STEP 2019 Solutions

Contents

Author's notes:

1. *At time of writing, I am not affiliated with Cambridge Assessment Admissions Testing. I did an undergraduate maths degree at Cambridge, so I sat the STEP II and III papers as an A-level student (in 2015), and I have also been one of a team of markers for the STEP exams (in 2019 and 2020). Any opinions given here are entirely my own, based on my own experiences of STEP.*

2. *These 'solutions' are not intended to be used as any sort of mark scheme. In terms of method, often there will be more than one correct way to answer a STEP question, and it is certainly not the case that the answers presented here are the only correct approaches to these questions. The worked solutions here were typed up after attempting the questions myself, and I have checked them against the official mark schemes published online. However, there is no guarantee that the solutions typed up here would achieve full marks. In particular, I have not provided diagrams for all questions due to the difficulties of typesetting them neatly. Many questions may ask the student to draw a diagram, and in these instances marks are often awarded for this. Another point of consideration is explanation: sometimes marks are awarded for explicitly justifying an assumption used. I have tried to justify these as I think necessary, but there is no guarantee that these solutions justify all assumptions to the standards of the mark schemes.*

3. *If you are preparing to sit the STEP exams, I hope these can be of some help.*

STEP I

Section A: Pure Mathematics

Question 1

The equation of the line is

$$y - k = -\tan\theta(x-1) \quad ,$$

hence the x-coordinate of the point X is

$$-\tan\theta(x_X - 1) = -k \quad \Longrightarrow \quad x_X = 1 + k\cot\theta \quad ,$$

and the y-coordinate of the point Y is

$$y_Y - k = \tan\theta \quad \Longrightarrow \quad y_Y = k + \tan\theta \quad ,$$

that is, the coordinates of X and Y are

$$X: \ (1 + k\cot\theta, 0) \quad , \quad Y: \ (0, k + \tan\theta) \quad .$$

(i) The area of the triangle is

$$\begin{aligned} A &= \frac{1}{2}x_X y_Y = \frac{1}{2}(1 + k\cot\theta)(k + \tan\theta) \\ &= \frac{1}{2}(k + \tan\theta + k^2\cot\theta + k) \\ &= \frac{1}{2}k^2\cot\theta + \frac{1}{2}\tan\theta + k \quad . \end{aligned}$$

To find the minimum as θ varies, we differentiate with respect to θ:

$$\frac{dA}{d\theta} = -\frac{1}{2}k^2\csc^2\theta + \frac{1}{2}\sec^2\theta \quad .$$

Thus the minimum occurs when

$$\begin{aligned} k^2\csc^2\theta &= \sec^2\theta \\ k^2 &= \tan^2\theta \\ \Longrightarrow \quad \tan\theta &= k \quad , \end{aligned}$$

where we take the positive root since $0 < \theta < \frac{\pi}{2}$ and $k > 0$. This gives the minimum area

$$A_{\text{min}} = \frac{1}{2}k^2 \cdot \frac{1}{k} + \frac{1}{2}k + k = 2k \quad .$$

(ii) The length of the hypotenuse XY is

$$\frac{y_Y}{\sin\theta} = \frac{k}{\sin\theta} + \frac{1}{\cos\theta} = k\csc\theta + \sec\theta \quad ,$$

giving the total perimeter

$$\begin{aligned}L &= x_X + y_Y + k\csc\theta + \sec\theta = 1 + k\cot\theta + \tan\theta + k + k\csc\theta + \sec\theta \\ &= 1 + \tan\theta + \sec\theta + k(1 + \cot\theta + \csc\theta) \quad .\end{aligned}$$

To find the minimum, we again differentiate with respect to θ:

$$\frac{dL}{d\theta} = \sec\theta + \sec\theta\tan\theta - k\csc^2\theta - k\csc\theta\cot\theta \quad ,$$

hence at the minimum we have $\theta = \alpha$, where α satisfies

$$\begin{aligned}\sec\alpha + \sec\alpha\tan\alpha &= k(\csc^2\alpha + \csc\alpha\cot\alpha) \\ \sin^2\alpha + \sin^3\alpha &= k(\cos^2\alpha + \cos^3\alpha) \\ \frac{\sin^2\alpha(1+\sin\alpha)}{\cos^2\alpha(1+\cos\alpha)} &= k \\ \frac{(1-\cos^2\alpha)(1+\sin\alpha)}{(1-\sin^2\alpha)(1+\cos\alpha)} &= k \\ \frac{(1-\cos\alpha)(1+\cos\alpha)(1+\sin\alpha)}{(1-\sin\alpha)(1+\sin\alpha)(1+\cos\alpha)} &= k \\ \frac{1-\cos\alpha}{1-\sin\alpha} &= k \quad .\end{aligned}$$

Substituting this into the expression for L, the minimum perimeter is

$$\begin{aligned}L_{\min} &= 1 + \tan\alpha + \sec\alpha + \frac{1-\cos\alpha}{1-\sin\alpha}(1 + \cot\alpha + \csc\alpha) \\ (1-\sin\alpha)L_{\min} &= (1-\sin\alpha)(1+\tan\alpha+\sec\alpha) + (1-\cos\alpha)(1+\cot\alpha+\csc\alpha) \\ &= 1 - \sin\alpha + \tan\alpha - \sin\alpha\tan\alpha + \sec\alpha - \tan\alpha \\ &\qquad + 1 - \cos\alpha + \cot\alpha - \cos\alpha\cot\alpha + \csc\alpha - \cot\alpha \\ &= 2 - \sin\alpha - \frac{\sin^2\alpha}{\cos\alpha} + \frac{1}{\cos\alpha} - \cos\alpha - \frac{\cos^2\alpha}{\sin\alpha} + \frac{1}{\sin\alpha} \\ &= 2 - \sin\alpha + \frac{\cos^2\alpha}{\cos\alpha} - \cos\alpha + \frac{\sin^2\alpha}{\sin\alpha} \\ &= 2 \\ L_{\min} &= \frac{2}{1-\sin\alpha} \quad .\end{aligned}$$

Question 2

Differentiating, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2}{6t} = t \quad .$$

At the point $(x, y) = (3p^2, 2p^3)$, the y-coordinate gives $2t^3 = 2p^3$, thus at this point $t = p$. Hence the equation of the required tangent is

$$\begin{aligned} y - 2p^3 &= p(x - 3p^2) \\ y &= 2p^3 + px - 3p^3 \\ y &= px - p^3 \quad . \end{aligned}$$

Similarly, the equation of the tangent at the point $(x, y) = (3q^2, 2q^3)$ is

$$y = qx - q^3 \quad .$$

At the intersection of these tangents we have

$$\begin{aligned} px - p^3 &= qx - q^3 \\ (p - q)x &= p^3 - q^3 \\ &= (p - q)(p^2 + pq + q^2) \\ x &= p^2 + pq + q^2 \quad . \end{aligned}$$

Substituting this into either tangent equation, the y-coordinate of the intersection is

$$y = p(p^2 + pq + q^2) - p^3 = p^2q + pq^2 = pq(p + q) \quad .$$

If the tangents are perpendicular, then their gradients multiply to -1:

$$\begin{aligned} pq = -1 \quad \implies \quad (p^2 + pq + q^2, pq(p + q)) &= ((p + q)^2 - pq, pq(p + q)) \\ &= (u^2 - (-1), -u) \\ &= (u^2 + 1, -u) \quad , \end{aligned}$$

where $u = p + q$. To find the intersection of C and $\tilde{C}$ we equate the parametric expressions for the x-coordinates and the y-coordinates:

$$\begin{aligned} 3t^2 = u^2 + 1 \text{ and } 2t^3 = -u \quad \implies \quad 3t^2 &= (-2t^3)^2 + 1 \\ 4t^6 - 3t^2 + 1 &= 0 \\ (t^2 + 1)(4t^4 - 4t^2 + 1) &= 0 \\ (t^2 + 1)(2t^2 - 1)^2 &= 0 \quad , \end{aligned}$$

we neglect $t^2 = -1$, since t is real, so we have $t^2 = \frac{1}{2}$.

Thus at the points of intersection $t = \pm\frac{1}{\sqrt{2}}$. Hence there are two points of intersection, with coordinates

$$(x, y) = \left(\frac{3}{2}, \pm\frac{1}{\sqrt{2}}\right) \quad .$$

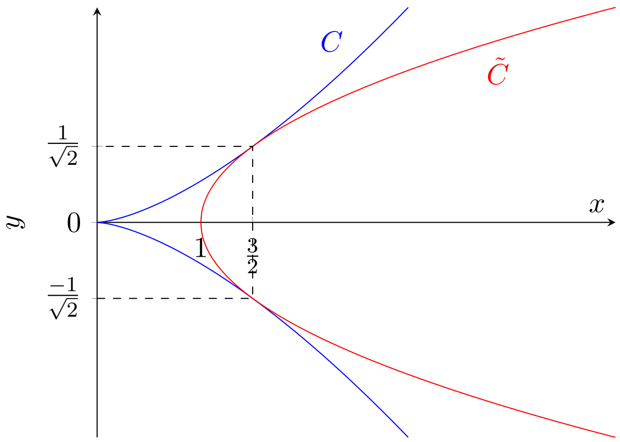
We can find the curve C in non-parametric form:

$$y^2 = 4t^6 \quad \text{and} \quad x^3 = 27t^6 \qquad \Longrightarrow \qquad y^2 = \frac{4}{27}x^3 \quad ,$$

which is a bi-cubic with $x \geqslant 0$, and a cusp at the origin. The curve $\tilde{C}$ is

$$x = (-y)^2 + 1 = y^2 + 1 \quad ,$$

a parabola with the vertex at $(1, 0)$. Sketching both curves on the same axes, we have the following picture.



Question 3

Following the given hint, we compute

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx &= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - \tan x \sec x)dx \\ &= [\tan x - \sec x]_0^{\frac{\pi}{4}} \\ &= (1-\sqrt{2}) - (0-1) \\ &= 2-\sqrt{2} \quad .\end{aligned}$$

By a similar method we also evaluate

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{1+\sec x}dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{1+\cos x}dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x - \cos^2 x}{1-\cos^2 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x - 1 + \sin^2 x}{\sin^2 x}dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cot x \csc x - \csc^2 x + 1)dx \\ &= [-\csc x + \cot x + x]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \left(-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{\pi}{3}\right) - \left(-\sqrt{2} + 1 + \frac{\pi}{4}\right) \\ &= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} + \sqrt{2} - 1 \quad .\end{aligned}$$

Finally we also compute

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{1}{(1+\sin x)^2}dx &= \int_0^{\frac{\pi}{3}} \frac{(1-\sin x)^2}{(1-\sin^2 x)^2}dx \\ &= \int_0^{\frac{\pi}{3}} \frac{1-2\sin x + \sin^2 x}{\cos^4 x}dx \\ &= \int_0^{\frac{\pi}{3}} \frac{2-2\sin x - \cos^2 x}{\cos^4 x}dx \\ &= \int_0^{\frac{\pi}{3}} \left(2(\sec x)^4 - 2\sin x(\cos x)^{-4} - \sec^2 x\right) dx \\ &= 2\int_0^{\frac{\pi}{3}} (\sec x)^4 dx - \left[\frac{2}{3}(\cos x)^{-3} + \tan x\right]_0^{\frac{\pi}{3}} \quad .\end{aligned}$$

We have

$$
\begin{aligned}
\int (\sec x)^4 dx = \int \sec^2 x \sec^2 x dx &= \tan x \sec^2 x - 2\int \sec^2 x \tan^2 x dx \\
&= \tan x \sec^2 x - 2\int \sec^2 x (\sec^2 x - 1) dx \\
&= \tan x \sec^2 x - 2\int (\sec x)^4 dx + 2\int \sec^2 x dx \\
&= \tan x \sec^2 x + 2\tan x - 2\int (\sec x)^4 dx
\end{aligned}
$$

$$
\begin{aligned}
\implies \quad 3\int (\sec x)^4 dx &= \tan x \sec^2 x + 2\tan x + c \\
\int (\sec x)^4 dx &= \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x + c \quad ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\int_0^{\frac{\pi}{3}} \frac{1}{(1+\sin x)^2} dx &= \left[\frac{2}{3}\tan x \sec^2 x + \frac{4}{3}\tan x - \frac{2}{3}(\cos x)^{-3} - \tan x\right]_0^{\frac{\pi}{3}} \\
&= \left(\frac{2}{3}\sqrt{3}\cdot 4 + \frac{4}{3}\sqrt{3} - \frac{2}{3}\cdot 8 - \sqrt{3}\right) - \left(0 + 0 - \frac{2}{3} - 0\right) \\
&= \left(\frac{8}{3} + \frac{4}{3} - 1\right)\sqrt{3} - \frac{16}{3} + \frac{2}{3} \\
&= 3\sqrt{3} - \frac{14}{3} \quad .
\end{aligned}
$$

Question 4

(i) Squaring both sides, we have

$$3 + 2\sqrt{2} = m^2 + 2n^2 + 2mn\sqrt{2} \quad ,$$

comparing integer components and coefficients of $\sqrt{2}$ then, we must have

$$m^2 + 2n^2 = 3 \quad \text{and} \quad 2mn = 2 \quad ;$$

since m and n are integers, this requires $(m,n) = (1,1)$ or $(m,n) = (-1,-1)$. We ignore $(-1,-1)$, since we need $m + n\sqrt{2} > 0$, and conclude that

$$\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2} \quad .$$

(ii) Since we are given that $f(x) = 0$ has four real roots, we can write

$$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \quad ,$$

for some real numbers $\alpha, \beta, \gamma, \delta$. Slightly expanding this factorisation, we have

$$f(x) = (x^2 - (\alpha+\beta)x + \alpha\beta)(x^2 - (\gamma+\delta)x + \gamma\delta) \quad .$$

