ &= \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)} \quad .\end{aligned}$$

(iii) As above, we use $\mathbb{P}_A$ to denote the probability that A wins, and $\mathbb{P}_{A,H}$ and $\mathbb{P}_{A,T}$ to denote the probabilities that A wins given that the first toss is a H or T respectively. By similar reasoning to the above for $a > 2$ we have

$$\begin{aligned}\mathbb{P}_{A,H} &= \mathbb{P}\{(a-1) \text{ consecutive Hs}\} + \sum_{r=0}^{a-2} \mathbb{P}\{r \text{ consecutive Hs}\}\mathbb{P}\{T\}\mathbb{P}_{A,T} \\ &= p^{a-1} + \left(\sum_{r=0}^{a-2} p^r\right) q\mathbb{P}_{A,T} = p^{a-1} + \frac{1-p^{a-1}}{1-p} q\mathbb{P}_{A,T} \\ &= p^{a-1} + \left(1-p^{a-1}\right)\mathbb{P}_{A,T} \quad ,\end{aligned}$$

and for $b > 2$ we have

$$\begin{aligned}\mathbb{P}_{A,T} &= \sum_{r=0}^{b-2} \mathbb{P}\{r \text{ consecutive Ts}\}\mathbb{P}\{H\}\mathbb{P}_{A,H} \\ &= \left(\sum_{r=0}^{b-2} q^r\right) p\mathbb{P}_{A,H} = \frac{1-q^{b-1}}{1-q} p\mathbb{P}_{A,H} \\ &= \left(1-q^{b-1}\right)\mathbb{P}_{A,H} \quad .\end{aligned}$$

These together give

$$\begin{aligned}\mathbb{P}_{A,H} &= p^{a-1} + \left(1-p^{a-1}\right)\mathbb{P}_{A,T} \\ &= p^{a-1} + \left(1-p^{a-1}\right)\left(1-q^{b-1}\right)\mathbb{P}_{A,H} \\ \iff \left(1-\left(1-p^{a-1}\right)\left(1-q^{b-1}\right)\right)\mathbb{P}_{A,H} &= p^{a-1} \\ \mathbb{P}_{A,H} &= \frac{p^{a-1}}{1-\left(1-p^{a-1}\right)\left(1-q^{b-1}\right)} \quad ,\end{aligned}$$

and

$$\mathbb{P}_{A,T} = \left(1-q^{b-1}\right)\mathbb{P}_{A,H} = \frac{p^{a-1}\left(1-q^{b-1}\right)}{1-\left(1-p^{a-1}\right)\left(1-q^{b-1}\right)} \quad .$$

Thus we have

$$\begin{aligned}\mathbb{P}_A = p\mathbb{P}_{A,H} + q\mathbb{P}_{A,T} &= \frac{p^{a-1}(p+q(1-q^{b-1}))}{1-(1-p^{a-1})(1-q^{b-1})} \\ &= \frac{p^{a-1}(1-q^b)}{1-(1-p^{a-1})(1-q^{b-1})} \quad .\end{aligned}$$

Substituting in $a = b = 2$ we get

$$\mathbb{P}_A = \frac{p(1-q^2)}{1-(1-p)(1-q)} = \frac{p(1-q)(1+q)}{1-qp} = \frac{p^2(1+q)}{1-pq} \quad ,$$

matching the result from earlier.

Question 12

(i) We note that

$$\begin{aligned}\mathbb{P}\{X = i\} \geqslant 0 \ \text{ for each } i \quad &\iff \quad \frac{1}{n} + \epsilon_i \geqslant 0 \ \text{ for each } i \\ &\iff \quad \epsilon_i \geqslant -\frac{1}{n} \ \text{ for each } i \quad ,\end{aligned}$$

and also

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{P}\{X = i\} = 1 \quad &\iff \quad \sum_{i=1}^{n} \left(\frac{1}{n} + \epsilon_i \right) = 1 \\ &\iff \quad 1 + \sum_{i=1}^{n} \epsilon_i = 1 \\ &\iff \quad \sum_{i=1}^{n} \epsilon_i = 0 \quad .\end{aligned}$$

Let X_1, X_2 be the scores on independent rolls of the die. The probability that X_1 and X_2 show the same number is

$$\begin{aligned}\mathbb{P}\{X_1 = X_2\} &= \sum_{i=1}^{n} \mathbb{P}\{X_1 = i\} \cdot \mathbb{P}\{X_2 = i\} \\ &= \sum_{i=1}^{n} \left(\frac{1}{n} + \epsilon_i \right)^2 \\ &= \sum_{i=1}^{n} \left(\frac{1}{n^2} + \frac{2}{n}\epsilon_i + \epsilon_i^2 \right) \\ &= \frac{n}{n^2} + \frac{2}{n} \sum_{i=1}^{n} \epsilon_i + \sum_{i=1}^{n} \epsilon_i^2 \\ &= \frac{1}{n} + \sum_{i=1}^{n} \epsilon_i^2 \quad .\end{aligned}$$

In the case of an unbiased die, $\epsilon_i = 0$ for each i, giving $\mathbb{P}\{X_1 = X_2\} = \frac{1}{n}$.

Hence a biased die is more likely than an unbiased die to show the same score on two successive rolls.

(ii) Consider the probability that $X_1 > X_2$. By symmetry (and independence) this is equal to the probability that $X_2 > X_1$, thus (using (i))

$$\begin{aligned}\mathbb{P}\{X_1 > X_2\} &= \frac{1}{2}\mathbb{P}\{X_1 > X_2 \text{ or } X_2 > X_1\} \\ &= \frac{1}{2}\left(1 - \mathbb{P}\{X_1 = X_2\}\right) \\ &= \frac{1}{2} - \frac{1}{2n} - \frac{1}{2}\sum_{i=1}^{n}\epsilon_i^2 \quad .\end{aligned}$$

In the unbiased case ($\epsilon_i = 0$, for each i), this gives $\mathbb{P}\{X_1 > X_2\} = \frac{1}{2} - \frac{1}{2n} = \frac{n-1}{2n}$. Note that this probability in the unbiased case is greater than in the biased case.

Given a set of n positive numbers $x_1, x_2, \ldots, x_n$, let $T = \sum_{i=1}^{n} x_i$ and set $\epsilon_i = \frac{x_i}{T} - \frac{1}{n}$. Note that

$$\epsilon_i = \frac{x_i}{T} - \frac{1}{n} \text{ and } x_i \geqslant 0 \implies \epsilon_i \geqslant -\frac{1}{n} \text{ for each } i \quad ,$$

and

$$\sum_{i=1}^{n}\epsilon_i = \sum_{i=1}^{n}\frac{x_i}{T} - \sum_{i=1}^{n}\frac{1}{n} = \frac{1}{T}\left(\sum_{i=1}^{n}x_i\right) - 1 = 1 - 1 = 0 \quad .$$

Using the above probability inequality with these ϵ_i:

$$\begin{aligned}\mathbb{P}\{X_1 > X_2\} &\leqslant \frac{n-1}{2n} \\ \sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathbb{P}\{X_1 = i\}\cdot\mathbb{P}\{X_2 = j\} &\leqslant \frac{n-1}{2n} \\ \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(\frac{1}{n} + \epsilon_i\right)\left(\frac{1}{n} + \epsilon_j\right) &\leqslant \frac{n-1}{2n} \\ \sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{x_i}{T}\frac{x_j}{T} &\leqslant \frac{n-1}{2n} \\ \sum_{i=2}^{n}\sum_{j=1}^{i-1}x_i x_j &\leqslant \frac{n-1}{2n}\left(\sum_{i=1}^{n}x_i\right)^2 \quad .\end{aligned}$$

(iii) Let X_3 be the score on a third independent roll of the die, then in the biased case

$$\begin{aligned}\mathbb{P}\{X_1 = X_2 = X_3\} &= \sum_{i=1}^{n} \left(\frac{1}{n} + \epsilon_i\right)^3 \\ &= \sum_{i=1}^{n} \left(\frac{1}{n^3} + \frac{3\epsilon_i}{n^2} + \frac{3\epsilon_i^2}{n} + \epsilon_i^3\right) \\ &= \frac{n}{n^3} + \frac{3}{n^2}\sum_{i=1}^{n} \epsilon_i + \sum_{i=1}^{n} \left(\frac{3\epsilon_i^2}{n} + \epsilon_i^3\right) \\ &= \frac{1}{n^2} + \sum_{i=1}^{n} \epsilon_i^2 \left(\frac{3}{n} + \epsilon_i\right) \quad ,\end{aligned}$$

in the case of an unbiased die this probability is just $\frac{1}{n^2}$ (which we see by setting $\epsilon_i = 0$ for each i). Since $\epsilon_i \geqslant -\frac{1}{n}$ for each i, we have

$$\begin{aligned}\mathbb{P}\{X_1 = X_2 = X_3\} - \frac{1}{n^2} &= \sum_{i=1}^{n} \epsilon_i^2 \left(\frac{3}{n} + \epsilon_i\right) \\ &\geqslant \sum_{i=1}^{n} \epsilon_i^2 \left(\frac{3}{n} - \frac{1}{n}\right) \\ &\geqslant \frac{2}{n}\sum_{i=1}^{n} \epsilon_i^2 \\ &\geqslant 0\end{aligned}$$

(with equality if and only if $\epsilon_i = 0$ for each i). That is, a biased die is more likely than an unbiased die to show the same score on three consecutive rolls.