By Vieta's formulae, considering the coefficient of x^3 in $f(x)$, we must have

$$\alpha + \beta + \gamma + \delta = 0 \quad \Longrightarrow \quad -(\alpha+\beta) = \gamma + \delta = s \quad ,$$

for some real number s. This then gives

$$f(x) = (x^2 + sx + p)(x^2 - sx + q) \quad ,$$

where $p = \alpha\beta$ and $q = \gamma\delta$ are also real numbers. Fully expanding this form now

$$f(x) = x^4 + (p + q - s^2)x^2 + s(q-p)x + pq \quad ,$$

thus we must have

$$\begin{cases} p + q - s^2 = -10 \\ s(q-p) = 12 \\ pq = -2 \end{cases} \quad .$$

Rewriting these as

$$\begin{cases} p + q = s^2 - 10 \\ p - q = -\frac{12}{s} \\ pq = -2 \end{cases} \quad ,$$

we can get an equation in s only:

$$\begin{aligned} (p+q)^2 - (p-q)^2 &= 4pq \\ \Longrightarrow \quad (s^2-10)^2 - \frac{144}{s^2} &= -8 \\ s^2(s^2-10)^2 + 8s^2 - 144 &= 0 \quad . \end{aligned}$$

To solve this, let $t = s^2$:

$$
\begin{aligned}
t(t-10)^2 + 8t - 144 &= 0 \\
t^3 - 20t^2 + 108t - 144 &= 0 \\
(t-2)(t^2 - 18t + 72) &= 0 \\
(t-2)(t-6)(t-12) &= 0 \quad .
\end{aligned}
$$

Hence we have $s^2 = 2$, $s^2 = 6$, or $s^2 = 12$. Using $s^2 = 2$ (we choose the positive root, $s = \sqrt{2}$ arbitrarily), we have

$$
\begin{cases} p + q = -8 \\ p - q = -6\sqrt{2} \quad , \end{cases}
$$

hence

$$
p = -4 - 3\sqrt{2} \quad , \quad q = -4 + 3\sqrt{2} \quad .
$$

We thus have

$$
f(x) = (x^2 + \sqrt{2}x - (4 + 3\sqrt{2}))(x^2 - \sqrt{2}x - (4 - 3\sqrt{2})) \quad ,
$$

and the solutions to $f(x) = 0$ are

$$
x = \frac{-\sqrt{2} \pm \sqrt{2 + 16 + 12\sqrt{2}}}{2} \quad \text{and} \quad x = \frac{\sqrt{2} \pm \sqrt{2 + 16 - 12\sqrt{2}}}{2} \quad ,
$$

which are

$$
x = \frac{-\sqrt{2} \pm \sqrt{6}\sqrt{3 + 2\sqrt{2}}}{2} \quad \text{and} \quad x = \frac{\sqrt{2} \pm \sqrt{6}\sqrt{3 - 2\sqrt{2}}}{2} \quad .
$$

We can simplify $\sqrt{3 + 2\sqrt{2}}$ by the result of part (i). Similarly, we can find

$$
m + n\sqrt{2} = \sqrt{3 - 2\sqrt{2}} \quad \Longrightarrow \quad m^2 + 2n^2 = 3 \;\; \text{and} \;\; 2mn = -2 \quad ;
$$

thus $(m, n) = (-1, 1)$ and $\sqrt{3 - 2\sqrt{2}} = -1 + \sqrt{2}$ (since $(m, n) = (1, -1)$ gives $1 - \sqrt{2} < 0$). Hence we simplify our roots of $f(x) = 0$ to

$$
x = \frac{-\sqrt{2} \pm \sqrt{6}(1 + \sqrt{2})}{2} \quad \text{and} \quad x = \frac{\sqrt{2} \pm \sqrt{6}(-1 + \sqrt{2})}{2} \quad ;
$$

that is:

$$
\begin{aligned}
x &= \frac{-\sqrt{2} + \sqrt{6} + 2\sqrt{3}}{2} \; , \; \frac{-\sqrt{2} - \sqrt{6} - 2\sqrt{3}}{2} \\
\text{and} \quad x &= \frac{\sqrt{2} - \sqrt{6} + 2\sqrt{3}}{2} \; , \; \frac{\sqrt{2} + \sqrt{6} - 2\sqrt{3}}{2} \quad .
\end{aligned}
$$

Question 5

(i) If $\overrightarrow{PQ} = \overrightarrow{SR}$, then $PQRS$ is a parallelogram.
If $\overrightarrow{PQ} = \overrightarrow{SR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$, then $PQRS$ is a rhombus.

(ii) If the four points lie in the same plane, then the diagonal PR intersects the diagonal QS. The line PR is given by the parametric equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{P} + t(\vec{R} - \vec{P}) = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} r - p \\ 1 \\ 1 \end{pmatrix} \quad ,$$

and the line QS is given by the parametric equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{Q} + u(\vec{S} - \vec{Q}) = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ s - q \\ 1 \end{pmatrix} \quad ;$$

hence they intersect when

$$\begin{cases} p + t(r - p) = 1 - u \\ t = q + u(s - q) \\ t = u \end{cases} \quad .$$

Substituting $u = t$ into the first two equations:

$$\begin{cases} t(r - p + 1) = 1 - p \\ t(1 - s + q) = q \end{cases} \quad ,$$

then eliminating t, we have

$$\begin{aligned} \frac{1-p}{r-p+1} &= \frac{q}{1-s+q} \\ (1-p)(1-s+q) &= q(r-p+1) \\ 1 - p - s + ps + q - pq &= qr - pq + q \\ qr &= 1 - p - s + ps \\ rq &= (1-s)(1-p) \quad . \end{aligned}$$

(a) From the expressions for $\vec{P}$, $\vec{Q}$, $\vec{R}$, and $\vec{S}$, the centroid is

$$\vec{C} = \begin{pmatrix} \frac{1}{4}(p + r + 1) \\ \frac{1}{4}(q + s + 1) \\ \frac{1}{2} \end{pmatrix} \quad .$$

We have that $\overrightarrow{PQ} = \overrightarrow{SR}$ if and only if

$$\vec{Q} - \vec{P} = \vec{R} - \vec{S} \iff \begin{cases} 1 - p = r \\ q = 1 - s \\ 0 = 0 \end{cases}$$

$$\iff \begin{cases} p + r + 1 = 2 \\ q + s + 1 = 2 \end{cases}$$

$$\iff \begin{cases} \frac{1}{4}(p + r + 1) = \frac{1}{2} \\ \frac{1}{4}(q + s + 1) = \frac{1}{2} \\ \frac{1}{2} = \frac{1}{2} \end{cases} ,$$

which are exactly the equations for the centroid $\vec{C}$ being at the centre of the cube $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

(b) Now we are given that $\overrightarrow{PQ} = \overrightarrow{SR}$, hence from above

$$\begin{cases} r = 1 - p \\ q = 1 - s \end{cases} ,$$

and $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$, thus

$$\begin{aligned} |\vec{Q} - \vec{P}|^2 &= |\vec{S} - \vec{P}|^2 \\ (p-1)^2 + q^2 &= p^2 + s^2 + 1 \\ p^2 - 2p + 1 + q^2 &= p^2 + s^2 + 1 \\ -2p + q^2 &= s^2 \\ -2p + (1-s)^2 &= s^2 \\ -2p + 1 - 2s + s^2 &= s^2 \\ s &= \frac{1}{2} - p \ , \end{aligned}$$

thus

$$\begin{cases} q = 1 - \left(\frac{1}{2} - p\right) = \frac{1}{2} + p \\ r = 1 - p \\ s = \frac{1}{2} - p \end{cases} .$$

Using the cosine rule on the triangle PQR:

$$|\overrightarrow{PR}|^2 = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 - 2|\overrightarrow{PQ}||\overrightarrow{QR}| \cos PQR \ ,$$

and since we know $PQRS$ is a rhombus, we have $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$:

$$
\begin{aligned}
\frac{|\overrightarrow{PR}|^2}{|\overrightarrow{PQ}|^2} &= 2 - 2\cos PQR \\
\cos PQR &= \frac{1}{2}\left(2 - \frac{|\overrightarrow{PR}|^2}{|\overrightarrow{PQ}|^2}\right) \\
&= \frac{1}{2}\left(2 - \frac{(p-r)^2+2}{(p-1)^2+q^2}\right) = \frac{1}{2}\left(2 - \frac{(2p-1)^2+2}{(p-1)^2+(p+\frac{1}{2})^2}\right) \\
&= \frac{1}{2}\left(2 - \frac{4p^2-4p+3}{2p^2-p+\frac{5}{4}}\right) \\
&= \frac{1}{2}\cdot\frac{4p^2-2p+\frac{5}{2}-4p^2+4p-3}{\frac{5}{4}-p+2p^2} = \frac{2p-\frac{1}{2}}{\frac{5}{2}-2p+4p^2} \\
&= \frac{4p-1}{5-4p+8p^2} \quad .
\end{aligned}
$$

If $PQRS$ is a square, then $\cos PQR = 0$ (PQR is a right angle), hence $p = \frac{1}{4}$, giving

$$
\begin{cases} q = \frac{3}{4} \\ r = \frac{3}{4} \\ s = \frac{1}{4} \end{cases} \quad ,
$$

and the side length is

$$
\begin{aligned}
|\overrightarrow{PQ}|^2 = (p-1)^2 + q^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{18}{16} \\
\implies |\overrightarrow{PQ}| = \frac{3\sqrt{2}}{4} = \frac{3\sqrt{50}}{20} > \frac{3\sqrt{49}}{20} = \frac{21}{20} \quad .
\end{aligned}
$$

Question 6

(i) We have

$$\begin{aligned} 9x^2 - 12x\cos\theta + 4 &= 9x^2 - 12x\cos\theta + 4\cos^2\theta - 4\cos^2\theta + 4 \\ &= (3x - 2\cos\theta)^2 - 4\cos^2\theta + 4 \\ &= (3x - 2\cos\theta)^2 + 4\sin^2\theta \quad , \end{aligned}$$

hence the minimum value is $4\sin^2\theta$, and this occurs at $x = \frac{2}{3}\cos\theta$. Similarly we have that

$$\begin{aligned} 12x^2\sin\theta - 9x^4 &= -(9x^4 - 12x^2\sin\theta + 4\sin^2\theta) + 4\sin^2\theta \\ &= 4\sin^2\theta - (3x^2 - 2\sin\theta)^2 \quad , \end{aligned}$$

hence the maximum value of this is $4\sin^2\theta$, and this occurs at $x^2 = \frac{2}{3}\sin\theta$. We can now solve

$$\begin{aligned} & 9x^4 + (9 - 12\sin\theta)x^2 - 12x\cos\theta + 4 = 0 \\ \iff \quad & (9x^2 - 12x\cos\theta + 4) - (12x^2\sin\theta - 9x^4) = 0 \quad ; \end{aligned}$$

since the left-hand bracket has a minimum value of $4\sin^2\theta$, and the right-hand bracket has a maximum value of $4\sin^2\theta$, the only solutions are those satisfying

$$x = \frac{2}{3}\cos\theta \quad \text{and} \quad x^2 = \frac{2}{3}\sin\theta \quad .$$

Eliminating x^2, we get an equation for θ only:

$$\begin{aligned} \frac{4}{9}\cos^2\theta &= \frac{2}{3}\sin\theta \\ 2(1 - \sin^2\theta) &= 3\sin\theta \\ 2\sin^2\theta + 3\sin\theta - 2 &= 0 \\ (2\sin\theta - 1)(\sin\theta + 2) &= 0 \quad , \end{aligned}$$

hence $\sin\theta = \frac{1}{2}$, thus $0 < \theta < \pi$ gives $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, and our solutions are

$$\begin{aligned} \theta = \tfrac{\pi}{6} \quad &\implies \quad x = \frac{2}{3}\cos\left(\tfrac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \\ \text{and} \quad \theta = \tfrac{5\pi}{6} \quad &\implies \quad x = \frac{2}{3}\cos\left(\tfrac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}} \quad . \end{aligned}$$

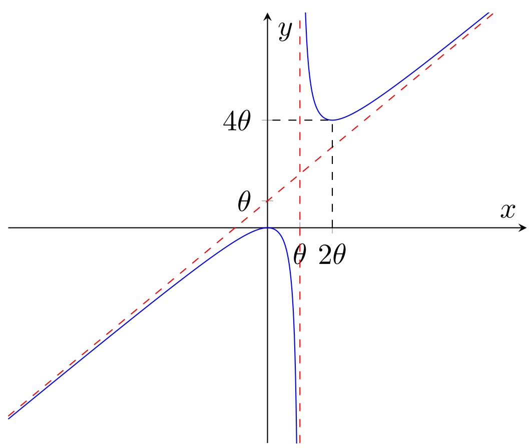
(ii) We have

$$\frac{x^2}{x-\theta} = \frac{x^2-\theta^2+\theta^2}{x-\theta} = x+\theta+\frac{\theta^2}{x-\theta} \quad .$$

Hence the curve has a vertical asymptote $x=\theta$, and an oblique asymptote $y=x+\theta$. Differentiating:

$$\frac{dy}{dx} = 1-\frac{\theta^2}{(x-\theta)^2} \quad \text{hence} \quad \frac{dy}{dx}=0 \iff (x-\theta)^2=\theta^2$$
$$x^2-2\theta x=0$$
$$x(x-2\theta)=0 \quad ,$$

thus there are stationary points at $x=0$, $y=0$, and at $x=2\theta$, $y=4\theta$. Our sketch is as follows.



We see that for $x>\theta$ we have $y\geqslant 4\theta$, and for $x<\theta$ we have $y\leqslant 0$.

The numerator of the fraction has a minimum value of 0, if $\sin\theta=0$ (but this has no solutions for $0<\theta<\pi$) or $\cos x=0$, and a maximum value of 1, if both $\sin^2\theta=1$ (so $\theta=\frac{\pi}{2}$) and $\cos^2 x=1$. The denominator of the fraction has minimum value 1, if $\cos\theta=0$ (so $\theta=\frac{\pi}{2}$) or $\sin x=0$, and maximum value 2, if both $\cos^2\theta=1$ (which has no solutions for $0<\theta<\pi$) and $\sin^2 x=1$. Taking the maximum value for the numerator and the minimum value for the denominator then, we have

$$\frac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \leqslant \frac{1}{1}=1 \quad ,$$

with equality if and only if $\sin^2\theta=1$, $\cos^2 x=1$, and at least one of $\cos\theta=0$ or $\sin x=0$. But $\sin^2\theta=1 \iff \cos\theta=0$ and $\cos^2 x=1 \iff \sin x=0$, so these are just the conditions $\sin^2\theta=1$, $\cos^2 x=1$, which are equivalent to $\theta=\frac{\pi}{2}$, $\sin x=0$. Similarly, the fraction has a minimum value of 0, achievable if and only if $\cos x=0$.

We want to solve

$$\frac{x^2}{x-\theta} = 4\theta \frac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \quad ;$$

by the inequalities deduced above, the right-hand side has a maximum value of 4θ, and a minimum value of 0. By our sketch graph then, the possible solutions are

$$\begin{aligned} (1):\qquad & \frac{x^2}{x-\theta} = 0 \quad \text{and} \quad \cos x = 0 \\ \Longrightarrow \quad & x = 0 \quad \text{and} \quad \cos x = 0 \quad , \end{aligned}$$

this is a contradiction so it gives no solutions, leaving only

$$\begin{aligned} (2):\qquad & \frac{x^2}{x-\theta} = 4\theta \quad \text{and} \quad \theta = \frac{\pi}{2} \quad , \quad \sin x = 0 \\ \Longrightarrow \quad & x = 2\theta \quad \text{and} \quad \theta = \frac{\pi}{2} \quad , \quad \sin x = 0 \end{aligned}$$

giving the (only) solution $\theta = \frac{\pi}{2}$, $x = \pi$.

Question 7

(i) **Step 1**: If a is not divisible by 3, then it must be one more than or one less than a multiple of 3. Thus $a = 3k \pm 1$ for some integer k, and squaring gives

$$a^2 = 9k^2 \pm 6k + 1 = 3(3k^2 \pm 2k) + 1 \quad ,$$

which is 1 more than a multiple of 3.

Step 3: Starting from $\sqrt{2} + \sqrt{3} = \frac{a}{b}$, we have

$$\begin{aligned} \frac{a^2}{b^2} &= (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 \\ a^2 - 5b^2 &= 2\sqrt{6}b^2 \\ a^4 - 10a^2b^2 + 25b^2 &= 24b^4 \\ a^4 + b^4 &= 10a^2b^2 \quad . \end{aligned}$$

Step 4: If a is divisible by 3, then $a = 3k$ for some integer k, giving

$$\begin{aligned} 81k^4 + b^4 &= 90k^2b^2 \\ b^4 &= 3(30k^2b^2 - 27k^3) \quad . \end{aligned}$$

So if a is divisible by 3, then b^4 must also be divisible by 3. If b is not divisible by 3, then by Step 1, $b^2 = 3k + 1$ for some integer k, but then squaring gives

$$b^4 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 \quad ,$$

which is not a multiple of 3. It follows that if a is divisible by 3 then b must also be (since if b is not divisible by 3, then b^4 is also not).

Step 5: We can now deduce that a is not divisible by 3, since a and b cannot share a common factor greater than 1. Thus $a^2 \equiv 1 \pmod 3$ and $a^4 \equiv 1 \pmod 3$. We then have

$$1 + b^4 \equiv 10b^2 \equiv b^2 \pmod 3 \quad .$$

By Step 1, either $b^2 \equiv 0 \pmod 3$ (if b is divisible by 3) or $b^2 \equiv 1 \pmod 3$. Trying $b^2 \equiv 0 \pmod 3$ gives

$$1 \equiv 0 \pmod 3 \quad ,$$

a contradiction. Trying $b^2 \equiv 1 \pmod 3$ gives

$$2 \equiv 1 \pmod 3 \quad ,$$

also a contradiction. Hence $\sqrt{2} + \sqrt{3}$ is not a rational number.

(ii) If an integer a is not a multiple of 5, either $a = 5k \pm 1$ or $a = 5k \pm 2$ for some integer k. Then

$$\begin{aligned} & a \equiv \pm 1 \pmod 5 \implies a^2 \equiv 1 \pmod 5 \implies a^4 \equiv 1 \pmod 5 \\ \text{or} \quad & a \equiv \pm 2 \pmod 5 \implies a^2 \equiv 4 \equiv -1 \pmod 5 \implies a^4 \equiv 1 \pmod 5 \quad . \end{aligned}$$

Suppose $\sqrt{6} + \sqrt{7}$ is rational, and equal to $\frac{a}{b}$, where a and b share no common factors other than 1. Then

$$\begin{aligned} \frac{a^2}{b^2} &= 6 + 2\sqrt{42} + 7 \\ a^2 - 13b^2 &= 2\sqrt{42}b^2 \\ a^4 - 26a^2b^2 + 169b^4 &= 168b^4 \\ a^4 + b^4 &= 26a^2b^2 \quad . \end{aligned}$$

If a is divisible by 5, then $a = 5k$ gives

$$b^4 = 26 \cdot 25b^2k^2 - 5^4k^4 = 5(130b^2k^2 - 125k^4) \quad ,$$

by the above, this requires b to be a multiple of 5, hence by contradiction with a and b being co-prime, a cannot be a multiple of 5. Thus $a^2 \equiv \pm 1 \pmod 5$ and $a^4 \equiv 1 \pmod 5$. This gives

$$1 + b^4 \equiv \pm 26b^2 \equiv \pm b^2 \pmod 5 \quad .$$

By the above, we can have $b^2 \equiv 0 \pmod 5$ (if b is divisible by 5) or $b^2 \equiv \pm 1 \pmod 5$. Trying $b^2 \equiv 0 \pmod 5$ gives

$$1 \equiv 0 \pmod 5 \quad ,$$

a contradiction; trying $b^2 \equiv 1 \pmod 5$ gives

$$2 \equiv \pm 1 \pmod 5 \quad ,$$

also a contradiction; finally trying $b^2 \equiv -1 \pmod 5$ gives

$$2 \equiv \mp 1 \pmod 5 \quad ,$$

also a contradiction. So $\sqrt{6} + \sqrt{7}$ is not rational.

Trying to use divisibility by 3 would lead to the conclusion that a is not divisible by 3, as before. but then $a^2 \equiv a^4 \equiv 1 \pmod 3$ gives

$$1 + b^4 \equiv 26b^2 \equiv 2b^2 \pmod 3 \quad ,$$

which could be satisfied, for example if $b^2 \equiv 1 \pmod 3$ (which is the case for any integer b that is not a multiple of 3).

Question 8

(i) Using the substitution $u = 2t$ we have

$$\int_2^x \sqrt{\frac{u-2}{u+2}}du = \int_1^{\frac{x}{2}} \sqrt{\frac{2t-2}{2t+2}} \cdot 2dt = 2\int_1^{\frac{x}{2}} \sqrt{\frac{t-1}{t+1}}dt = 2f\left(\tfrac{1}{2}x\right) \quad .$$

(ii) Using the substitution $u = v - 2$, we get

$$\int_0^x \sqrt{\frac{u}{u+4}}du = \int_2^{x+2} \sqrt{\frac{v-2}{v+2}}dv = 2f\left(\tfrac{1}{2}(x+2)\right) = 2f\left(\tfrac{1}{2}x+1\right) \quad ,$$

using the result of part (i).

(iii) Using the substitution $u = v + 2$, we get

$$\int_5^x \sqrt{\frac{u-5}{u+1}}du = \int_3^{x-2} \sqrt{\frac{v-3}{v+3}}dv \quad ,$$

now using the substitution $v = 3t$, we get

$$\int_3^{x-2} \sqrt{\frac{v-3}{v+3}}dv = \int_1^{\frac{x}{3}-\frac{2}{3}} \sqrt{\frac{3t-3}{3t+3}} \cdot 3dt = 3\int_1^{\frac{x}{3}-\frac{2}{3}} \sqrt{\frac{t-1}{t+1}}dt = 3f\left(\tfrac{1}{3}x - \tfrac{2}{3}\right) \quad .$$

(iv) Using the substitution $u = \sqrt{v}$, we get

$$\int_1^2 \frac{u^2}{\sqrt{u^2+4}}du = \int_1^4 \frac{v}{\sqrt{v+4}} \cdot \frac{1}{2}\frac{1}{\sqrt{v}}dv = \frac{1}{2}\int_1^4 \sqrt{\frac{v}{v+4}}dv \quad ,$$

and we can split this integral into two to get

$$\frac{1}{2}\int_1^4 \sqrt{\frac{v}{v+4}}dv = \frac{1}{2}\int_0^4 \sqrt{\frac{v}{v+4}}dv - \frac{1}{2}\int_0^1 \sqrt{\frac{v}{v+4}}dv = f(3) - f\left(\tfrac{3}{2}\right) \quad ,$$

using the result of part (ii).

Section B: Mechanics

Question 9

As ever, a diagram is particularly useful for the mechanics questions.

(i) Consider the moments on the ladder about its base, just before the box starts to topple. The length of the ladder is $\sqrt{b^2+h^2}$. In one direction, we have the normal reaction force R from the box on the ladder, at the top of the ladder acting perpendicular to the ladder; in the other direction, we have the painter's weight kW, acting vertically downwards at a fraction λ of the ladder's full height:

$$R \cdot \sqrt{b^2+h^2} = kW\cos\alpha \cdot \lambda\sqrt{b^2+h^2} \implies R = k\lambda W\cos\alpha \quad .$$

(ii) Now consider the moments on the box about the far edge of the base (the edge the box will topple about). At the top of the diagonal of the box, the normal reaction force from the ladder on the box acts at an angle α to the downwards vertical, which is an angle $2\alpha - \frac{\pi}{2}$ above the diagonal of the box. In the other direction to this moment, the weight of the box acts vertically downwards at a distance $\frac{1}{2}\sqrt{b^2+h^2}$ along the diagonal of the box:

$$\begin{aligned}
W\cos\alpha \cdot \frac{1}{2}\sqrt{b^2+h^2} &= R\sin\left(2\alpha - \tfrac{\pi}{2}\right) \cdot \sqrt{b^2+h^2} \\
W\cos\alpha &= 2k\lambda W\cos\alpha(-\sin\left(\tfrac{\pi}{2} - 2\alpha\right)) \\
1 &= -2k\lambda\sin\left(\tfrac{\pi}{2} - 2\alpha\right) \\
2k\lambda\sin\left(\tfrac{\pi}{2} - 2\alpha\right) + 1 &= 0 \\
2k\lambda\cos 2\alpha + 1 &= 0 \quad .
\end{aligned}$$

(iii) Resolving vertically on the box, the normal reaction force from the ground R_G is

$$R_G = R\cos\alpha + W = k\lambda W\cos^2\alpha + W = W(k\lambda\cos^2\alpha + 1) \quad .$$

Resolving horizontally on the box, the frictional force from the ground F is

$$F = R\sin\alpha = k\lambda W\sin\alpha\cos\alpha \quad .$$

Since the box topples before sliding, the frictional force just before toppling is less than or equal to the maximum possible frictional force μR_G, thus

$$\begin{aligned}
F \leqslant \mu R_G \implies \mu &\geqslant \frac{F}{R_G} = \frac{k\lambda\sin\alpha\cos\alpha}{k\lambda\cos^2\alpha + 1} \\
\mu &\geqslant \frac{2k\lambda\cos 2\alpha\sin\alpha\cos\alpha}{2k\lambda\cos 2\alpha\cos^2\alpha + 2\cos 2\alpha} = \frac{-\sin\alpha\cos\alpha}{-\cos^2\alpha + 2\cos 2\alpha} \\
\mu &\geqslant \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha - 4\cos 2\alpha} = \frac{\sin 2\alpha}{4\sin^2\alpha - 2\cos^2\alpha} \\
\mu &\geqslant \frac{\sin 2\alpha}{\sin^2\alpha + \cos^2\alpha + 3\sin^2\alpha - 3\cos^2\alpha} = \frac{\sin 2\alpha}{1 - 3\cos 2\alpha} \quad .
\end{aligned}$$

Question 10

(i) By parabolic motion, the coordinates of the particle at time t are

$$(x(t), y(t)) = (u\sin\alpha\, t, u\cos\alpha\, t - \tfrac{1}{2}gt^2) \quad .$$

At some time t_1, we have $(x(t_1), y(t_1)) = (h\tan\beta, h)$:

$$h\tan\beta = u\sin\alpha\, t_1 \quad \text{and} \quad h = u\cos\alpha\, t_1 - \tfrac{1}{2}gt_1^2 \quad ,$$

thus $t_1 = \frac{h\tan\beta}{u\sin\alpha}$, and

$$\begin{aligned} h &= u\cos\alpha \frac{h\tan\beta}{u\sin\alpha} - \tfrac{1}{2}g\frac{h^2\tan^2\beta}{u^2\sin^2\alpha} \\ h\cot^2\beta &= h\cot\alpha\cot\beta - \frac{gh^2}{2u^2}\frac{1}{\sin^2\alpha} \\ \frac{2u^2}{gh}\cot^2\beta &= \frac{2u^2}{gh}\cot\alpha\cot\beta - (1+\cot^2\alpha) \\ c^2 - ck\cot\beta + 1 + k\cot^2\beta &= 0 \quad , \end{aligned}$$

where $c = \cot\alpha$ and $k = \frac{2u^2}{gh}$.