STEP III

Section A: Pure Mathematics

Question 1

(i) We have

$$\begin{aligned} I(a,b) - I(a-1,b-1) &= \int_0^{\frac{\pi}{2}} \cos^{a-1}(x) \left(\cos(x)\cos(bx) - \cos((b-1)x)\right) dx \\ &= -\int_0^{\frac{\pi}{2}} \cos^{a-1}(x)\sin(bx)\sin(x) dx \quad , \end{aligned}$$

now integrating by parts gives

$$\begin{aligned} I(a,b) - I(a-1,b-1) &= \left[\frac{1}{a}\cos^a(x)\sin(bx)\right]_0^{\frac{\pi}{2}} - \frac{b}{a}\int_0^{\frac{\pi}{2}} \cos^a(x)\cos(bx) dx \\ &= 0 - \frac{b}{a} I(a,b) \quad . \end{aligned}$$

Rearranging gives

$$\begin{aligned} aI(a,b) - aI(a-1,b-1) &= -bI(a,b) \\ (a+b)I(a,b) &= aI(a-1,b-1) \\ I(a,b) &= \frac{a}{a+b} I(a-1,b-1) \quad , \end{aligned}$$

as required.

(ii) First consider the base case $n = 0$: for any non-negative integer m, we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos((2m+1)x) dx &= \left[\frac{1}{2m+1}\sin((2m+1)x)\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2m+1}\sin\left((2m+1)\tfrac{\pi}{2}\right) \\ &= \frac{1}{2m+1}\sin(m\pi + \tfrac{\pi}{2}) \\ &= \frac{(-1)^m}{2m+1}\sin(\tfrac{\pi}{2}) \\ &= \frac{(-1)^m}{2m+1} \quad . \end{aligned}$$

We also have

$$(-1)^m \frac{2^0 \cdot 0!(2m)!(0+m)!}{m!(0+2m+1)!} = \frac{(-1)^m (2m)! m!}{m!(2m+1)!} = \frac{(-1)^m}{2m+1} \quad ,$$

so the induction hypothesis holds for $n = 0$.

Now suppose that for non-negative integer $n = k - 1$ we have that

$$\int_0^{\frac{\pi}{2}} \cos^{k-1}(x)\cos(((k-1)+2m+1)x)dx = (-1)^m \frac{2^{k-1}(k-1)!(2m)!(k-1+m)!}{m!(2(k-1)+2m+1)!}$$

for all non-negative integers m. Fix any particular non-negative integer m, then by the result of (i) and the above assumption, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^{k}(x)\cos((k+2m+1)x)dx &= I(k, k+2m+1) \\
&= \frac{k}{2k+2m+1} I(k-1, k-1+2m+1) \\
&= \frac{k}{2k+2m+1} \int_0^{\frac{\pi}{2}} \cos^{k-1}(x)\cos(((k-1)+2m+1)x)dx \\
&= \frac{k}{2k+2m+1}(-1)^m \frac{2^{k-1}(k-1)!(2m)!(k-1+m)!}{m!(2(k-1)+2m+1)!} \\
&= \frac{1}{2k+2m+1}(-1)^m \frac{2^{k-1}k!(2m)!(k-1+m)!}{m!(2k+2m-1)!} \\
&= \frac{2k+2m}{2k+2m+1}(-1)^m \frac{2^{k-1}k!(2m)!(k-1+m)!}{m!(2k+2m)!} \\
&= (k+m)(-1)^m \frac{2^{k}k!(2m)!(k-1+m)!}{m!(2k+2m+1)!} \\
&= (-1)^m \frac{2^{k}k!(2m)!(k+m)!}{m!(2k+2m+1)!} \quad ,
\end{aligned}$$

proving the induction hypothesis for $n = k$. By induction, the result holds for all non-negative integers n.

Question 2

(i) Differentiating with respect to x, we find

$$\begin{aligned} \cosh x + \cosh y \frac{dy}{dx} &= 0 \\ \implies \quad \frac{dy}{dx} &= -\frac{\cosh x}{\cosh y} \quad . \end{aligned}$$

Since $\cosh t \geqslant 1$ for all real t, there are no points at which $\frac{dy}{dx} = 0$. Similarly, inverting the fraction, there are no points at which $\frac{dx}{dy} = 0$.

Differentiating again, we have

$$\begin{aligned} \sinh x + \sinh y \left(\frac{dy}{dx}\right)^2 + \cosh y \frac{d^2y}{dx^2} &= 0 \\ \sinh x + \sinh y \frac{\cosh^2 x}{\cosh^2 y} + \cosh y \frac{d^2y}{dx^2} &= 0 \quad . \end{aligned}$$

Looking for points of inflection $\frac{d^2y}{dx^2} = 0$, since $\cosh y \geqslant 1$ for all real y, we have

$$\begin{aligned} \frac{d^2y}{dx^2} = 0 \quad \iff \quad \sinh x + \sinh y \frac{\cosh^2 x}{\cosh^2 y} &= 0 \\ \sinh x(1 + \sinh^2 y) + \sinh y(1 + \sinh^2 x) &= 0 \\ \sinh x + \sinh x \sinh^2 y + \sinh y + \sinh y \sinh^2 x &= 0 \\ (\sinh x + \sinh y)(1 + \sinh x \sinh y) &= 0 \\ 2k(1 + \sinh x \sinh y) &= 0 \quad , \end{aligned}$$

and since $k > 0$, we conclude that

$$\frac{d^2y}{dx^2} = 0 \quad \iff \quad 1 + \sinh x \sinh y = 0 \quad .$$

Solving for the coordinates of the points of inflection:

$$\begin{aligned} & 1 + \sinh x(2k - \sinh x) = 0 \\ & \sinh^2 x - 2k \sinh x - 1 = 0 \\ & \qquad \sinh x = k \pm \sqrt{k^2 + 1} \quad \implies \quad \sinh y = k \mp \sqrt{k^2 + 1} \\ & \implies (x, y) = \left(\operatorname{arcsinh}(k \pm \sqrt{k^2 + 1}), \operatorname{arcsinh}(k \mp \sqrt{k^2 + 1})\right) \quad . \end{aligned}$$

(ii) Substituting $y = a - x$, we have

$$\begin{aligned}
\sinh x + \sinh(a-x) &= 2k \\
\frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^a e^{-x} - e^{-a}e^x) &= 2k \\
e^{2x} - 1 + e^a - e^{-a}e^{2x} &= 4k \\
(1-e^{-a})e^{2x} - 4ke^x + (e^a - 1) &= 0 \quad .
\end{aligned}$$

This is a quadratic in e^x, and for it to have real solutions, the discriminant must be non-negative:

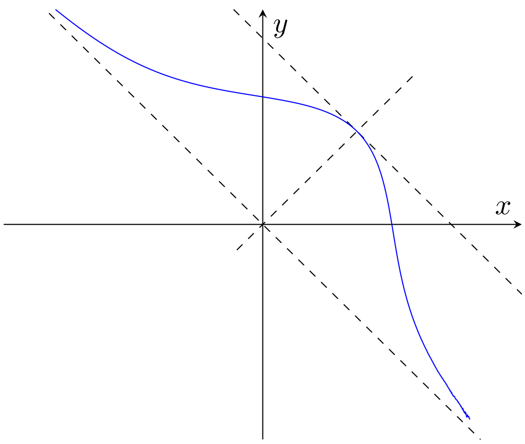
$$\begin{aligned}
16k^2 - 4(1-e^{-a})(e^a - 1) &\geqslant 0 \\
4k^2 - (e^a - 2 + e^{-a}) &\geqslant 0 \\
e^a + e^{-a} &\leqslant 4k^2 + 2 \\
\cosh a &\leqslant 2k^2 + 1 \quad .
\end{aligned}$$

We note that $\cosh a = 1$ if and only if $a = 0$, but that $a = 0$ would give $y = -x$, and thus $\sinh x + \sinh y = 0$. Since $k > 0$, we conclude that $\cosh a > 1$. Giving

$$1 < \cosh a \leqslant 2k^2 + 1 \quad ,$$

as required.

(iii) In order to sketch C, we note that the curve is symmetric in the line $y = x$, that $\frac{dy}{dx} < 0$ for all points on C, and that as $|x| \to \infty$ we have $y \to -x$. Our sketch is as follows.



Both the y-axis and x-axis intercepts are at $\text{arcsinh}(2k)$. The points of inflection lie just to the left of the y-axis, and just below the x-axis. The bounding lines are $x + y = 0$ (lower) and $x + y = \text{arccosh}(2k^2 + 1)$ (upper). The symmetry axis $y = x$ is also drawn.

Question 3

(i) Let K be represented in the complex plane by k, then

$$\begin{aligned} k - a &= e^{-\frac{i\pi}{3}}(b-a) \\ k &= \left(1 - e^{-\frac{i\pi}{3}}\right)a + e^{-\frac{i\pi}{3}}b \quad , \end{aligned}$$

thus

$$g_{ab} = \frac{1}{3}(a+b+k) = \frac{1}{3}\left(\left(2 - e^{-\frac{i\pi}{3}}\right)a + \left(1 + e^{-\frac{i\pi}{3}}\right)b\right) \quad ,$$

we compute that

$$\omega = e^{\frac{i\pi}{6}} = \frac{1}{2}(\sqrt{3}+i) \quad , \quad e^{\frac{-i\pi}{3}} = \frac{1}{2}(1-\sqrt{3}i) \quad ,$$

giving

$$\begin{aligned} g_{ab} &= \frac{1}{3}\left(\left(2 - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)a + \left(1 + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)b\right) \\ &= \frac{1}{\sqrt{3}}\left(\frac{1}{2}\left(\sqrt{3}+i\right)a + \frac{1}{2}\left(\sqrt{3}-i\right)b\right) \\ &= \frac{1}{\sqrt{3}}\left(\omega a + \omega^* b\right) \quad , \end{aligned}$$

as required.

(ii) Let A, B, C, D be represented in the complex plane by a, b, c, d, and G_{AB}, G_{BC}, G_{CD}, G_{DA} be represented in the complex plane by g_{ab}, g_{bc}, g_{cd}, g_{da} respectively. We have that Q_1 is a parallelogram if and only if $b - a = c - d$, while Q_2 is a parallelogram if and only if $g_{bc} - g_{ab} = g_{cd} - g_{da}$. Consider then, by (i),

$$\begin{aligned} g_{bc} - g_{ab} - (g_{cd} - g_{da}) &= \frac{1}{\sqrt{3}}\left(\omega(b-a) + \omega^*(c-b) - (\omega(c-d) + \omega^*(d-a))\right) \\ &= \frac{1}{\sqrt{3}}\left(\omega(b-a-(c-d)) + \omega^*(c-b-(d-a))\right) \\ &= \frac{\omega - \omega^*}{\sqrt{3}}\left(b-a-(c-d)\right) \quad . \end{aligned}$$

Since $\omega \neq \omega^*$, we have that

$$b - a = c - d \iff g_{bc} - g_{ab} = g_{cd} - g_{da} \quad ,$$

that is, Q_1 is a parallelogram if and only if Q_2 is a parallelogram.

(iii) Let G_{AB}, G_{BC}, G_{CA} be represented in the complex plane by g_{ab}, g_{bc}, g_{ca} respectively. Consider the edges $G_{BC}G_{AB}$, $G_{CA}G_{AB}$ which we represent by complex numbers x, y respectively. By (i),

$$\begin{aligned} x := g_{bc} - g_{ab} &= \frac{1}{\sqrt{3}}\left(\omega(b-a) + \omega^*(c-b)\right) \\ \text{and} \quad y := g_{ca} - g_{ab} &= \frac{1}{\sqrt{3}}\left(\omega(c-a) + \omega^*(a-b)\right) \quad . \end{aligned}$$

We show that $G_{AB}G_{BC}G_{CA}$ is an equilateral triangle by showing that the edge $G_{CA}G_{AB}$ is the image of the edge $G_{BC}G_{AB}$ under an anticlockwise rotation through an angle of $\frac{\pi}{3}$. In terms of complex numbers, this is equivalent to the equation $y = \omega^2 x$. Indeed, we have that

$$\begin{aligned} \omega^2 x &= \frac{\omega^2}{\sqrt{3}}\big(\omega(b-a) + \omega^{-1}(c-b)\big) && (\text{since } \omega^* = \omega^{-1}) \\ &= \frac{1}{\sqrt{3}}\big(\omega^3(b-a) + \omega(c-b)\big) \\ &= \frac{1}{\sqrt{3}}\left(i(b-a) + \omega(c-b)\right) && (\text{since } \omega^3 = e^{\frac{i\pi}{2}} = i) \\ &= \frac{1}{\sqrt{3}}\left((\omega - \omega^*)(b-a) + \omega(c-b)\right) && (\text{since } \omega - \omega^* = 2\mathrm{Im}(\omega) = i) \\ &= \frac{1}{\sqrt{3}}\left(\omega(c-a) + \omega^*(a-b)\right) = y \quad , \end{aligned}$$

as required.

Question 4

Note that the position vector of Q, $\boldsymbol{q} = \boldsymbol{x} - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$, satisfies

$$\boldsymbol{q} \cdot \boldsymbol{n} = \boldsymbol{x} \cdot \hat{\boldsymbol{n}} - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}) = \boldsymbol{x} \cdot \hat{\boldsymbol{n}} - \boldsymbol{x} \cdot \hat{\boldsymbol{n}} = 0 \quad ,$$

thus Q lies on Π. The vector corresponding to PQ is given by

$$\boldsymbol{x} - \boldsymbol{q} = \boldsymbol{x} - (\boldsymbol{x} - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}) = (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \quad .$$

This is parallel to $\hat{\boldsymbol{n}}$, hence it is perpendicular to Π.

(i) We have $\hat{\boldsymbol{n}} = (a, b, c)$. Taking a general point $\boldsymbol{x}$, we can decompose it as

$$\boldsymbol{x} = (\boldsymbol{x} - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}) + (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \quad ,$$

the first component here lies within the plane $ax + by + cz = 0$, while the second component is perpendicular to the plane. The effect of the transformation T is to reflect this second component, hence

$$\begin{aligned} T(\boldsymbol{x}) &= (\boldsymbol{x} - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}) - (\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \\ &= \boldsymbol{x} - 2(\boldsymbol{x} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \quad . \end{aligned}$$

In particular, applying this to the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$:

$$T(\hat{\boldsymbol{\imath}}) = \hat{\boldsymbol{\imath}} - 2a\hat{\boldsymbol{n}} = \begin{pmatrix} 1 - 2a^2 \\ -2ab \\ -2ac \end{pmatrix} = \begin{pmatrix} b^2 + c^2 - a^2 \\ -2ab \\ -2ac \end{pmatrix} \quad ,$$

and

$$T(\hat{\boldsymbol{\jmath}}) = \hat{\boldsymbol{\jmath}} - 2b\hat{\boldsymbol{n}} = \begin{pmatrix} -2ab \\ 1 - 2b^2 \\ -2bc \end{pmatrix} = \begin{pmatrix} -2ab \\ a^2 + c^2 - b^2 \\ -2bc \end{pmatrix} \quad ,$$

and

$$T(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{k}} - 2c\hat{\boldsymbol{n}} = \begin{pmatrix} -2ac \\ -2bc \\ 1 - 2c^2 \end{pmatrix} = \begin{pmatrix} -2ac \\ -2bc \\ a^2 + b^2 - c^2 \end{pmatrix} \quad ,$$

giving the matrix for this transformation:

$$\boldsymbol{M} = \begin{pmatrix} b^2 + c^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 + c^2 - b^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix} \quad .$$

(ii) Subtracting off the identity from our expression and the given matrix, we find

$$\begin{pmatrix} -2a^2 & -2ab & -2ac \\ -2ab & -2b^2 & -2bc \\ -2ac & -2bc & -2c^2 \end{pmatrix} = \begin{pmatrix} -0.36 & 0.48 & 0.6 \\ 0.48 & -0.64 & -0.8 \\ 0.6 & -0.8 & -1 \end{pmatrix}$$
$$\implies \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} = \begin{pmatrix} 0.18 & -0.24 & -0.3 \\ -0.24 & 0.32 & 0.4 \\ -0.3 & 0.4 & 0.5 \end{pmatrix} .$$

Now we consider this as component-wise equations:

$$a^2 = 0.18 \implies a = \pm\frac{\sqrt{18}}{10} = \pm\frac{3\sqrt{2}}{10} ,$$

we choose the positive root arbitrarily – this is purely a choice of sign in $\hat{\boldsymbol{n}}$, and does not change the equation of the plane. Then

$$ab = -0.24 \implies b = -\frac{24}{100}\cdot\frac{10}{3\sqrt{2}} = -\frac{2\sqrt{2}}{5} ,$$

and

$$ac = -0.3 \implies c = -\frac{3}{10}\cdot\frac{10}{3\sqrt{2}} = \frac{\sqrt{2}}{2} .$$

These give that the Cartesian equation of the plane is

$$\frac{3\sqrt{2}}{10}x - \frac{2\sqrt{2}}{5}y - \frac{\sqrt{2}}{2}z = 0$$
$$\text{or, equivalently} \qquad 3x - 4y - 5z = 0 .$$

(iii) Again taking $\hat{\boldsymbol{n}} = (a, b, c)$ and decomposing a general vector

$$\boldsymbol{x} = (\boldsymbol{x} - (\boldsymbol{x}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}) + (\boldsymbol{x}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} ,$$

the effect of $\boldsymbol{N}$ is to rotate the first component through an angle π, hence

$$\begin{aligned}\boldsymbol{N}\boldsymbol{x} &= -(\boldsymbol{x} - (\boldsymbol{x}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}) + (\boldsymbol{x}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \\ &= -\boldsymbol{x} + 2(\boldsymbol{x}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} .\end{aligned}$$

In particular, we recognise

$$\boldsymbol{N}\boldsymbol{x} = -T(\boldsymbol{x}) ,$$

thus:

$$\boldsymbol{N} = -\boldsymbol{M} = \begin{pmatrix} a^2 - b^2 - c^2 & 2ab & 2ac \\ 2ab & b^2 - a^2 - c^2 & 2bc \\ 2ac & 2bc & c^2 - a^2 - b^2 \end{pmatrix} .$$

(iv) We have $\boldsymbol{N}\boldsymbol{M} = -\boldsymbol{M}^2 = -\boldsymbol{I}$ ($\boldsymbol{M}^2 = \boldsymbol{I}$, since T is a reflection, so performing it twice is just the identity). Thus $\boldsymbol{N}\boldsymbol{M}$ is an enlargement with scale factor -1, and centre of enlargement at the origin.

Question 5

Starting with the right-hand side, we have

$$\begin{aligned}(x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1} &= \sum_{r=1}^{n} x^{n-r+1}y^{r-1} - \sum_{r=1}^{n} x^{n-r}y^{r} \\ &= \sum_{r=0}^{n-1} x^{n-r}y^{r} - \sum_{r=1}^{n} x^{n-r}y^{r} \\ &= x^{n-0}y^{0} - x^{n-n}y^{n} = x^n - y^n \quad ,\end{aligned}$$

where the sums have telescoped to cancel all but the first and last terms.