(a) We have a quadratic in $\cot\alpha$, hence by Vieta's formulae, if $\cot\alpha_1$ and $\cot\alpha_2$ are the roots, then

$$\cot\alpha_1 + \cot\alpha_2 = k\cot\beta \quad ,$$

and

$$\cot\alpha_1\cot\alpha_2 = 1 + k\cot^2\beta \quad .$$

By the tan addition formula, we have

$$\cot(\alpha_1+\alpha_2) = \frac{1-\tan\alpha_1\tan\alpha_2}{\tan\alpha_1+\tan\alpha_2} = \frac{\cot\alpha_1\cot\alpha_2 - 1}{\cot\alpha_2+\cot\alpha_1} = \frac{k\cot^2\beta}{k\cot\beta} = \cot\beta \quad .$$

Since α_1, α_2, and β are acute, we conclude that $\alpha_1+\alpha_2=\beta$.

(b) Since the quadratic has two distinct roots, the discriminant gives

$$\begin{aligned} (k\cot\beta)^2 &> 4(1+k\cot^2\beta) \\ k^2 &> 4\tan^2\beta + 4k \\ k^2 - 4k + 4 &> 4\tan^2\beta + 4 \\ (k-2)^2 &> 4\sec^2\beta \\ k-2 &> 2\sec\beta \\ k &> 2(1+\sec\beta) \end{aligned}$$

(we ignore $k-2 < -2\sec\beta$ since $k-2 > -2$ and $-2\sec\beta < -2$).

(ii) The greatest height in the trajectory of the particle occurs at $\frac{dy}{dt} = 0$:

$$u \cos\alpha - gt = 0 \implies t = \frac{u}{g}\cos\alpha \quad .$$

At this point, the height of the particle is

$$y\left(\tfrac{u}{g}\cos\alpha\right) = \frac{u^2}{g}\cos^2\alpha - \frac{u^2}{2g}\cos^2\alpha = \frac{u^2}{2g}\cos^2\alpha = \frac{kh}{4}\cos^2\alpha \quad .$$

Since this maximum height must be at least h, we have

$$\frac{kh}{4}\cos^2\alpha \geqslant h$$
$$k \geqslant 4\sec^2\alpha \quad .$$

Section C: Probability and Statistics

Question 11

(i) On any given toss, the probability that a decision is made is

$$\mathbb{P}\{\text{two heads}\} + \mathbb{P}\{\text{two tails}\} = p^2 + q^2 \quad ,$$

and the probability that a decision is not made is

$$\mathbb{P}\{1 \text{ head, } 1 \text{ tail}\} = 2pq \quad .$$

Hence the probability that the decision is made on the n-th round is

$$\mathbb{P}\{\text{no decision for } n-1 \text{ rounds}\} \cdot (p^2 + q^2) = (p^2 + q^2)(2pq)^{n-1} \quad .$$

The probability that a decision is made on or before the n-th round is

$$\begin{aligned}
\mathbb{P} &= \sum_{k=1}^{n} (p^2 + q^2)(2pq)^{k-1} \\
&= (p^2 + q^2) \sum_{k=0}^{n-1} (2pq)^k \\
&= (p^2 + (1-p)^2) \frac{1 - (2pq)^n}{1 - 2pq} \\
&= (2p^2 - 2p + 1) \frac{1 - (2p(1-p))^n}{1 - 2p + 2p^2} \\
&= 1 - (2p(1-p))^n \quad .
\end{aligned}$$

This probability is minimised when $p(1-p)$ is maximised.

$$\frac{d}{dp}(p(1-p)) = 0 \implies 1 - 2p = 0 \implies p = \frac{1}{2} \quad , \quad p(1-p) = \frac{1}{4}$$

Hence the probability that a decision is made on or before the n-th round is at least

$$1 - \left(2 \cdot \frac{1}{4}\right)^n = 1 - \frac{1}{2^n} \quad .$$

(ii) The probability that a decision is made on or before the second round is the sum of the probabilities that (1): they toss three heads on the first round, (2): they toss three tails on the first round, (3): they toss 2 heads and one tail on the first round, and the two heads show heads again when re-tossed on the second round, (4): they toss 2 tails and one head on the first round, and the two tails show tails again when re-tossed on the second round. Thus the required probability is

$$\begin{aligned}\mathbb{P} &= p^3 + q^3 + 3p^2q \cdot p^2 + 3pq^2 \cdot q^2 \\ &= p^3 + (1-p)^3 + 3p(1-p)(p^3 + (1-p)^3) \\ &= (1 + 3p(1-p))(p^3 + 1 - 3p + 3p^2 - p^3) \\ &= (1 + 3p(1-p))(1 - 3p(1-p)) \\ &= 1 - 9(p(1-p))^2 \quad .\end{aligned}$$

This is again minimised when $p(1-p)$ is maximised (at $p = \frac{1}{2}$, $p(1-p) = \frac{1}{4}$), so the probability is at least

$$1 - \frac{9}{16} = \frac{7}{16} \quad .$$

STEP II

Section A: Pure Mathematics

Question 1

We have

$$f'(x) = g(x) + (x-p)g'(x) \quad ,$$

thus the tangent to the curve at $x = a$ is

$$\begin{aligned} y - f(a) &= (g(a) + (a-p)g'(a))(x-a) \\ y - (a-p)g(a) &= (g(a) + (a-p)g'(a))(x-a) \quad . \end{aligned}$$

This tangent passes through the point $(p, 0)$ if and only if

$$\begin{aligned} 0 - (a-p)g(a) &= (g(a) + (a-p)g'(a))(p-a) \\ (p-a)g(a) &= (p-a)g(a) - (p-a)^2 g'(a) \\ \iff g'(a)(p-a)^2 = 0 \quad , \end{aligned}$$

thus, given $p \neq a$, this occurs if and only if $g'(a) = 0$.

(i) C has the given form of $f(x)$ with $g(x) = A(x-q)(x-r)$. By the above, we know that $g'(a) = 0$. We have

$$g'(x) = \frac{d}{dx}(A(x-q)(x-r)) = A(2x - (q+r)) \quad ,$$

hence

$$2a - (q+r) = 0 \implies 2a = q + r \quad .$$

The gradient of the tangent is

$$\begin{aligned} f'(a) &= g(a) + (a-p)g'(a) = g(a) \\ &= A(a-q)(a-r) \\ &= A\left(\tfrac{1}{2}(q+r) - q\right)\left(\tfrac{1}{2}(q+r) - r\right) \\ &= \frac{A}{4}(-q+r)(q-r) \\ &= -\frac{A}{4}(r-q)^2 \quad . \end{aligned}$$

(ii) By symmetry with part (i), we know that $2c = p + q$, and thus the tangent to C at $x = c$ has gradient $-\frac{A}{4}(q-p)^2$. Hence the two tangents are parallel if and only if

$$\begin{aligned} (q-p)^2 &= (r-q)^2 \\ q^2 - 2pq + p^2 &= r^2 - 2qr + q^2 \\ 2q(r-p) &= r^2 - p^2 \\ \iff 2q &= p + r \quad , \end{aligned}$$

since we know $r \neq p$.

The tangent to the curve at $x = q$ is given by

$$y - (q-p)g(q) = (g(q) + (q-p)g'(q))(x-q) \quad ;$$

using $g(q) = 0$, $g'(q) = A(2q - (q+r)) = A(q-r)$, this simplifies to

$$y = A(q-p)(q-r)(x-q) \quad .$$

Looking for intersections of this tangent and the curve C, we find

$$\begin{aligned} A(x-p)(x-q)(x-r) &= A(q-p)(q-r)(x-q) \\ \iff (x-q)((x-p)(x-r) - (q-p)(q-r)) &= 0 \\ (x-q)(x^2 - (p+r)x + pr - q^2 + (p+r)q - pr) &= 0 \\ (x-q)(x^2 - q^2 - (p+r)x + (p+r)q) &= 0 \\ (x-q)^2(x + q - (p+r)) &= 0 \\ (x-q)^2(x - (p-q+r)) &= 0 \quad . \end{aligned}$$

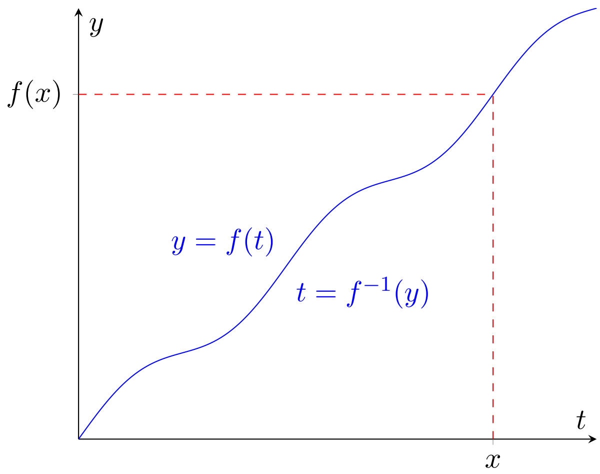
Hence $x = q$ is the only intersection if and only if $x = q$ is the only root of the above polynomial: that is, if and only if

$$\begin{aligned} p - q + r &= q \\ \iff 2q &= p + r \quad . \end{aligned}$$

Thus the tangent at $x = c$ is parallel to the tangent in part (i) if and only if the tangent at $x = q$ has no intersection with C other than at $x = q$.

Question 2

We use the following sketch:



The area under the curve up to the vertical dashed line is $\int_0^x f(t)dt$, and the area above the curve up to the horizontal dashed line is $\int_0^{f(x)} f^{-1}(y)dy$; combined, these areas make up a rectangle of area $xf(x)$, thus

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(y)dy = xf(x) \quad .$$

(i) Substituting $t = 0$, we have

$$g(0)^3 + g(0) = 0$$
$$g(0)(g(0)^2 + 1) = 0 \quad ,$$

since we are given that g is real, we discount $g^2(0) = -1$, to get $g(0) = 0$. Differentiation gives

$$(3g(t)^2 + 1)\frac{dg}{dt} = 1$$
$$g'(t) = \frac{1}{3g(t)^2 + 1} > 0 \quad ,$$

thus $g'(t) > 0$ for all t.

Setting $g(t) = y$, we have

$$g^{-1}(y) = y^3 + y \quad .$$

Substituting $t = 2$, we have

$$\begin{aligned} g(2)^3 + g(2) &= 2 \\ g(2)^3 + g(2) - 2 &= 0 \\ (g(2) - 1)(g(2)^2 + g(2) + 2) &= 0 \quad , \end{aligned}$$

and the discriminant of the quadratic bracket here is $1^2 - 4 \cdot 2 = -7 < 0$, thus for $g(2)$ to be real, we must have $g(2) = 1$. Then the identity shown above gives

$$\begin{aligned} \int_0^2 g(t)dt &= 2g(2) - \int_0^{g(2)} g^{-1}(y)dy \\ &= 2 \cdot 1 - \int_0^1 (y^3 + y)dy \\ &= 2 - \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_0^1 \\ &= 2 - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{5}{4} \quad . \end{aligned}$$

(ii) We note that

$$h(t)^3 + h(t) = t + 2 = g(t+2)^3 + g(t+2) \quad ,$$

thus for all t we have

$$\begin{aligned} h(t)^3 - g(t+2)^3 + h(t) - g(t) &= 0 \\ (h(t) - g(t+2))(h(t)^2 + g(t+2)h(t) + g(t+2))^2 + 1) &= 0 \quad , \end{aligned}$$

treating this as a polynomial in $h(t)$, the discriminant of the quadratic bracket is $g(t+2)^2 - 4(g(t+2)^2 + 1) = -3g(t+2)^2 - 4 < 0$, thus the only real root is

$$h(t) \equiv g(t+2) \quad .$$

Hence, we compute

$$\begin{aligned} \int_0^8 h(t)dt = \int_0^8 g(t+2)dt &= \int_2^{10} g(t)dt \\ &= \int_0^{10} g(t)dt - \int_0^2 g(t)dt \quad . \end{aligned}$$

Substituting $t = 10$ into the functional equation for $g(t)$:

$$\begin{aligned} g(10)^3 + g(10) - 10 &= 0 \\ (g(10) - 2)(g(10)^2 + 2g(10) + 5) &= 0 \quad , \end{aligned}$$

and the discriminant of the quadratic bracket is $2^2 - 4 \cdot 5 = -16 < 0$, so we conclude that $g(10) = 2$.