(i) (a) We have

$$\frac{1}{x^n(x-k)} = \frac{A}{x-k} + \frac{f(x)}{x^n} \iff \frac{1}{(x-k)} = \frac{Ax^n}{x-k} + f(x)$$
$$\iff f(x) = \frac{1-Ax^n}{x-k} \quad .$$

Since we know $f(x)$ is a polynomial, we must have that $(x-k)$ is a factor of $1-Ax^n$. That is, when $x-k=0$, we must have $1-Ax^n=0$, thus

$$1 - Ak^n = 0 \implies A = \frac{1}{k^n} \quad ,$$

giving

$$f(x) = \frac{1-\frac{x^n}{k^n}}{x-k} = \frac{1}{x-k}\left(1-\left(\frac{x}{k}\right)^n\right) \quad ,$$

as required. This also then gives

$$\begin{aligned}F(x) &= \frac{1}{k^n(x-k)} + \frac{1}{x^n(x-k)}\left(1-\left(\frac{x}{k}\right)^n\right) \\ &= \frac{1}{k^n(x-k)} - \frac{1}{x^n k^n(x-k)}(x^n-k^n)\end{aligned}$$

Now using the root result, replacing y with k, we get

$$\begin{aligned}F(x) &= \frac{1}{k^n(x-k)} - \frac{1}{x^n k^n(x-k)}(x-k)\sum_{r=1}^{n} x^{n-r}k^{r-1} \\ &= \frac{1}{k^n(x-k)} - \frac{1}{x^n k^n}\sum_{r=1}^{n} x^{n-r}k^{r-1} \\ &= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n} x^{-r}k^{r-n} = \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r} \quad .\end{aligned}$$

(b) We now have two forms for $F(x)$, giving

$$x^n F(x) = \frac{1}{x-k} \quad \text{and} \quad x^n F(x) = \frac{x^n}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n} \frac{x^{n-r}}{k^{n-r}} \quad ,$$

hence we have

$$\frac{d}{dx}\left(x^n F(x)\right) = -\frac{1}{(x-k)^2} \quad ,$$

and, letting $z = \frac{x}{k}$,

$$\begin{aligned}
\frac{d}{dx}\left(x^n F(x)\right) &= \frac{dz}{dx}\frac{d}{dz}\left(\frac{z^n}{k(z-1)} - \frac{1}{k}\sum_{r=1}^{n} z^{n-r}\right) \\
&= \frac{1}{k}\left(-\frac{z^n}{k(z-1)^2} + \frac{nz^{n-1}}{k(z-1)} - \frac{1}{k}\sum_{r=1}^{n}(n-r)z^{n-r-1}\right) \\
&= -\frac{x^n}{k^{n+2}(\frac{x}{k}-1)^2} + \frac{nx^{n-1}}{k^{n+1}(\frac{x}{k}-1)} - \frac{1}{k^2}\sum_{r=1}^{n}(n-r)\frac{x^{n-r-1}}{k^{n-r-1}} \\
&= -\frac{x^n}{k^n(x-k)^2} + \frac{nx^{n-1}}{k^n(x-k)} - \sum_{r=1}^{n}\frac{(n-r)x^{n-r-1}}{k^{n-r+1}} \quad .
\end{aligned}$$

Equating our two results, we have

$$\begin{aligned}
-\frac{1}{(x-k)^2} &= -\frac{x^n}{k^n(x-k)^2} + \frac{nx^{n-1}}{k^n(x-k)} - \sum_{r=1}^{n}\frac{(n-r)x^{n-r-1}}{k^{n-r+1}} \\
\implies \quad \frac{1}{x^n(x-k)^2} &= \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n}\frac{(n-r)x^{-r-1}}{k^{n-r+1}} \\
&= \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}} \quad .
\end{aligned}$$

(ii) Employing the result in (b) with $n = 3$ and $k = 1$, we have

$$\begin{aligned}
\int_2^N \frac{1}{x^3(x-1)^2}dx &= \int_2^N \left(\frac{1}{(x-1)^2} - \frac{3}{x(x-1)} + \sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\right)dx \\
&= \int_2^N \left(\frac{1}{(x-1)^2} + \frac{3}{x} - \frac{3}{x-1} + \frac{2}{x^2} + \frac{1}{x^3}\right)dx \\
&= \left[\frac{-1}{x-1} + 3\ln(x) - 3\ln(x-1) - \frac{2}{x} - \frac{1}{2x^2}\right]_2^N \\
= \left(\frac{-1}{N-1} + 3\ln\left(\frac{N}{N-1}\right) - \frac{2}{N} - \frac{1}{2N^2}\right) &- \left(-1 + 3\ln(2) - 1 - \frac{1}{8}\right) \quad .
\end{aligned}$$

Now we wish to take the limit as $N \to \infty$. We note that

$$\lim_{N\to\infty} \frac{1}{N-1} = \lim_{N\to\infty} \frac{1}{N} = 0 \quad , \quad \lim_{N\to\infty} \frac{1}{N^2} = 0 \quad , \quad \text{and} \quad \lim_{N\to\infty} \frac{N}{N-1} = 1$$
$$\text{thus, since ln is continuous,} \quad \lim_{N\to\infty} \ln\left(\frac{N}{N-1}\right) = \ln(1) = 0 \quad .$$

Together, these imply that all the N dependent terms on the right-hand side tend to zero as $N \to \infty$, leaving

$$\begin{aligned} \lim_{N\to\infty} \int_2^N \frac{1}{x^3(x-1)^2} dx &= -\left(-1 + 3\ln(2) - 1 - \frac{1}{8}\right) \\ &= 2 + \frac{1}{8} - 3\ln(2) \quad . \end{aligned}$$

Question 6

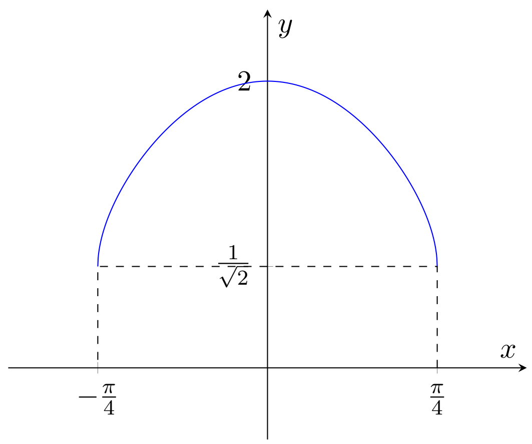
(i) We note that over the given domain, $\cos x$ and $\cos(2x)$ are concave, likewise $\sqrt{x}$ is a concave function, thus by composition and addition of concave functions, this graph is concave. We also note that $\cos x + \sqrt{\cos(2x)}$ is an even function of x, thus the graph will be symmetric across the y-axis. Also consider

$$\begin{aligned} y &= \cos x + \sqrt{\cos(2x)} \\ \implies \quad \frac{dy}{dx} &= -\sin x + \frac{1}{2} \cdot \frac{-2\sin(2x)}{(\cos(2x))^{1/2}} \\ &= -\sin x \left(1 + \frac{2\cos x}{\sqrt{\cos(2x)}}\right) \quad . \end{aligned}$$

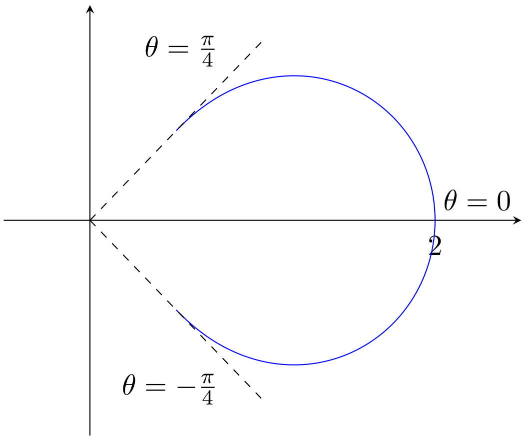
From this we see that $\frac{dy}{dx} = 0$ only at $x = 0$ (where $y = 2$), and that as $x \to \pm\frac{\pi}{4}$ we have $\frac{dy}{dx} \to \mp\infty$, thus the curve is vertical at its endpoints. The coordinates of the endpoints are

$$x = \pm\frac{\pi}{4} \quad \implies \quad y = \frac{1}{\sqrt{2}} + \sqrt{0} = \frac{1}{\sqrt{2}} \quad .$$

Our graph is as follows.



(ii) Now sketching this as a polar curve, we will have symmetry across $\theta = 0$, the graph will have a maximum at $\theta = 0$, $r = 2$. The endpoints of the curve will be $\theta = \pm\frac{\pi}{4}$, $r = \frac{1}{\sqrt{2}}$, where we will have $\frac{dr}{d\theta} \to \mp\infty$, thus the curve will be tangent to the radii $\theta = \pm\frac{\pi}{4}$. Our graph is as follows.



(iii) Substituting $\theta = \pm\frac{\pi}{4}$, we have

$$\begin{aligned} r^2 - \frac{2}{\sqrt{2}}r + \left(\pm\frac{1}{\sqrt{2}}\right)^2 &= 0 \\ r^2 - \frac{2}{\sqrt{2}}r + \left(\frac{1}{\sqrt{2}}\right)^2 &= 0 \\ \left(r - \frac{1}{\sqrt{2}}\right)^2 &= 0 \quad , \end{aligned}$$

hence at both $\theta = \pm\frac{\pi}{4}$, we have $r = \frac{1}{\sqrt{2}}$. Attempting to find where r is small, we substitute $r = 0$ to get

$$\sin^2\theta = 0 \quad \Longrightarrow \quad \theta = 0 \quad ,$$

since this is the only θ in the domain that gives $\sin\theta = 0$. Hence r is small only when θ is small, and we may proceed by Taylor expanding $\cos\theta$ and $\sin\theta$ for small θ:

$$r^2 - 2r + \theta^2 \approx 0 \quad ,$$

we also neglect the r^2 term, since this is necessarily smaller than the r term:

$$-2r + \theta^2 \approx 0 \quad \Longrightarrow \quad r \approx \frac{1}{2}\theta^2 \quad .$$

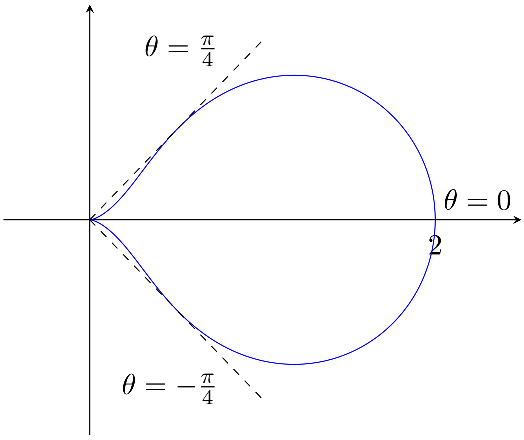
This shows that there is a cusp in the graph of C_2 at $\theta = 0$, $r = 0$. Solving the equation of C_2 for general r in terms of θ:

$$\begin{aligned} r^2 - 2r\cos\theta + \sin^2\theta &= 0 \\ r^2 - 2r\cos\theta + \cos^2\theta &= \cos^2\theta - \sin^2\theta \\ (r - \cos\theta)^2 &= \cos(2\theta) \\ r &= \cos\theta \pm \sqrt{\cos(2\theta)} \quad . \end{aligned}$$

Hence the curve C_2 has one branch given by C_1, and another branch given by

$$r = \cos\theta - \sqrt{\cos(2\theta)} \qquad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4} \quad .$$

By similarity with part (i), we see that this inner branch will also have endpoints at $\theta = \pm\frac{\pi}{4}$, $r = \frac{1}{\sqrt{2}}$, where the curve will be tangent to the radii $\theta = \pm\frac{\pi}{4}$; thus the two branches of C_2 join smoothly. Our graph of C_2 is as follows.



Using the formula for areas enclosed by radial curves, the required area is

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/4} (\cos\theta + \sqrt{\cos(2\theta)})^2 d\theta - \frac{1}{2}\int_0^{\pi/4} (\cos\theta - \sqrt{\cos(2\theta)})^2 d\theta \\ &= 2\int_0^{\pi/4} \cos\theta\sqrt{\cos(2\theta)} d\theta \quad . \end{aligned}$$

Making the change of variables $u = \sin\theta$ ($\cos(2\theta) = 1 - 2u^2$, $\cos\theta d\theta = du$):

$$A = 2\int_0^{\frac{1}{\sqrt{2}}} \sqrt{1 - 2u^2} du \quad .$$

Now making the change of variables $v = \sqrt{2}u$:

$$A = 2\int_0^1 \sqrt{1 - v^2} \cdot \frac{1}{\sqrt{2}} dv = \sqrt{2}\int_0^1 \sqrt{1 - v^2} dv \quad .$$

This last integral is the integral under a semicircular arc of radius 1: $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$, giving the area of a quarter-circle of radius 1, thus

$$A = \sqrt{2} \cdot \frac{\pi}{4} = \frac{\pi}{2\sqrt{2}} \quad ,$$

as required.

Question 7

(i) Differentiating the differential equation for y, we have

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y = \frac{du}{dx} \quad ,$$

now using the differential equation for u to substitute for $\frac{du}{dx}$

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y = h(x) - f(x)u \quad ,$$

and finally using the differential equation for y to substitute for u:

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y = h(x) - f(x)\left(\frac{dy}{dx} + g(x)y\right)$$

$$\implies \quad \frac{d^2y}{dx^2} + (g(x) + f(x))\frac{dy}{dx} + (g'(x) + f(x)g(x))y = h(x) \quad .$$

(ii) Comparing coefficients, we note that the coefficients of $\frac{d^2y}{dx^2}$ in (1) and (2) are both 1, so we don't need to worry about dividing through by any function of x, and we can read off that

$$g(x) + f(x) = 1 + 4x^{-1} \quad ,$$
$$g'(x) + f(x)g(x) = 2x^{-1} + 2x^{-2} \quad ,$$
$$\text{and} \quad h(x) = 4x + 12 \quad .$$

Combining the first two equations here, we have

$$g'(x) + (1 + 4x^{-1} - g(x))g(x) = 2x^{-1} + 2x^{-2}$$
$$g'(x) + (1 + 4x^{-1})g(x) - g(x)^2 = 2x^{-1} + 2x^{-2} \quad .$$

Substituting $g(x) = kx^n$ (noting $k \neq 0$, $n \neq 0$, since $g(x) \equiv$ constant is -by inspection- not a solution)

$$nkx^{n-1} + kx^n + 4kx^{n-1} - k^2x^{2n} = 2x^{-1} + 2x^{-2}$$
$$k(n+4)x^{n-1} + kx^n - k^2x^{2n} = 2x^{-1} + 2x^{-2} \quad .$$

We have three potentially distinct powers of x on the left-hand side, which must match up with the two powers of x on the right-hand side. First we consider $n = -4$ (in which case the left-most term vanishes). If $n = -4$ the powers of x on the left are $n = -4$ and $2n = -8$, which do not match x^{-1} and x^{-2}. Thus we must have that two terms on the left-hand side actually give the same power of x.

Since we cannot have $n = 2n$ (as noted above, $n = 0$ does not give a solution), and $n = n - 1$ cannot be satisfied for any finite n, we must have $2n = n - 1$; that is $n = -1$. This gives

$$3kx^{-2} + kx^{-1} - k^2x^{-2} = 2x^{-1} + 2x^{-2} \quad ,$$

from which we conclude that $k = 2$ (so that $3k - k^2 = 6 - 4 = 2$, as needed).

Now knowing $g(x)$, we use the equations from before to find

$$f(x) = 1 + 4x^{-1} - g(x) = 1 + 2x^{-1} \quad \text{and} \quad h(x) = 4x + 12 \quad .$$

We can now solve the differential equation for u, using an integrating factor:

$$\frac{du}{dx} + (1 + 2x^{-1})u = 4x + 12 \quad ,$$

and

$$\exp\left(\int^x \left(1 + 2t^{-1}\right) dt\right) = \exp\left(x + 2\ln(x)\right) = x^2 e^x \quad ,$$

gives

$$\begin{aligned} \frac{d}{dx}\left(x^2 e^x u\right) &= \left(4x^3 + 12x^2\right) e^x = \left(4x^3 + \frac{d}{dx}(4x^3)\right) e^x \\ x^2 e^x u &= 4x^3 e^x + c \\ u(x) &= 4x + Ax^{-2} e^{-x} \quad . \end{aligned}$$

Now substituting into the differential equation for y:

$$\frac{dy}{dx} + 2x^{-1} y = 4x + Ax^{-2} e^{-x} \quad ,$$

we use the fact that at $x = 1$, $y = 5$ and $\frac{dy}{dx} = -3$ to get

$$-3 + 2 \cdot 5 = 4 + Ae^{-1} \quad \Longrightarrow \quad A = 3e \quad .$$

Now we can solve for y (again using an integrating factor):

$$\frac{dy}{dx} + 2x^{-1} y = 4x + x^{-2} e^{3-x} \quad ,$$

and

$$\exp\left(\int^x 2t^{-1} dt\right) = \exp\left(2\ln(x)\right) = x^2 \quad ,$$

gives

$$\begin{aligned} \frac{d}{dx}\left(x^2 y\right) &= 4x^3 + e^{3-x} \\ x^2 y - 1^2 \cdot 5 &= x^4 - e^{3-x} - \left(1^4 - e^{3-1}\right) \\ x^2 y - 10 &= x^4 - e^{3-x} - 1 + e^2 \\ y &= x^2 + (9 + e^2) x^{-2} - x^{-2} e^{3-x} \quad . \end{aligned}$$

Question 8

(i) First off, we note that all terms u_k will be positive integers, since each term is equal to a previous term, or the sum of two previous terms and the basis case $u_1 = 1$ is a positive integer. From this we can deduce that for $k \geqslant 1$ we have

$$u_{k+1} > 0 \quad \Longrightarrow \quad u_{2k+1} > u_{2k} \quad ;$$

so every odd-subscript term after -and including- u_3 is strictly greater than the even-subscript term immediately before it. Then we may deduce that for $k \geqslant 1$

$$\begin{aligned} u_{2k+2} = u_{k+1} &= u_{2k+1} - u_k \\ \Longrightarrow \qquad u_{2k+2} &< u_{2k+1} \quad ; \end{aligned}$$

so every odd-subscript term after -and including- u_3 is strictly greater than the even-subscript term immediately *after* it. Hence for every pair of consecutive terms, except the first pair, the term with the odd subscript is larger than the term with the even subscript.

(ii) Suppose that u_{2k} has a factor $q > 1$ for some $k \geqslant 1$. Let $u_{2k} = qa$ where $a \in \mathbb{N}$. Since $u_{2k} = u_k = qa$ we have that u_k is also divisible by q. We now consider two cases:

- If u_{2k+1} is also divisible by q, let $u_{2k+1} = qb$ where $b \in \mathbb{N}$. Then

  $$u_{k+1} = u_{2k+1} - u_k = qb - qa = q(b-a) \quad ,$$

  so both u_k and u_{k+1} share the common factor q.
- If u_{2k-1} is also divisible by q, let $u_{2k-1} = qc$ where $c \in \mathbb{N}$. Then

  $$u_{k-1} = u_{2(k-1)+1} - u_{(k-1)+1} = u_{2k-1} - u_k = qc - qa = q(c-a) \quad ,$$

  so both u_k and u_{k-1} share the common factor q.