Thus, using the result of part (i):

$$
\begin{aligned}
\int_0^8 h(t)dt &= 10g(10) - \int_0^{g(10)} g^{-1}(y)dy - \frac{5}{4} \\
&= 20 - \int_0^2 (y^3 + y)dy - \frac{5}{4} \\
&= 20 - \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_0^2 - \frac{5}{4} \\
&= 20 - (4 + 2) - \frac{5}{4} \\
&= 14 - \frac{5}{4} = \frac{51}{4} \quad .
\end{aligned}
$$

Question 3

The quantity $|x_1 + x_2|$ is maximised when x_1 and x_2 have the same sign, in which case $|x_1 + x_2| = |x_1| + |x_2|$, hence

$$|x_1 + x_2| \leqslant |x_1| + |x_2| \quad ,$$

for any two real numbers x_1, x_2. By induction, we have

$$\begin{aligned}|x_1 + x_2 + \cdots + x_n| &\leqslant |x_1 + x_2 + \cdots + x_{n-1}| + |x_n| \\ &\leqslant |x_1 + x_2 + \cdots + x_{n-2}| + |x_{n-1}| + |x_n| \\ &\leqslant \ldots \\ &\leqslant |x_1| + |x_2| + \cdots + |x_n| \quad ,\end{aligned}$$

for any real numbers x_1, x_2, ..., x_n.

(i) (a) We have

$$f(x) - 1 = a_1 x + a_2 x^2 + \cdots + x^n \quad ,$$

so taking moduli:

$$\begin{aligned}|f(x) - 1| &= |a_1 x + a_2 x^2 + \cdots + x^n| \\ &\leqslant |a_1 x| + |a_2 x^2| + \ldots |x^n| \\ &= |a_1||x| + |a_2||x|^2 + \ldots |x|^n \\ &\leqslant A|x| + A|x|^2 + \ldots A|x|^n \\ &= A\left(|x| + |x|^2 + \ldots |x|^n\right) \\ &= \frac{A|x|}{1 - |x|}(1 - |x|^n) \\ &\leqslant \frac{A|x|}{1 - |x|} \quad .\end{aligned}$$

(b) If ω is a root of f, the above inequality gives

$$\begin{aligned} 1 &\leqslant \frac{A|\omega|}{1 - |\omega|} \\ \implies 1 - |\omega| &\leqslant A|\omega| && \text{since } 1 - |\omega| > 0 \\ \implies (A+1)|\omega| &\geqslant 1 \\ \implies |\omega| &\geqslant \frac{1}{A+1} && \text{since } 1 + A > 0 \quad ,\end{aligned}$$

and we can also note

$$|\omega| < 1 < 1 + A \quad ,$$

hence we have

$$\frac{1}{1 + A} \leqslant |\omega| \leqslant 1 + A \quad .$$

(c) For the case $|\omega| \geqslant 1$, consider $g(x)$ given by $g(x) = x^n f(x^{-1})$:

$$g(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1 \quad ;$$

this is the same form as $f(x)$, but with the coefficients labelled in reverse order. Applying the result of (i)(a) to $g(x)$ then, we have

$$|g(x) - 1| \leqslant \frac{A|x|}{1 - |x|} \quad .$$

We have $f(x) = x^n g(x^{-1})$, thus if $f(\omega) = 0$ and $|\omega| \geqslant 1$ (so $\omega \neq 0$), then $g(\omega^{-1}) = 0$. Hence

$$\begin{aligned} |g(\omega^{-1}) - 1| &\leqslant \frac{A|\omega^{-1}|}{1 - |\omega^{-1}|} \\ 1 &\leqslant \frac{A\frac{1}{|\omega|}}{1 - \frac{1}{|\omega|}} = \frac{A}{|\omega| - 1} \\ \implies \quad |\omega| - 1 &\leqslant A \qquad\qquad \text{since } |\omega| - 1 > 0 \\ |\omega| &\leqslant A + 1 \quad , \end{aligned}$$

and we can also note

$$|\omega| \geqslant 1 = \frac{1}{1} > \frac{1}{1 + A} \quad ,$$

hence (again) we have

$$\frac{1}{1+A} \leqslant |\omega| \leqslant 1 + A \quad .$$

(ii) We are solving

$$x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1 = 0 \quad ,$$

thus we can use the results of part (i) with $A = 1$. Accordingly, any root x of this equation satisfies

$$\frac{1}{2} \leqslant |x| \leqslant 2 \quad ,$$

so for integer roots, we need only check $x = \pm 2$ and $x = \pm 1$. Note that substituting $x = \pm 2$ into

$$135x^5 - 135x^4 - 100x^3 - 91x^2 - 126x + 135 \quad ,$$

all terms except the last will be even; hence the entire expression will evaluate to an odd integer and thus cannot equal zero. Trying $x = 1$:

$$135 - 135 - 100 - 91 - 126 + 135 = -191 + 9 = -182 \quad ,$$

hence $x = 1$ is not a solution. Trying $x = 1$:

$$-135 - 135 + 100 - 91 + 126 + 135 = -135 + 9 + 126 = 0 \quad ,$$

hence $x = -1$ is a solution, and is the only integer solution.

Question 4

(i) We have

$$\begin{aligned}\sin\left(\tfrac{\pi}{9}\right)\cos\left(\tfrac{\pi}{9}\right)\cos\left(\tfrac{2\pi}{9}\right)\cos\left(\tfrac{4\pi}{9}\right) &= \frac{1}{2}\sin\left(\tfrac{2\pi}{9}\right)\cos\left(\tfrac{2\pi}{9}\right)\cos\left(\tfrac{4\pi}{9}\right)\\ &= \frac{1}{4}\sin\left(\tfrac{4\pi}{9}\right)\cos\left(\tfrac{4\pi}{9}\right)\\ &= \frac{1}{8}\sin\left(\tfrac{8\pi}{9}\right)\\ &= \frac{1}{8}\sin\left(\pi-\tfrac{8\pi}{9}\right) = \frac{1}{8}\sin\left(\tfrac{\pi}{9}\right) \quad ,\end{aligned}$$

hence (since $\sin\left(\tfrac{\pi}{9}\right) \neq 0$)

$$\cos\left(\tfrac{\pi}{9}\right)\cos\left(\tfrac{2\pi}{9}\right)\cos\left(\tfrac{4\pi}{9}\right) = \frac{1}{8} \quad .$$

(ii) By a similar method to the above, we have that for $0 < x < \frac{\pi}{2}$

$$\begin{aligned}\sin\left(\tfrac{x}{2^n}\right)\prod_{k=0}^{n}\cos\left(\tfrac{x}{2^k}\right) &= \frac{1}{2}\sin\left(\tfrac{x}{2^{n-1}}\right)\prod_{k=0}^{n-1}\cos\left(\tfrac{x}{2^k}\right)\\ &= \frac{1}{4}\sin\left(\tfrac{x}{2^{n-2}}\right)\prod_{k=0}^{n-2}\cos\left(\tfrac{x}{2^k}\right)\\ &= \dots\\ &= \frac{1}{2^n}\sin(x)\cos(x)\\ &= \frac{1}{2^{n+1}}\sin(2x) \quad ,\end{aligned}$$

hence (since $\sin\left(\tfrac{x}{2^n}\right) \neq 0$)

$$\prod_{k=0}^{n}\cos\left(\tfrac{x}{2^k}\right) = \frac{\sin(2x)}{2^{n+1}\sin\left(\tfrac{x}{2^n}\right)} \quad .$$

Taking logs of both sides of this identity:

$$\sum_{k=0}^{n}\log\left(\cos\left(\tfrac{x}{2^k}\right)\right) = \log(\sin(2x)) - (n+1)\log 2 - \log\left(\sin\left(\tfrac{x}{2^n}\right)\right) \quad .$$

Now differentiating both sides with respect to x, we get

$$\begin{aligned}\sum_{k=0}^{n}\frac{-\frac{1}{2^k}\sin\left(\frac{x}{2^n}\right)}{\cos\left(\frac{x}{2^k}\right)} &= \frac{2\cos(2x)}{\sin(2x)} - \frac{\frac{1}{2^n}\cos\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)}\\ \sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\tfrac{x}{2^n}\right) &= \frac{1}{2^n}\cot\left(\tfrac{x}{2^n}\right) - 2\cot(2x) \quad .\end{aligned}$$

(iii) By the above result, we have

$$\prod_{k=1}^{\infty}\cos\left(\tfrac{x}{2^k}\right) = \lim_{n\to\infty}\left(\frac{1}{\cos(x)}\prod_{k=0}^{n}\cos\left(\tfrac{x}{2^k}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{\cos(x)}\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}\right)$$
$$= \lim_{n\to\infty}\left(\frac{\sin(x)}{2^n\sin\left(\frac{x}{2^n}\right)}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\right)$$

If we let $\theta = \frac{x}{2^n}$ with $x \neq 0$ fixed, then the limit $\theta \to 0$ is the limit $n \to \infty$ and so

$$\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1 \implies \lim_{n\to\infty}\left(\frac{\sin\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}\right) = 1$$
$$\implies \lim_{n\to\infty}\left(2^n\sin\left(\tfrac{x}{2^n}\right)\right) = x$$
$$\implies \lim_{n\to\infty}\left(\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\right) = \frac{1}{x}$$
$$\implies \lim_{n\to\infty}\left(\frac{\sin(x)}{2^n\sin\left(\frac{x}{2^n}\right)}\right) = \frac{\sin(x)}{x} \quad .$$

Thus

$$\prod_{k=1}^{\infty}\cos\left(\tfrac{x}{2^k}\right) = \frac{\sin(x)}{x} \quad .$$

(This holds also at $x = 0$ in the sense that $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = 1$, and the product evaluated at $x = 0$ is an infinite product of ones).

We also compute that

$$\sum_{j=2}^{\infty}\frac{1}{2^{j-2}}\tan\left(\tfrac{\pi}{2^j}\right) = \sum_{j=0}^{\infty}\frac{1}{2^j}\tan\left(\tfrac{1}{2^j}\tfrac{\pi}{4}\right) = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\tfrac{1}{2^n}\tfrac{\pi}{4}\right)\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{2^n}\cot\left(\tfrac{1}{2^n}\tfrac{\pi}{4}\right) - 2\cot\left(\tfrac{\pi}{2}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{2^n}\cot\left(\tfrac{1}{2^n}\tfrac{\pi}{4}\right)\right) \quad .$$

If we let $\theta = \frac{1}{2^n}\frac{\pi}{4}$, then the limit $\theta \to 0$ is the limit $n \to \infty$ and so

$$\lim_{\theta\to 0}\left(\frac{\tan(\theta)}{\theta}\right) = 1 \implies \lim_{n\to\infty}\left(\frac{\tan\left(\frac{1}{2^n}\frac{\pi}{4}\right)}{\frac{1}{2^n}\frac{\pi}{4}}\right) = 1$$
$$\implies \lim_{n\to\infty}\left(2^n\tan\left(\tfrac{1}{2^n}\tfrac{\pi}{4}\right)\right) = \frac{\pi}{4}$$
$$\implies \lim_{n\to\infty}\left(\frac{1}{2^n}\cot\left(\tfrac{1}{2^n}\tfrac{\pi}{4}\right)\right) = \frac{4}{\pi} \quad .$$

Thus

$$\sum_{j=2}^{\infty}\frac{1}{2^{j-2}}\tan\left(\tfrac{\pi}{2^j}\right) = \frac{4}{\pi} \quad .$$

Question 5

(i) The sequence is constant if and only if $f(a) = a$, that is

$$\begin{aligned} p + (a-p)a &= a \\ (a-p)a - a + p &= 0 \\ (a-p)(a-1) &= 0 \quad , \end{aligned}$$

hence the sequence is constant for $a = 1$ or $a = p$. The sequence is of period two if and only if $f(f(a)) = a$ and $f(a) \neq a$. We have

$$\begin{aligned} f(f(a)) = a \quad \iff \quad p + (p + (a-p)a - p)(p + (a-p)a) &= a \\ a(a-p)(p + (a-p)a) - a + p &= 0 \\ (a-p)(a(p + (a-p)a) - 1) &= 0 \\ (a-p)(a(a^2 - pa + p) - 1) &= 0 \\ (a-p)(a^3 - pa^2 + pa - 1) &= 0 \\ (a-p)(a-1)(a^2 - (p-1)a + 1) &= 0 \quad . \end{aligned}$$

Since $a = p$ and $a = 1$ give constant sequences, we neglect these. For a period two sequence to exist, the residual quadratic here must have (real) roots that are not $a = 1$ or $a = p$. For real roots, the discriminant must be positive:

$$\begin{aligned} (p-1)^2 - 4 &\geqslant 0 \\ p^2 - 2p - 3 &\geqslant 0 \\ (p-3)(p+1) &\geqslant -0 \\ \iff \quad p \geqslant 3 \quad &\text{or} \quad p \leqslant -1 \quad . \end{aligned}$$

If the roots are distinct (so the discriminant is strictly positive), then the roots cannot be $(1, p)$, since by Vieta's formulae their sum is $p - 1$ (and $p - 1 \neq p + 1$). If $p = -1$, then the residual quadratic is

$$a^2 + 2a + 1 = 0 \quad ,$$

with a repeated root $a = -1 = p$, hence this case does not give a period two sequence. If $p = 3$, then the residual quadratic is

$$a^2 - 2a + 1 = 0 \quad ,$$

with a repeated root $a = 1$, hence this case also does not give a period two sequence. Thus there exists a period two sequence for some values of a if and only if $p < -1$ or $p > 3$.