Thus if two consecutive terms share a common factor greater than one then there are two *earlier* consecutive terms that share the same common factor. By induction, if two consecutive terms share a common factor greater than one then the *first* two terms share that same common factor. Since the first two terms are $u_1 = 1$ and $u_2 = u_1 = 1$, this is a contradiction, and so all pairs of consecutive terms are co-prime.

(iii) Suppose $u_{2k} = m$ (which implies $u_k = m$) and $u_{2k+j} = m$ for some $k, j \geqslant 1$. Again we consider two cases.

- Suppose $u_{2k+1} = n$ and $u_{2k+j+1} = n$. By (i) we must have $m < n$, and so (also by (i)) we must have that j is even, let $j = 2i$. This then yields

  $$\begin{aligned} & u_{2k} = m \quad , \quad u_{2k+1} = n \\ \Longrightarrow \qquad & u_k = m \quad , \quad u_{k+1} = n - m \end{aligned}$$

but also

$$\begin{aligned} u_{2k+2i} = m \quad &, \quad u_{2k+2i+1} = n \\ \implies \quad u_{k+i} = m \quad &, \quad u_{k+i+1} = n - m \quad . \end{aligned}$$

- Suppose $u_{2k-1} = n$ and $u_{2k+j-1} = n$. Again by (i) we must have $m < n$ and j is even, again let $j = 2i$. This yields

$$\begin{aligned} u_{2k} = m \quad &, \quad u_{2k-1} = n \\ \implies \quad u_k = m \quad &, \quad u_{k-1} = n - m \end{aligned}$$

  but also

$$\begin{aligned} u_{2k+2i} = m \quad &, \quad u_{2k+2i-1} = n \\ \implies \quad u_{k+i} = m \quad &, \quad u_{k+i-1} = n - m \quad . \end{aligned}$$

So if the integer pair $(u_{2k}, u_{2k\pm1}) = (m, n)$ appears consecutively in the same order again later in the sequence, we can find an earlier example of an integer pair which also appears in the same order later in the sequence. By induction, if any integer pair appears consecutively more than once, then the first pair of terms must do so. But if a pair of terms after (u_1, u_2) is equal to $(1, 1)$ then this contradicts (i). Hence no integer pair (m, n) appears consecutively in the same order more than once in the sequence.

(iv) Suppose $a > b$ are two co-prime positive integers which do not occur consecutively in the sequence with b following a. First, if a and b are co-prime then b and $a - b$ must be co-prime (if some integer $c > 1$ divides b and $a - b$, then it must divide a). Trivially, the sum of $a - b$ and b is a, which is less than $a + b$ since b is a positive integer. Suppose that the pair $(a - b, b)$ *do* appear consecutively in the sequence with b following $a - b$. That is, suppose $u_k = a - b$ and $u_{k+1} = b$ for some $k \geqslant 1$. This yields $u_{2k+1} = a$ and $u_{2k+2} = u_{k+1} = b$, contradiction the original supposition.

Now suppose $a < b$ are two co-prime positive integers which do not occur consecutively in the sequence with b following a. Again, if a and b are co-prime then a and $b - a$ must be co-prime, and the sum of a and $b - a$ is b, which is less than $a + b$ since a is a positive integer. Suppose that the pair $(b - a, a)$ *do* appear consecutively in the sequence with $b - a$ following a. That is, suppose $u_k = a$ and $u_{k+1} = b - a$ for some $k \geqslant 1$. This yields $u_{2k} = u_k = a$ and $u_{2k+1} = b$, contradiction the original supposition.

(v) By the fact that all terms are positive integers, we have that the range of f is contained by the positive rational numbers. By (ii) we have that f is injective ($f(n_1) = f(n_2)$ if and only if $n_1 = n_2$). By combining the two results of (iv) and inducting, if there is any pair of co-prime integers (a, b) which do not appear consecutively in the sequence, then we can find a different pair of co-prime integers, whose sum is smaller, which do not appear consecutively in the sequence. Since this process cannot be repeated indefinitely (the sum cannot keep getting strictly smaller since it must be a positive integer), we must be able to find *every* pair of co-prime integers (a, b) appearing consecutively (in both orders) somewhere in the sequence. By (iii) each ordering appears exactly once. Therefore, for each rational number $p = a/b > 0$ (expressed in reduced form) there exists a unique k such that $u_k = a$ and $u_{k+1} = b$, and so $p = u_k/u_{k+1}$. Hence the range of f is exactly the positive rational numbers, and the uniqueness of the index k ensures f has a well-defined inverse.

Section B: Mechanics

Question 9

As with most mechanics questions, a diagram is useful.

(i) Let R be the normal reaction force on the rod from Π_1 and S be the normal reaction force on the rod from Π_2. Resolving forces horizontally and vertically, we have

$$\begin{aligned} R\sin\alpha &= S\sin\beta \quad , \\ \text{and} \qquad R\cos\alpha + S\cos\beta &= mg \quad . \end{aligned}$$

Let the length of the rod be $2l$, then taking moments about Q we find

$$\begin{aligned} mgl\cos\theta &= 2Rl\cos(\alpha-\theta) \\ \Longrightarrow \qquad mg\cos\theta &= 2R\cos(\alpha-\theta) \quad . \end{aligned}$$

Eliminating the weight mg, we have

$$\begin{aligned} 2R\cos(\alpha-\theta) &= (R\cos\alpha + S\cos\beta)\cos\theta \\ 2R\cos\alpha\cos\theta + 2R\sin\alpha\sin\theta &= R\cos\alpha\cos\theta + S\cos\beta\cos\theta \\ 2R\sin\alpha\sin\theta &= S\cos\beta\cos\theta - R\cos\alpha\cos\theta \\ 2R\sin\alpha\tan\theta &= (S\sin\beta)\cot\beta - R\cos\alpha \\ 2R\sin\alpha\tan\theta &= (R\sin\alpha)\cot\beta - R\cos\alpha \qquad (S\sin\beta = R\sin\alpha) \\ 2\tan\theta &= \cot\beta - \cot\alpha \quad . \end{aligned}$$

(ii) Keeping the same labels as before, there is now a frictional force $F = \mu S$ directed down Π_2. Now resolving horizontally and vertically, we have

$$\begin{aligned} R\sin\alpha &= S\sin\beta + \mu S\cos\beta \quad , \\ \text{and} \qquad R\cos\alpha + S\cos\beta &= mg + \mu S\sin\beta \quad , \end{aligned}$$

and taking moments about Q we have

$$mg\cos\phi = 2R\cos(\alpha-\phi) \quad .$$

Rearranging yields

$$mg = R\cos\alpha + S(\cos\beta - \mu\sin\beta) \qquad \text{and} \qquad S = \frac{R\sin\alpha}{\sin\beta + \mu\cos\beta} \quad .$$

Substitution then gives

$$2R\cos(\alpha-\phi) = \left(R\cos\alpha + \frac{R\sin\alpha}{\sin\beta+\mu\cos\beta}(\cos\beta - \mu\sin\beta)\right)\cos\phi \quad ,$$

which we re-arrange and simplify:

$$
\begin{aligned}
2\cos\alpha\cos\phi + 2\sin\alpha\sin\phi &= \left(\cos\alpha + \frac{\sin\alpha}{\sin\beta + \mu\cos\beta}(\cos\beta - \mu\sin\beta)\right)\cos\phi \\
\cos\alpha\cos\phi + 2\sin\alpha\sin\phi &= \frac{\sin\alpha}{\sin\beta + \mu\cos\beta}(\cos\beta - \mu\sin\beta)\cos\phi \\
\cot\alpha + 2\tan\phi &= \frac{\cos\beta - \mu\sin\beta}{\sin\beta + \mu\cos\beta} \\
(\cot\beta - 2\tan\theta) + 2\tan\phi &= \frac{1 - \mu\tan\beta}{\tan\beta + \mu} \\
2(\tan\theta - \tan\phi) &= \frac{\mu\tan\beta - 1}{\mu + \tan\beta} + \cot\beta \\
&= \frac{\mu\tan\beta - 1 + \mu\cot\beta + 1}{\mu + \tan\beta} \\
&= \frac{\mu}{\mu + \tan\beta}(\tan\beta + \cot\beta) \\
&= \frac{\mu}{\mu + \tan\beta}\frac{\sin^2\beta + \cos^2\beta}{\sin\beta\cos\beta} \\
\tan\theta - \tan\phi &= \frac{\mu}{(\mu + \tan\beta)2\sin\beta\cos\beta} \\
\tan\theta - \tan\phi &= \frac{\mu}{(\mu + \tan\beta)\sin(2\beta)} \quad ,
\end{aligned}
$$

where we have used the substitution $2\tan\theta = \cot\beta - \cot\alpha$.

Question 10

Again, a diagram is useful.

Let the extension of the spring at the equilibrium position be d, then resolving the vertical forces we have

$$mg = \frac{d}{a}kmg \quad ,$$

hence $d = a/k$. Let the extension of the spring when the particle is released be $d + z(t)$, then we get the equation of motion:

$$\begin{aligned} m\frac{d^2z}{dt^2} &= mg - \frac{d+z}{a}kmg = mg - mg - \frac{kmg}{a}z \\ \implies \qquad \frac{d^2z}{dt^2} &= -\frac{kg}{a}z \quad . \end{aligned}$$

This is a simple harmonic oscillator with frequency Ω given by $\Omega^2 = \frac{kg}{a}$. Thus $kg = a\Omega^2$.

Let $y(t)$ be the displacement of the platform below the centre of its oscillation. Then $y(t) = b - x(t)$ and $\frac{d^2y}{dt^2} = -\omega^2 y = -\omega^2(b - x)$. When the particle is in contact with the platform, we get the equation of motion

$$m\frac{d^2y}{dt^2} = mg - R - \frac{h-a-x}{a}kmg \quad ,$$

where R is the upward force on the particle from the platform. Thus

$$-m\omega^2(b-x) = mg - R - (h-a-x)m\Omega^2 \quad .$$

Rearranging, we get

$$R = mg + (a + x - h)m\Omega^2 + m\omega^2(b - x) \quad ,$$

as required. For the particle to remain in contact with the platform through its motion, we must have $R \geqslant 0$ for $0 \leqslant x \leqslant 2b$. We have

$$R = mg + (a-h)m\Omega^2 + m\omega^2 b + m(\Omega^2 - \omega^2)x \quad ,$$

hence if $\Omega > \omega$, then the minimum occurs at $x = 0$, and the inequality is satisfied only if

$$\begin{aligned} & mg + (a-h)m\Omega^2 + m\omega^2 b \geqslant 0 \\ \iff \qquad & g + a\Omega^2 + \omega^2 b \geqslant h\Omega^2 \\ \iff \qquad & h \leqslant \frac{g}{\Omega^2} + a + \frac{\omega^2}{\Omega^2}b \\ & h \leqslant \frac{a}{k} + a + \frac{\omega^2}{\Omega^2}b \\ & h \leqslant a\left(1 + \frac{1}{k}\right) + \frac{\omega^2}{\Omega^2}b \quad . \end{aligned}$$

Alternatively, if $\omega > \Omega$ the minimum occurs at $x = 2b$, and the inequality is satisfied only if

$$\begin{aligned} mg + (a-h)m\Omega^2 + m\omega^2 b + 2m(\Omega^2 - \omega^2)b &\geqslant 0 \\ \iff \quad g + a\Omega^2 + (2\Omega^2 - \omega^2)b &\geqslant h\Omega^2 \\ \iff \quad h &\leqslant \frac{g}{\Omega^2} + a + \frac{2\Omega^2 - \omega^2}{\Omega^2} b \\ h &\leqslant \frac{a}{k} + a + \left(2 - \frac{\omega^2}{\Omega^2}\right) b \\ h &\leqslant a\left(1 + \frac{1}{k}\right) - \frac{\omega^2}{\Omega^2} b + 2b \quad . \end{aligned}$$

If $\omega < \Omega$ we need

$$\begin{aligned} h &\leqslant a\left(1 + \frac{1}{k}\right) + \frac{\omega^2}{\Omega^2} b \\ \implies \quad h &\leqslant a\left(1 + \frac{1}{k}\right) + b \quad . \end{aligned}$$

If $\omega > \Omega$ we need

$$\begin{aligned} h &\leqslant a\left(1 + \frac{1}{k}\right) + \left(2 - \frac{\omega^2}{\Omega^2}\right) b \\ \implies \quad h &\leqslant a\left(1 + \frac{1}{k}\right) + b \quad . \end{aligned}$$

If $\omega = \Omega$, then

$$\begin{aligned} R = mg + (a-h)m\Omega^2 + m\Omega^2 b &\geqslant 0 \\ \iff \quad h &\leqslant \frac{g}{\Omega^2} + a + b \\ \iff \quad h &\leqslant a\left(1 + \frac{1}{k}\right) + b \quad . \end{aligned}$$

Thus a necessary condition for the particle to remain in contact with the plate is

$$h \leqslant a\left(1 + \frac{1}{k}\right) + b \quad .$$

Section C: Probability and Statistics

Question 11

Note that $\mathbb{P}\{X \leqslant x\} = \frac{x-a}{b-a}$ for $x \in [a,b]$.

(i) For $y \in [a,b]$, we have

$$\begin{aligned}
\mathbb{P}\{Y \leqslant y\} &= \mathbb{P}\{f(X) \leqslant y\} \\
&= \mathbb{P}\{f^{-1}(X) \leqslant y\} && \text{(since } f^{-1} \text{ exists, and } f = f^{-1}\text{)} \\
&= \mathbb{P}\{f(f^{-1}(X)) \geqslant f(y)\} && \text{(since } f \text{ is strictly decreasing)} \\
&= \mathbb{P}\{X \geqslant f(y)\} \\
&= 1 - \mathbb{P}\{X \leqslant f(y)\} = 1 - \frac{f(y)-a}{b-a} \\
&= \frac{b-f(y)}{b-a} \quad .
\end{aligned}$$

Let $g(y)$, $y \in [a,b]$ be the probability density function of Y. Differentiating the above result:

$$g(y) = \frac{-f'(y)}{b-a} \quad .$$

Taking the expectation of Y^2, we have

$$\mathbb{E}(Y^2) = \int_a^b y^2 g(y)dy = -\frac{1}{b-a}\int_a^b y^2 f'(y)dy \quad ,$$

integrating by parts:

$$\begin{aligned}
\mathbb{E}(Y^2) &= -\frac{1}{b-a}\left[y^2 f(y)\right]_a^b + \frac{2}{b-a}\int_a^b yf(y)dy \\
&= -\frac{b^2 f(b) - a^2 f(a)}{b-a} + \int_a^b \frac{2yf(y)}{b-a}dy \\
&= -\frac{ab^2 - a^2 b}{b-a} + \int_a^b \frac{2xf(x)}{b-a}dx \\
&= -ab\frac{b-a}{b-a} + \int_a^b \frac{2xf(x)}{b-a}dx = -ab + \int_a^b \frac{2xf(x)}{b-a}dx \quad .
\end{aligned}$$

(ii) Note that c is strictly positive and finite (since a and b are strictly positive and finite), and that $\frac{1}{c} > \frac{1}{a}$, and so $c < a$. Let $h(x) = \left(\frac{1}{c} - \frac{1}{x}\right)^{-1}$ (we have $Z = h(X)$), which is now necessarily well-defined for all $x \in [a,b]$. We see that

$$h(a) = \left(\frac{1}{c} - \frac{1}{a}\right)^{-1} = \left(\frac{1}{b}\right)^{-1} = b \quad ,$$

and similarly

$$h(b) = \left(\frac{1}{c} - \frac{1}{b}\right)^{-1} = \left(\frac{1}{a}\right)^{-1} = a \quad .$$

We also find that

$$h'(x) = -\frac{1}{x^2}\left(\frac{1}{c} - \frac{1}{x}\right)^{-2} = \frac{-c^2}{(x-c)^2} < 0 \quad ,$$

hence $h(x)$ is strictly decreasing for $x \in [a, b]$. Lastly, we have

$$\left(\frac{1}{c} - \frac{1}{h(x)}\right)^{-1} = \left(\frac{1}{c} - \left(\frac{1}{c} - \frac{1}{x}\right)\right)^{-1} = x \quad ,$$

hence $h(h(x)) = x$ for all $x \in [a, b]$, and so $h^{-1}(x) = h(x)$ for all $x \in [a, b]$.

By the above results, we know the probability density function of Z is

$$\frac{-h'(z)}{b-a} = \frac{c^2}{b-a}\frac{1}{(z-c)^2} \quad , \quad z \in [a, b] \quad .$$

We first find $\mathbb{E}(Z)$:

$$\begin{aligned}
\mathbb{E}(Z) &= \int_a^b \frac{c^2}{b-a}\frac{z}{(z-c)^2} dz = \frac{c^2}{b-a}\int_a^b \left(\frac{1}{z-c} + \frac{c}{(z-c)^2}\right) dz \\
&= \frac{c^2}{b-a}\left[\ln(z-c)\right]_a^b + c\int_a^b \frac{c^2}{b-a}\frac{1}{(z-c)^2} dz \\
&= \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right) + c \quad ,
\end{aligned}$$

since the last integral here is the integral over $z \in [a, b]$ of the probability density function of Z. Let $l = \ln\left(\frac{b-c}{a-c}\right)$, and note that $b - c > a - c > 0$, hence $l > 0$.