(ii) There is no value of a for which the sequence is constant if and only if $f(x) = x$ has no solutions. We have

$$f(x) = x \iff q + (x - p)x = x$$
$$x^2 - (p+1)x + q = 0 \quad .$$

This has no solutions if and only if the discriminant is negative, that is if and only if $(p+1)^2 - 4q < 0$. Completing the square, we have

$$\begin{aligned} f(x) - x &= x^2 - (p+1)x + q \\ &= \big(x - \tfrac{1}{2}(p+1)\big)^2 - \tfrac{1}{4}(p+1)^2 + q \\ &= \big(x - \tfrac{1}{2}(p+1)\big)^2 - \tfrac{1}{4}((p+1)^2 - 4q) \quad . \end{aligned}$$

Hence the minimum value of $f(x) - x$ is $-\frac{1}{4}((p+1)^2 - 4q)$, and $f(x) - x > 0$ for all x if and only if $(p+1)^2 - 4q < 0$. Thus there is no value of a for which the sequence is constant if and only if $f(x) - x > 0$ for all x.

We deduce that if there is no value of a for which the sequence is constant, then $f(x) > x$ for all x, and so

$$f(f(x)) > f(x) > x \quad \text{for all } x \quad ,$$

thus there are no solutions to $f(f(x)) = x$, and so there is no value of a for which the sequence has period two.

If we set $p = q = 0$, then $f(x) = x^2$ and $f(f(x)) = x^4$. In this case, the constant sequences are given by

$$a^2 = a \implies a = 0 \text{ or } a = 1 \quad ,$$

and we have

$$f(f(a)) = a \iff a^4 = a \iff a = 0 \text{ or } a = 1 \quad ,$$

thus there can be values of a for which the sequence is constant but no values of a for which the sequence has period two.

Question 6

(i) Substituting $y = mx + c$ into the differential equation, we have

$$\begin{aligned} m &= (mx + c) + x + 1 \\ (m+1)x + c - m + 1 &= 0 \quad ; \end{aligned}$$

since this must hold for all x, we must have both $m + 1 = 0$ and $c - m + 1 = 0$, hence $m = -1$, $c = -2$.

We have a stationary point if and only if $\frac{dy}{dx} = 0$, that is if and only if

$$\begin{aligned} y + x + 1 &= 0 \\ y &= -x - 1 \quad . \end{aligned}$$

We know that $y = -x - 2$ is a solution to the differential equation, and that stationary points exist only on the line $y = -x - 1$; thus, if a solution curve has $y(0) < -2$, it lies below $y = -x - 2$ for all x (since it cannot intersect $y = -x - 2$) and thus cannot intersect $y = -x - 1$, hence it cannot have a stationary point.

Differentiating, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 \quad ,$$

hence at any stationary point, we have $\frac{d^2y}{dx^2} = 1 > 0$, thus the stationary point is a minimum.

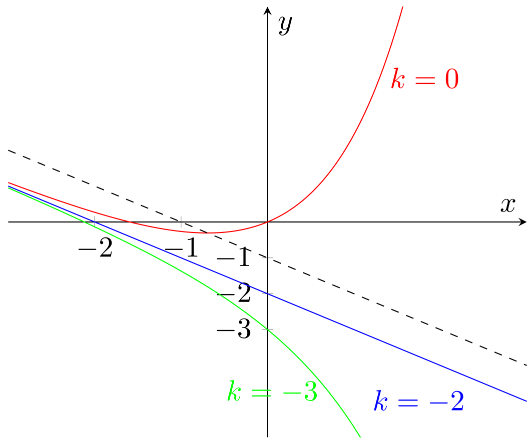
Substituting $Y = y + x$:

$$\begin{aligned} \frac{d}{dx}(Y - x) &= Y + 1 \\ \frac{dY}{dx} &= Y + 2 \\ \frac{1}{Y+2}\frac{dY}{dx} &= 1 \\ \log(Y + 2) &= x + c \\ Y &= -2 + Ae^x \\ y &= -x - 2 + Ae^x \quad . \end{aligned}$$

We have $y(0) = A - 2$, thus $A = y(0) + 2 = k + 2$, and our solutions with $k = 0, -2, -3$ are

$$\begin{aligned} k = 0 &\implies y = -x - 2 + 2e^x \quad , \\ k = -2 &\implies y = -x - 2 \quad , \\ \text{and} \quad k = -3 &\implies y = -x - 2 - e^x \quad . \end{aligned}$$

Sketching these on the same axes, we have:



where the dashed line is the line of stationary points, $y = -x - 1$.

(ii) Now substituting $y = mx + c$ gives

$$\begin{aligned} m &= x^2 + m^2x^2 + 2cmx + c^2 - 2mx^2 - 2cx - 4x + 4mx + 4c + 3 \\ 0 &= (1 + m^2 - 2m)x^2 + (2cm - 2c - 4 + 4m)x + (c^2 + 4c + 3) \\ &= (m-1)^2x^2 + (2c+4)(m-1)x + (c+2)^2 - 1 \quad , \end{aligned}$$

and since this must hold for all x, we must have $m = 1$ and

$$(c+2)^2 = 1 \quad \Longrightarrow \quad c = -2 \pm \sqrt{2} \quad .$$

Thus our two straight line solutions are $y = x - 2 + \sqrt{2}$ and $y = x - 2 - \sqrt{2}$.

We have stationary points if and only if $\frac{dy}{dx} = 0$, that is

$$\begin{aligned} x^2 + y^2 - 2xy - 4x + 4y + 3 &= 0 \\ (y-x)^2 + 4(y-x) + 3 &= 0 \\ (y-x+2)^2 - 1 &= 0 \\ (y-x+1)(y-x+3) &= 0 \quad , \end{aligned}$$

thus the lines $y = x - 1$ and $y = x - 3$ are the two lines of stationary points.

We have

$$\frac{d^2y}{dx^2} = 2x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 4 + 4\frac{dy}{dx} \quad ,$$

thus on either line of stationary points ($\frac{dy}{dx} = 0$)

$$\frac{d^2y}{dx^2} = 2x - 2y - 4 \quad ,$$

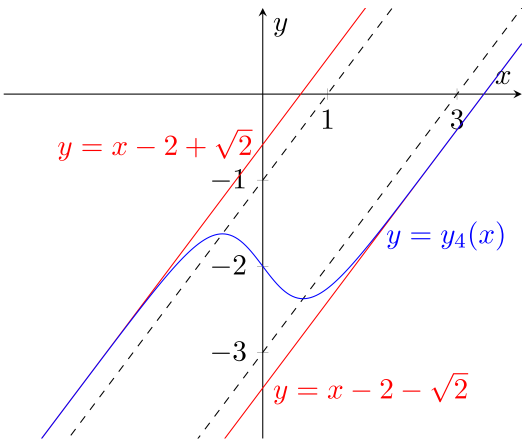
hence on $y = x - 1$ we have $\frac{d^2y}{dx^2} = -2$, so these are local maxima, and on $y = x - 3$ we have $\frac{d^2y}{dx^2} = 2$, so these are local minima.

We have

$$\frac{dy}{dx} = (y - (x-1))(y - (x-3)) \quad ,$$

thus between the two lines of stationary points $-x - 3 < y < -x - 1$ and so $\frac{dy}{dx} < 0$.

Sketching this then:



where the dashed lines are the lines of stationary points, $y = x - 1$ and $y = x - 3$.

Question 7

(i) We have

$$\begin{aligned} \boldsymbol{a}+\boldsymbol{b}=-\boldsymbol{c} \quad \Longrightarrow \quad (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) &= |\boldsymbol{c}|^2 = 1 \\ |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2 &= 1 \\ 2\boldsymbol{a}\cdot\boldsymbol{b} + 2 &= 1 \\ \boldsymbol{a}\cdot\boldsymbol{b} &= -\frac{1}{2} \quad . \end{aligned}$$

By symmetry we also have $\boldsymbol{b}\cdot\boldsymbol{c} = -\frac{1}{2}$ and $\boldsymbol{a}\cdot\boldsymbol{c} = -\frac{1}{2}$. We can compute

$$|AB|^2 = (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) = 1 - 2\boldsymbol{a}\cdot\boldsymbol{b} + 1 = 3 \quad \Longrightarrow \quad |AB| = \sqrt{3} \quad ,$$

and, identically

$$\begin{aligned} |AC|^2 = (\boldsymbol{c}-\boldsymbol{a})\cdot(\boldsymbol{c}-\boldsymbol{a}) = 1 - 2\boldsymbol{a}\cdot\boldsymbol{c} + 1 = 3 \quad \Longrightarrow \quad |AC| = \sqrt{3} \quad , \\ |BC|^2 = (\boldsymbol{c}-\boldsymbol{b})\cdot(\boldsymbol{c}-\boldsymbol{b}) = 1 - 2\boldsymbol{b}\cdot\boldsymbol{c} + 1 = 3 \quad \Longrightarrow \quad |BC| = \sqrt{3} \quad , \end{aligned}$$

hence ABC is an equilateral triangle.

(ii) Now we have

$$\begin{aligned} (\boldsymbol{a}_1+\boldsymbol{a}_2) &= -(\boldsymbol{a}_3+\boldsymbol{a}_4) \\ \Longrightarrow \quad (\boldsymbol{a}_1+\boldsymbol{a}_2)\cdot(\boldsymbol{a}_1+\boldsymbol{a}_2) &= (\boldsymbol{a}_3+\boldsymbol{a}_4)\cdot(\boldsymbol{a}_3+\boldsymbol{a}_4) \\ |\boldsymbol{a}_1|^2 + 2\boldsymbol{a}_1\cdot\boldsymbol{a}_2 + |\boldsymbol{a}_2|^2 &= |\boldsymbol{a}_3|^2 + 2\boldsymbol{a}_3\cdot\boldsymbol{a}_4 + |\boldsymbol{a}_4|^2 \\ 2\boldsymbol{a}_1\cdot\boldsymbol{a}_2 + 2 &= 2\boldsymbol{a}_3\cdot\boldsymbol{a}_4 + 2 \\ \boldsymbol{a}_1\cdot\boldsymbol{a}_2 &= \boldsymbol{a}_3\cdot\boldsymbol{a}_4 \quad . \end{aligned}$$

And similarly, $\boldsymbol{a}_1\cdot\boldsymbol{a}_3 = \boldsymbol{a}_2\cdot\boldsymbol{a}_4$, and $\boldsymbol{a}_1\cdot\boldsymbol{a}_4 = \boldsymbol{a}_2\cdot\boldsymbol{a}_3$.

(a) These give that

$$\begin{aligned} (\boldsymbol{a}_2-\boldsymbol{a}_1)\cdot(\boldsymbol{a}_2-\boldsymbol{a}_1) &= 1 - 2\boldsymbol{a}_1\cdot\boldsymbol{a}_2 + 1 \\ &= |\boldsymbol{a}_4|^2 - 2\boldsymbol{a}_3\cdot\boldsymbol{a}_4 + |\boldsymbol{a}_3|^2 \\ &= (\boldsymbol{a}_4-\boldsymbol{a}_3)\cdot(\boldsymbol{a}_4-\boldsymbol{a}_3) \\ \Longrightarrow \quad |A_1A_2| &= |A_3A_4| \quad , \end{aligned}$$

and by symmetry we also have $|A_1A_3| = |A_2A_4|$, and $|A_1A_4| = |A_2A_3|$. Thus the opposite sides of the quadrilateral are of equal length, and the diagonals are also of equal length. Hence $A_1A_2A_3A_4$ is a rectangle.

(b) By the symmetry of the regular tetrahedron, all dot products $\boldsymbol{a}_i \cdot \boldsymbol{a}_j$, $i \neq j$ are equal. We thus have

$$\begin{aligned}
\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 &= -\boldsymbol{a}_4 \\
\implies \quad (\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3) &= 1 \\
1 + 1 + 1 + 2(\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 + \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 + \boldsymbol{a}_2 \cdot \boldsymbol{a}_3) &= 1 \\
6\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 &= -2 \\
\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 &= -\tfrac{1}{3} \quad .
\end{aligned}$$

Using this, the side length of the tetrahedron l is

$$\begin{aligned}
l^2 = |A_1 A_2|^2 &= (\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\boldsymbol{a}_2 - \boldsymbol{a}_1) \\
&= |\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 - 2\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \\
&= 2 + \frac{2}{3} = \frac{8}{3} \\
\implies \quad l &= \frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{6}}{3} \quad .
\end{aligned}$$

Question 8

(i) We have

$$\begin{aligned} f(\boldsymbol{MI}) &= f(\boldsymbol{M})f(\boldsymbol{I}) \\ \iff \quad f(\boldsymbol{M}) &= f(\boldsymbol{M})f(\boldsymbol{I}) \quad ; \end{aligned}$$

since we are given $f(\boldsymbol{M}) \neq 0$, this requires $f(\boldsymbol{I}) = 1$.

(ii) We have

$$\boldsymbol{J}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I} \quad ,$$

thus

$$\begin{aligned} f(\boldsymbol{J}^2) &= f(\boldsymbol{I}) \\ f(\boldsymbol{JJ}) &= 1 \\ f(\boldsymbol{J})f(\boldsymbol{J}) &= 1 \\ f(\boldsymbol{J})^2 &= 1 \quad , \end{aligned}$$

and since we are given that $f(\boldsymbol{J}) \neq 1$, we deduce that $f(\boldsymbol{J}) = -1$.

We have that

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} \boldsymbol{J} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad ,$$

hence

$$f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix} \boldsymbol{J}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) f(\boldsymbol{J}) = -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) \quad .$$

Similarly,

$$\boldsymbol{J}\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ,$$

hence

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left(\boldsymbol{J}\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f(\boldsymbol{J}) f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) \quad .$$

(iii) If the second row of $\boldsymbol{A}$ is a multiple of the first row, then we have

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$$

for some real numbers a, b, k. We can write this as

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \boldsymbol{K} \begin{pmatrix} a & b \\ a & b \end{pmatrix} \quad .$$

Using the result of part (ii), setting $c = a$, $d = b$, we have

$$f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) \quad ,$$

hence

$$f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0 \quad .$$

This then gives

$$f(\boldsymbol{A}) = f\left(\boldsymbol{K}\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = f(\boldsymbol{K})f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0 \quad .$$

(iv) We have

$$\boldsymbol{P}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad , \quad \boldsymbol{P}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad ,$$

and for $k \neq 0$ (so that $\boldsymbol{K}^{-1}$ exists)

$$\begin{aligned} \boldsymbol{K}^{-1}\boldsymbol{P}\boldsymbol{K} &= \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & k^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \\ &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad . \end{aligned}$$

Note that

$$\begin{aligned} f(\boldsymbol{K}^{-1}\boldsymbol{P}\boldsymbol{K}) &= f(\boldsymbol{K}^{-1}\boldsymbol{P})f(\boldsymbol{K}) \\ &= f(\boldsymbol{K}^{-1})f(\boldsymbol{P})f(\boldsymbol{K}) \\ &= f(\boldsymbol{P})f(\boldsymbol{K}^{-1})f(\boldsymbol{K}) \\ &= f(\boldsymbol{P})f(\boldsymbol{K}^{-1}\boldsymbol{K}) \\ &= f(\boldsymbol{P})f(\boldsymbol{I}) \\ &= f(\boldsymbol{P}) \quad . \end{aligned}$$

Thus if we set $k = 2$, we have $\boldsymbol{K}^{-1}\boldsymbol{P}\boldsymbol{K} = \boldsymbol{P}^2$, giving

$$\begin{aligned} f(\boldsymbol{P}^2) &= f(\boldsymbol{P}) \\ f(\boldsymbol{P}\boldsymbol{P}) &= f(\boldsymbol{P}) \\ f(\boldsymbol{P})f(\boldsymbol{P}) &= f(\boldsymbol{P}) \\ f(\boldsymbol{P})^2 &= f(\boldsymbol{P}) \quad , \end{aligned}$$

hence we have either $f(\boldsymbol{P}) = 0$ or $f(\boldsymbol{P}) = 1$.

If we instead set $k = -1$, we have $\boldsymbol{K}^{-1}\boldsymbol{P}\boldsymbol{K} = \boldsymbol{P}^{-1}$, giving

$$\begin{aligned} f(\boldsymbol{P}^{-1}) &= f(\boldsymbol{P}) \\ \implies f(\boldsymbol{P})f(\boldsymbol{P}^{-1}) &= f(\boldsymbol{P})^2 \\ f(\boldsymbol{P}\boldsymbol{P}^{-1}) &= f(\boldsymbol{P})^2 \\ f(\boldsymbol{I}) &= f(\boldsymbol{P})^2 \\ 1 &= f(\boldsymbol{P})^2 \quad , \end{aligned}$$

hence we have either $f(\boldsymbol{P}) = 1$ or $f(\boldsymbol{P}) = -1$. Thus we conclude $f(\boldsymbol{P}) = 1$.

Section B: Mechanics

Question 9

(i) The trajectory of the particle can be described parametrically by

$$\begin{aligned} x(t) &= u\cos(\alpha)t \\ y(t) &= u\sin(\alpha)t - \tfrac{1}{2}gt^2 \quad , \end{aligned}$$

thus we have

$$\begin{aligned} |OP|^2 = x(t)^2 + y(t)^2 &= u^2\cos^2(\alpha)t^2 + u^2\sin^2(\alpha)t^2 - ug\sin(\alpha)t^3 + \frac{1}{4}g^2t^4 \\ &= u^2t^2 - ug\sin(\alpha)t^3 + \frac{1}{4}g^2t^4 \quad . \end{aligned}$$

Note that $|OP|$ increases if and only if $|OP|^2$ increases, thus we don't need to square root this quantity. Differentiating with respect to time:

$$\begin{aligned} \frac{d}{dt}(|OP|^2) &= 2u^2t - 3ug\sin(\alpha)t^2 + g^2t^3 \\ &= t(g^2t^2 - 3ug\sin(\alpha)t + 2u^2) \\ &= t\left(g^2t^2 - 3ug\sin(\alpha)t + \frac{9}{4}u^2\sin^2(\alpha) + 2u^2 - \frac{9}{4}u^2\sin^2(\alpha)\right) \\ &= t\left(\left(gt - \frac{3}{2}u\sin(\alpha)\right)^2 + \frac{9}{4}u^2\left(\frac{8}{9} - \sin^2(\alpha)\right)\right) \quad . \end{aligned}$$

If $\sin^2(\alpha) < \frac{8}{9}$, then $\frac{d}{dt}(|OP|^2)$ is always positive for $t > 0$. Thus if $\sin(\alpha) < \frac{2\sqrt{2}}{3}$ (since α is acute), then $|OP|$ is increasing throughout the flight.

If $\sin(\alpha) > \frac{2\sqrt{2}}{3}$, then at $t = \frac{3u}{2g}\sin(\alpha)$, $\frac{d}{dt}(|OP|^2)$ will be negative, hence $|OP|$ will be decreasing. The particle lands at

$$y(t) = 0 \ , \quad t \neq 0 \implies \tfrac{1}{2}gt = u\sin(\alpha) \quad ,$$

thus the particle lands at $t = \frac{2u}{g}\sin(\alpha)$, which is greater than $\frac{3u}{2g}\sin(\alpha)$. Hence if $\sin(\alpha) > \frac{2\sqrt{2}}{3}$, then at some time before the particle lands OP will be decreasing.

(ii) The position of the projected particle relative to Q can be described parametrically by

$$\begin{aligned} \tilde{x}(t) &= u\cos(\alpha)t + vt \\ \tilde{y}(t) &= u\sin(\alpha)t - \tfrac{1}{2}gt^2 \quad , \end{aligned}$$

thus we have

$$\begin{aligned} |QP|^2 &= (u^2t^2 - ug\sin(\alpha)t^3 + \frac{1}{4}g^2t^4) + 2uv\cos(\alpha)t^2 + v^2t^2 \\ &= (u^2 + 2uv\cos(\alpha) + v^2)t^2 - ug\sin(\alpha)t^3 + \frac{1}{4}g^2t^4 \quad . \end{aligned}$$

Differentiating with respect to time:

$$\begin{aligned}\frac{d}{dt}|QP|^2 &= g^2t^3 - 3ug\sin(\alpha)t^2 + 2(u^2 + 2uv\cos(\alpha) + v^2)t \\ &= t(g^2t^2 - 3ug\sin(\alpha)t + 2u^2 + 4uv\cos(\alpha) + 2v^2) \\ &= t\left(\left(gt - \frac{3}{2}u\sin(\alpha)\right)^2 - \frac{9}{4}u^2\sin^2(\alpha) + 2u^2 + 4uv\cos(\alpha) + 2v^2\right) \quad .\end{aligned}$$

For $|QP|$ to be increasing throughout the flight, we need

$$\begin{aligned}2v^2 + 4uv\cos(\alpha) + 2u^2 - \frac{9}{4}u^2\sin^2(\alpha) &> 0 \\ 8v^2 + 16uv\cos(\alpha) + 8u^2 - 9u^2\sin^2(\alpha) &> 0 \\ (2\sqrt{2}v + 2\sqrt{2}u\cos(\alpha))^2 - 8u^2\cos^2\alpha + 8u^2 - 9u^2\sin^2(\alpha) &> 0 \\ (2\sqrt{2}v + 2\sqrt{2}u\cos(\alpha))^2 + 8u^2\sin^2\alpha - 9u^2\sin^2(\alpha) &> 0 \\ (2\sqrt{2}v + 2\sqrt{2}u\cos(\alpha))^2 - u^2\sin^2(\alpha) &> 0 \\ (2\sqrt{2}v + 2\sqrt{2}u\cos(\alpha) - u\sin(\alpha))(2\sqrt{2}v + 2\sqrt{2}u\cos(\alpha) + u\sin(\alpha)) &> 0 \\ (2\sqrt{2}v - (\sin(\alpha) - 2\sqrt{2}\cos(\alpha))u)(2\sqrt{2}v + (\sin(\alpha) + 2\sqrt{2}\cos(\alpha))u) &> 0\end{aligned}$$

thus, if $2\sqrt{2}v > (\sin(\alpha) - 2\sqrt{2}\cos(\alpha))u$, then this inequality is satisfied, and $|QP|$ is increasing throughout the flight of P.

Question 10

(i) Taking moments about A, we have

$$T\sin(2\theta)\cdot 2a = kW\cos(\theta)\cdot 2a + W\cos(\theta)\cdot a \quad ,$$

where T is the tension in the string. Thus

$$\begin{aligned} 4T\sin(\theta)\cos(\theta) &= W\cos(\theta)(2k+1) \\ T &= \frac{2k+1}{4\sin(\theta)}W \quad , \end{aligned}$$

and (assuming the ring does not slip) the string will break (that is $T > W$) if

$$\begin{aligned} & \frac{2k+1}{4\sin\theta} > 1 \\ \iff \quad & 2k+1 > 4\sin\theta \quad . \end{aligned}$$

(ii) Resolving forces on the rod vertically:

$$R = (k+1)W - T\sin(\theta) \quad ,$$

where R is the normal reaction force from the rail on the rod (via contact with the ring). Thus

$$R = (k+1)W - \frac{1}{4}(2k+1)W = \frac{1}{4}(2k+3)W \quad .$$

Resolving forces on the rod horizontally:

$$F = T\cos(\theta) \quad ,$$

where F is the frictional force from the rail on the ring. Thus

$$F = \frac{1}{4}(2k+1)\cot(\theta)W \quad .$$

Hence (assuming that the string does not break) the ring will slip if

$$\begin{aligned} & F > \mu R \\ \iff \quad & \frac{1}{4}(2k+1)W\cot(\theta) > \mu\cdot\frac{1}{4}(2k+3)W \\ \iff \quad & 2k+1 > (2k+3)\mu\tan(\theta) \quad . \end{aligned}$$

(iii) We can compute that (if the ring does not slip first) the string will break at

$$\begin{aligned} 2k+1 &= 4\sin(\theta) \\ k &= 2\sin(\theta) - \frac{1}{2} \quad , \end{aligned}$$

and that (if the string does not break first) the ring will slip at

$$\begin{aligned} 2k+1 &= (2k+3)\mu\tan(\theta) \\ (2-2\mu\tan(\theta))k &= 3\mu\tan(\theta) - 1 \\ k &= \frac{3\mu\tan(\theta)-1}{2(1-\mu\tan(\theta))} \quad . \end{aligned}$$

Thus, the ring will slip before the string breaks if

$$\begin{aligned} \frac{3\mu\tan(\theta)-1}{2(1-\mu\tan(\theta))} &< 2\sin(\theta) - \frac{1}{2} \\ 3\mu\tan(\theta) - 1 &< (4\sin(\theta)-1)(1-\mu\tan\theta) \qquad \text{(since } 1-\mu\tan(\theta) > 0\text{)} \\ 3\mu\sin(\theta) - \cos(\theta) &< (4\sin(\theta)-1)(\cos(\theta)-\mu\sin(\theta)) \qquad \text{(since } \cos(\theta) > 0\text{)} \\ 3\mu\sin(\theta) - \cos(\theta) &< 4\sin(\theta)\cos(\theta) - \cos(\theta) - 4\mu\sin^2(\theta) + \mu\sin(\theta) \\ (2\sin(\theta) + 4\sin^2(\theta))\mu &< 4\sin(\theta)\cos(\theta) \\ \mu &< \frac{4\sin(\theta)\cos(\theta)}{2\sin(\theta)+4\sin^2(\theta)} \qquad \text{(since } \sin(\theta) > 0\text{)} \\ \mu &< \frac{2\cos(\theta)}{1+2\sin(\theta)} \quad . \end{aligned}$$

Section C: Probability and Statistics

Question 11

(i) We find the possibilities for n_1 and n_2 by inspection.

$n_3 = 9$: We have the following possibilities:

n_1	n_2
1	no possibilities satisfying $n_1 + n_2 > 9$
2	8
3	$7, 8$
4	$6, 7, 8$
5	$6, 7, 8$
6	$7, 8$
7	8
8	no possibilities satisfying $n_1 < n_2 < 9$

Thus there are 12 ways to choose n_1 and n_2 satisfying the given inequalities.

$n_3 = 10$: We have the following possibilities:

n_1	n_2
1	no possibilities satisfying $n_1 + n_2 > 10$
2	9
3	$8, 9$
4	$7, 8, 9$
5	$6, 7, 8, 9$
6	$7, 8, 9$
7	$8, 9$
8	9
9	no possibilities satisfying $n_1 < n_2 < 10$

Thus there are 16 ways to choose n_1 and n_2 satisfying the given inequalities.