Using the result of (i) then

$$
\begin{aligned}
\mathbb{E}(Z^2) &= -ab + \frac{2}{b-a}\int_a^b xh(x)dx = -ab + \frac{2}{b-a}\int_a^b \frac{x}{\left(\frac{1}{c}-\frac{1}{x}\right)}dx \\
&= -ab + \frac{2}{b-a}\int_a^b \frac{cx^2}{x-c}dx \\
&= -ab + \frac{2c}{b-a}\int_a^b \left(\frac{x^2-c^2}{x-c} + \frac{c^2}{x-c}\right)dx \\
&= -ab + \frac{2c}{b-a}\int_a^b \left(x+c+\frac{c^2}{x-c}\right)dx \\
&= -ab + \frac{c}{b-a}\left[x^2+2cx+2c^2\ln(x-c)\right]_a^b \\
&= -ab + \frac{c(b^2-a^2)}{(b-a)} + \frac{2c^2(b-a)}{b-a} + \frac{2c^3}{b-a}l \\
&= -ab + c(b+a) + 2c^2 + \frac{2c^3}{b-a}l = -ab + \frac{ab}{a+b}(b+a) + 2c^2 + \frac{2c^3}{b-a}l \\
&= 2c^2 + \frac{2c^3}{b-a}l \quad .
\end{aligned}
$$

Since Z is not a constant, we know that $\mathrm{Var}(Z) > 0$. Our expressions above give

$$
\begin{aligned}
\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 &> 0 \\
2c^2 + \frac{2c^3}{b-a}l - \left(\frac{c^2}{b-a}l + c\right)^2 &> 0 \\
\iff \qquad \frac{2}{c^2} + \frac{2}{c(b-a)}l - \left(\frac{1}{b-a}l + \frac{1}{c}\right)^2 &> 0 \\
\iff \qquad \frac{2(b-a)^2}{c^2} + \frac{2(b-a)}{c}l - \left(l + \frac{b-a}{c}\right)^2 &> 0
\end{aligned}
$$

since $c^4 > 0$ and $(b-a)^2 > 0$. Hence we have

$$
\begin{aligned}
\left(l + \frac{b-a}{c}\right)^2 &< \frac{2(b-a)^2}{c^2} + \frac{2(b-a)}{c}l \\
\iff \qquad l^2 + \frac{2(b-a)}{c}l + \frac{(b-a)^2}{c^2} &< \frac{2(b-a)^2}{c^2} + \frac{2(b-a)}{c}l \\
\iff \qquad l^2 &< \frac{(b-a)^2}{c^2} \quad .
\end{aligned}
$$

Since $l > 0$, $b-a > 0$, and $c > 0$, we can square root both sides without affecting the inequality, to at last get

$$
l < \frac{b-a}{c} \quad .
$$

Question 12

We start by noting that X and Y are geometric random variables:

$$\begin{aligned} \mathbb{P}\{X=x\} &= q^{x-1}p \quad , \quad x=1,2,3,\ldots \quad , \\ \text{and} \quad \mathbb{P}\{Y=y\} &= q^{y-1}p \quad , \quad y=1,2,3,\ldots \quad , \end{aligned}$$

and that X and Y and independent.

(i) We have for $s=2,3,4,\ldots$

$$\begin{aligned} \mathbb{P}\{S=s\} &= \sum_{r=1}^{s-1} \mathbb{P}\{X=r\}\mathbb{P}\{Y=s-r\} \\ &= \sum_{r=1}^{s-1} q^{r-1}p \cdot q^{s-r-1}p \\ &= \sum_{r=1}^{s-1} p^2 q^{s-2} \\ &= (s-1)p^2q^{s-2} \quad . \end{aligned}$$

And for $t=1,2,3,\ldots$

$$\begin{aligned} \mathbb{P}\{T=t\} &= \mathbb{P}\{X=t\}\mathbb{P}\{Y\leqslant t\} + \mathbb{P}\{X\leqslant t\}\mathbb{P}\{Y=t\} - \mathbb{P}\{X=t\}\mathbb{P}\{Y=t\} \\ &= q^{t-1}p\sum_{r=1}^{t}\mathbb{P}\{Y=r\} + q^{t-1}p\sum_{r=1}^{t}\mathbb{P}\{X=r\} - q^{t-1}p\cdot q^{t-1}p \\ &= q^{t-1}p\sum_{r=1}^{t} q^{r-1}p + q^{t-1}p\sum_{r=1}^{t} q^{r-1}p - p^2q^{2t-2} \\ &= 2p^2q^{t-2}\sum_{r=1}^{t} q^r - p^2q^{2t-2} \\ &= 2p^2q^{t-2}\frac{q-q^{t+1}}{1-q} - p^2q^{2t-2} \\ &= 2p^2q^{t-2}\frac{q-q^{t+1}}{p} - p(1-q)q^{2t-2} \\ &= pq^{t-2}\left(2(q-q^{t+1}) - (1-q)q^t\right) \\ &= pq^{t-1}\left(2-2q^t-q^{t-1}+q^t\right) \\ &= pq^{t-1}\left(2-q^{t-1}-q^t\right) \quad , \end{aligned}$$

as required.

(ii) For $u = 1, 2, 3, ...$, we have

$$\begin{aligned}
\mathbb{P}\{U = u\} &= \sum_{r=1}^{\infty} \mathbb{P}\{X = r\}\mathbb{P}\{Y = u + r\} + \sum_{r=1}^{\infty} \mathbb{P}\{Y = r\}\mathbb{P}\{X = u + r\} \\
&= 2\sum_{r=1}^{\infty} q^{r-1}p \cdot q^{u+r-1}p \\
&= 2p^2 q^{u-2} \sum_{r=1}^{\infty} q^{2r} \\
&= 2p^2 q^{u-2} \frac{q^2}{1-q^2} \\
&= \frac{2p^2 q^u}{(1-q)(1+q)} \\
&= \frac{2pq^u}{1+q} \quad ,
\end{aligned}$$

while for $u = 0$, we have

$$\begin{aligned}
\mathbb{P}\{U = 0\} &= \sum_{r=1}^{\infty} \mathbb{P}\{X = r\}\mathbb{P}\{Y = r\} = \sum_{r=1}^{\infty} q^{r-1}p \cdot q^{r-1}p \\
&= p^2 q^{-2} \sum_{r=1}^{\infty} q^{2r} \\
&= p^2 q^{-2} \frac{q^2}{1-q^2} \\
&= \frac{p^2}{(1-q)(1+q)} \\
&= \frac{p}{1+q} \quad .
\end{aligned}$$

For $w = 1, 2, 3, ...$, we have

$$\begin{aligned}
\mathbb{P}\{W = w\} &= \sum_{r=0}^{\infty} \mathbb{P}\{X = w\}\mathbb{P}\{Y = w + r\} + \sum_{r=0}^{\infty} \mathbb{P}\{Y = w\}\mathbb{P}\{X = w + r\} \\
&\qquad - \mathbb{P}\{X = w\}\mathbb{P}\{Y = w\} \\
&= 2\sum_{r=0}^{\infty} q^{w-1}p \cdot q^{w+r-1}p - q^{w-1}p \cdot q^{w-1}p \\
&= 2p^2 q^{2w-2} \sum_{r=0}^{\infty} \cdot q^r - p^2 q^{2w-2} = 2p^2 q^{2w-2} \frac{1}{1-q} - p^2 q^{2w-2} \\
&= 2pq^{2w-2} - p^2 q^{2w-2} \\
&= pq^{2w-2}(2-p) \\
&= pq^{2w-2}(1+q) \quad .
\end{aligned}$$

(iii) If $S = 2$, then we must have $X = Y = 1$ giving $T = 1$, and so

$$\mathbb{P}\{S = 2 \text{ and } T = 3\} = 0 \quad ,$$

while

$$\begin{aligned}\mathbb{P}\{S = 2\} \cdot \mathbb{P}\{T = 3\} &= (2-1)p^2q^{2-2} \cdot pq^{3-1}\left(2 - q^{3-1} - q^3\right) \\ &= p^3q^2\left(2 - q^2 - q^3\right) \\ &\neq 0 \quad ,\end{aligned}$$

(since for any $0 < q < 1$, we know $q^2 + q^3 < q + q < 2$). That is

$$\mathbb{P}\{S = 2 \text{ and } T = 3\} \neq \mathbb{P}\{S = 2\} \cdot \mathbb{P}\{T = 3\} \quad .$$

This shows that S and T are not independent.

(iv) For any $w = 1, 2, 3, ...$, we have

$$\begin{aligned}\mathbb{P}\{U = 0 \text{ and } W = w\} &= \mathbb{P}\{X = w \text{ and } Y = w\} = \mathbb{P}\{X = w\} \cdot \mathbb{P}\{Y = w\} \\ &= q^{w-1}p \cdot q^{w-1}p \\ &= p^2q^{2w-2} \\ &= \frac{p}{1+q} \cdot pq^{2w-2}(1+q) \\ &= \mathbb{P}\{U = 0\} \cdot \mathbb{P}\{W = w\} \quad ,\end{aligned}$$

and for any $u = 1, 2, 3, ...$ and $w = 1, 2, 3, ...$, we have

$$\begin{aligned}\mathbb{P}\{U = u \text{ and } W = w\} &= \mathbb{P}\{X = w \text{ and } Y = w + u\} \\ &\qquad + \mathbb{P}\{Y = w \text{ and } X = w + u\} \\ &= 2q^{w-1}p \cdot q^{w+u-1}p \\ &= 2p^2q^{2w-2+u} \\ &= \frac{2pq^u}{1+q} \cdot pq^{2w-2}(1+q) \\ &= \mathbb{P}\{U = u\} \cdot \mathbb{P}\{W = w\} \quad ,\end{aligned}$$

hence U and W are independent.

Note that S can take any given integer value $s \geqslant 2$ with non-zero probability,
T can take any given integer value $t \geqslant 1$ with non-zero probability,
U can take any given integer value $u \geqslant 0$ with non-zero probability,
and W can take any given integer value $w \geqslant 1$ with non-zero probability.

As noted above, the result in (iii) implies S and T are not independent.

Similar to in (iii) we note that we cannot have $S = 2$ and $U = 1$, since $S = 2$ requires $X = Y = 1$, giving $U = 0$. Hence S and U are not independent.

Likewise we cannot have $S = 2$ and $W = 2$, since $S = 2$ requires $X = Y = 1$, giving $W = 1$. Hence S and W are not independent.

Likewise we cannot have $T = 1$ and $U = 1$, since $T = 1$ requires $X = Y = 1$, giving $U = 0$. Hence T and U are not independent.

Likewise we cannot have $T = 1$ and $W = 2$, since $T = 1$ requires $X = Y = 1$, giving $W = 1$. Hence T and W are not independent.