When $n_3 = 2n + 1$, the number of ways to choose n_1, n_2 is

$$(1 + 2 + \cdots + (n-1)) \cdot 2 \quad ,$$

which can be simplified by summing the arithmetic series:

$$(1 + 2 + \cdots + (n-1)) \cdot 2 = \frac{1}{2}n(n-1) \cdot 2 = n(n-1) \quad .$$

When $n_3 = 2n$, the number of ways is

$$(1 + 2 + \cdots + (n-2)) \cdot 2 + (n-1) = (n-1)(n-2) + (n-1) = (n-1)^2 \quad .$$

(ii) We note that the three rods can form a triangle if and only if the other two rods chosen have lengths n_1, n_2 satisfying the inequalities in (i) with $n_3 = N$. Thus in the case $N = 2n + 1$, the number of distinct pairs of rods we can choose is

$$\binom{2n}{2} = \frac{1}{2}(2n)(2n-1) = n(2n-1) \quad ,$$

and (from above) the number of ways to choose rods that can form a triangle is $n(n-1)$, hence the probability that the chosen rods can form a triangle is

$$\frac{n(n-1)}{n(2n-1)} = \frac{n-1}{2n-1} \quad .$$

In the case $N = 2n$, the number of distinct pairs of rods we can choose is

$$\binom{2n-1}{2} = \frac{1}{2}(2n-1)(2n-2) = (n-1)(2n-1) \quad ,$$

and (from above) the number of ways to choose rods that can form a triangle is $(n-1)^2$, hence the probability that the chosen rods can form a triangle is

$$\frac{(n-1)^2}{(n-1)(2n-1)} = \frac{n-1}{2n-1} \quad .$$

(iii) The number of distinct ways of choosing three rods is

$$\binom{2M+1}{3} = \frac{1}{6}(2M+1)(2M)(2M-1) = \frac{1}{3}M(2M+1)(2M-1) \quad .$$

To calculate the number of ways of choosing rods that can form a triangle, we sum the number of ways this can happen if the longest rod has length $2n+1$ for each $n \in \{1, 2, \ldots, M\}$ or length $2n$ for each $n \in \{2, \ldots, M\}$ (we know the longest rod will have length at least 3):

$$\begin{aligned}
\sum_{n=1}^{M} n(n-1) + \sum_{n=2}^{M}(n-1)^2 &= \sum_{n=1}^{M} n(n-1) + \sum_{n=1}^{M}(n-1)^2 \\
&= \sum_{n=1}^{M}\left(n^2 - n + n^2 - 2n + 1\right) = 2\sum_{n=1}^{M} n^2 - 3\sum_{l=1}^{M} n + M \\
&= \frac{1}{3}M(M+1)(2M+1) - \frac{3}{2}M(M+1) + M \\
&= \frac{4}{6}M^3 - \frac{3}{6}M^2 - \frac{1}{6}M = \frac{1}{6}M(4M+1)(M-1) \quad .
\end{aligned}$$

Hence the probability that three randomly chosen rods can form a triangle is

$$\frac{\frac{1}{6}M(4M+1)(M-1)}{\frac{1}{3}M(2M+1)(2M-1)} = \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)} \quad .$$

Question 12

(i) The expectation is

$$\mu = \int_0^1 x \cdot nx^{n-1} dx = \int_0^1 nx^n dx = \frac{n}{n+1}\left[x^{n+1}\right]_0^1 = \frac{n}{n+1} \quad .$$

The variance is

$$\begin{aligned}
\sigma^2 &= \int_0^1 x^2 \cdot nx^{n-1} dx - \mu^2 \\
&= \int_0^1 nx^{n+1} dx - \frac{n^2}{(n+1)^2} \\
&= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)} \\
&= \frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)} \quad .
\end{aligned}$$

(ii) In the case $n = 2$, the variance is

$$\sigma^2 = \frac{2}{3^2 \cdot 4} = \frac{1}{18} \quad .$$

The upper quartile u is given by

$$\begin{aligned}
\int_0^u 2x dx = \frac{3}{4} \quad \Longrightarrow \quad \left[x^2\right]_0^u &= \frac{3}{4} \\
u^2 &= \frac{3}{4} \\
u &= \frac{\sqrt{3}}{2} \quad ,
\end{aligned}$$

and the lower quartile l is given by

$$\begin{aligned}
\int_0^l 2x dx = \frac{1}{4} \quad \Longrightarrow \quad \left[x^2\right]_0^l &= \frac{1}{4} \\
l^2 &= \frac{1}{4} \\
l &= \frac{1}{2} \quad ,
\end{aligned}$$

thus the interquartile range is

$$u - l = \frac{1}{2}(\sqrt{3} - 1) \quad .$$

We compare

$$(2\sigma)^2 = 4\sigma^2 = \frac{2}{9} \quad \text{and} \quad (u-l)^2 = \frac{1}{4}(3+1-2\sqrt{3}) = 1 - \frac{1}{2}\sqrt{3} \quad ;$$

we have

$$\begin{aligned}(2\sigma)^2 - (q-l)^2 &= \frac{2}{9} - 1 + \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3} - \frac{7}{9} \\ &= \frac{1}{18}(9\sqrt{3} - 14) \\ &= \frac{1}{18}(\sqrt{243} - \sqrt{196}) > 0 \\ \implies \quad & (q-l)^2 < (2\sigma)^2 \\ \implies \quad & q - l < 2\sigma \quad .\end{aligned}$$

(iii) We have

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + x^n \\ &= 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots + x^n \quad .\end{aligned}$$

In the general case, the median m is given by

$$\int_0^m nx^{n-1}dx = \frac{1}{2} \implies [x^n]_0^m = \frac{1}{2}$$
$$m^n = \frac{1}{2}$$
$$m = \left(\frac{1}{2}\right)^{\frac{1}{n}} \quad ,$$

and similarly, the lower quartile is given by

$$\int_0^l nx^{n-1}dx = \frac{1}{4} \implies l = \left(\frac{1}{4}\right)^{\frac{1}{n}} \quad .$$

Setting $x = \frac{1}{n}$ we have $1 + x = \frac{n+1}{n} = \frac{1}{\mu}$; thus

$$\begin{aligned}\left(\frac{1}{\mu}\right)^n &= 1 + \frac{1}{1!}\frac{n}{n} + \frac{1}{2!}\frac{n(n-1)}{n^2} + \cdots + \frac{1}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &\qquad + \ldots \frac{1}{n!}\frac{n(n-1)\cdots 1}{n^n} \\ &< 1 + \frac{1}{1!}\frac{n}{n} + \frac{1}{2!}\frac{n\cdot n}{n^2} + \cdots + \frac{1}{k!}\frac{n\cdot n\cdots n}{n^k} + \ldots \frac{1}{n!}\frac{n^n}{n^n} = \sum_{k=0}^{n}\frac{1}{k!} \\ &< \sum_{k=0}^{\infty}\frac{1}{k!} < 4 \quad ,\end{aligned}$$

hence $\mu^n > \frac{1}{4}$ and so $\mu > \left(\frac{1}{4}\right)^{\frac{1}{n}} = l$.

We can also bound this below by

$$
\begin{aligned}
\left(\frac{1}{\mu}\right)^n &= 1 + \frac{1}{1!}\frac{n}{n} + \frac{1}{2!}\frac{n(n-1)}{n^2} + \cdots + \frac{1}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k} \\
&\qquad + \dots \frac{1}{n!}\frac{n(n-1)\cdots 1}{n^n} \\
&> 1 + \frac{1}{1!}\frac{n}{n} = 2 \quad ,
\end{aligned}
$$

hence $\mu^n < \frac{1}{2}$ and so $\mu < \left(\frac{1}{2}\right)^{\frac{1}{n}} < m$.

STEP III

Section A: Pure Mathematics

Question 1

(i) We have

$$\begin{aligned}\dot{x} &= -x - y \\ \dot{y} &= x - y \quad ,\end{aligned}$$

hence $y = -x - \dot{x}$, and we have

$$\begin{aligned}\frac{d}{dt}(-x - \dot{x}) &= x - (-x - \dot{x}) \\ \dot{x} + \ddot{x} &= -x - x - \dot{x} \\ \ddot{x} + 2\dot{x} + 2x &= 0 \quad .\end{aligned}$$

We solve this by proposing the solution $x = Ce^{\lambda t}$, so λ satisfies

$$\begin{aligned}\lambda^2 + 2\lambda + 2 &= 0 \\ (\lambda + 1)^2 + 1 &= 0 \\ \implies \quad \lambda &= -1 \pm i \quad ,\end{aligned}$$

hence

$$\begin{aligned}x(t) &= e^{-1}(A\cos t + B\sin t) \\ \dot{x}(t) &= e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t) \quad .\end{aligned}$$

Now $x(0) = 1$, $y(0) = 0$ gives

$$\begin{aligned}x(0) = 1 \quad &\implies \quad A = 1 \\ \text{and} \quad -x(0) - \dot{x}(0) = 0 \quad &\implies \quad -A - (B - A) = 0 \quad \implies \quad B = 0 \quad .\end{aligned}$$

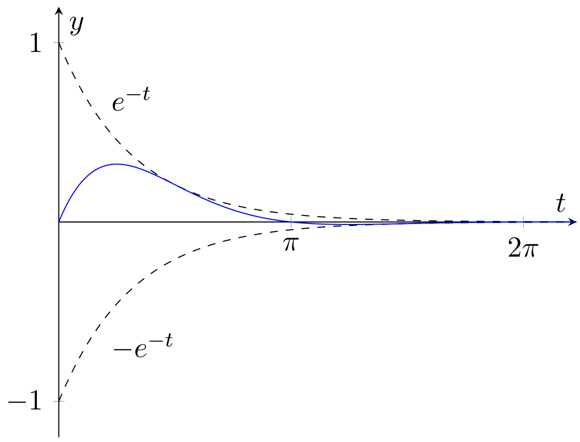
So

$$x(t) = e^{-t}\cos t \quad ,$$

and

$$\begin{aligned}y(t) &= -\dot{x}(t) - x(t) \\ &= -(-e^{-t}\sin t - e^{-t}\cos t) - e^{-t}\cos t \\ y(t) &= e^{-t}\sin t \quad .\end{aligned}$$

Sketching y as a function of t:

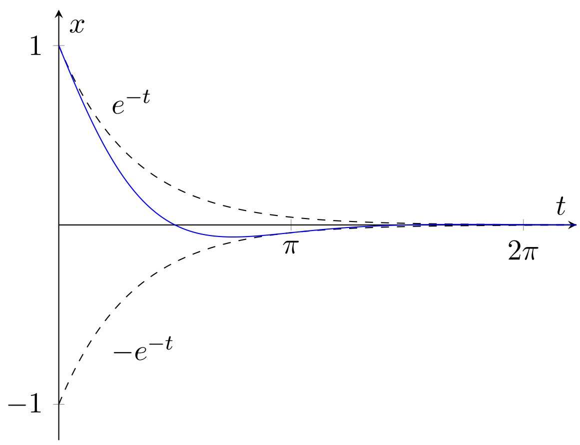


To find when y achieves its maximum, we use

$$\begin{aligned} \dot{y} = 0 \iff x(t) = y(t) &\iff \cos t = \sin t \\ &\iff \tan t = 1 \\ &\iff t = \tfrac{\pi}{4} + n\pi \quad , \end{aligned}$$

for some integer n. From our sketch, the maximum value of y occurs at the first instance $\dot{y} = 0$, which is $t = \frac{\pi}{4} \implies x = y = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}}$.

For completeness, we also sketch x as a function of t:

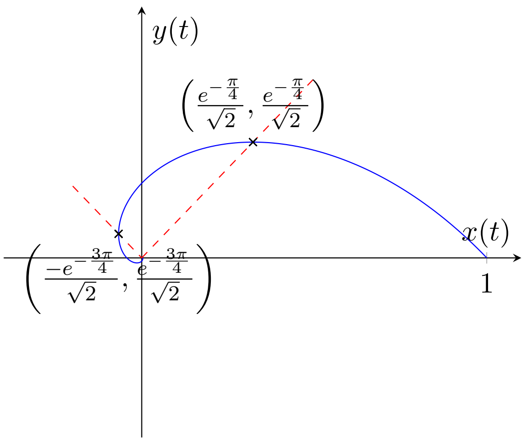


To find when x achieves its minimum, we use

$$\dot{x} = 0 \iff x(t) = -y(t) \iff \cos t = -\sin t$$
$$\iff \tan t = -1$$
$$\iff t = \tfrac{3\pi}{4} + n\pi \quad ,$$

for some integer n. From our sketch, the minimum value of x occurs at the first instance $\dot{x} = 0$, which is $t = \frac{3\pi}{4} \implies x = -\frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}, y = \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$.

We now sketch the trajectory of the particle. By comparing the forms of $x(t)$ and $y(t)$, to that of motion in a circle, we can recognise that the particle is travelling in a spiral about the origin, traversing anticlockwise, and that the radius of the spiral is given by e^{-t}:



(ii) Now we have

$$\dot{x} = -x$$
$$\dot{y} = x - y \quad ,$$

hence we can find $x(t)$ immediately:

$$\dot{x}(t) = -x(t) \implies x(t) = x(0)e^{-t} = e^{-t} \quad .$$

Then

$$\dot{y} = x - y = e^{-t} - y \implies \dot{y} + y = e^{-t}$$
$$e^t\dot{y} + e^t y = 1$$
$$\frac{d}{dt}\left(e^t y\right) = 1$$
$$e^t y(t) - y(0) = t - 0$$
$$y(t) = te^{-t} \quad .$$

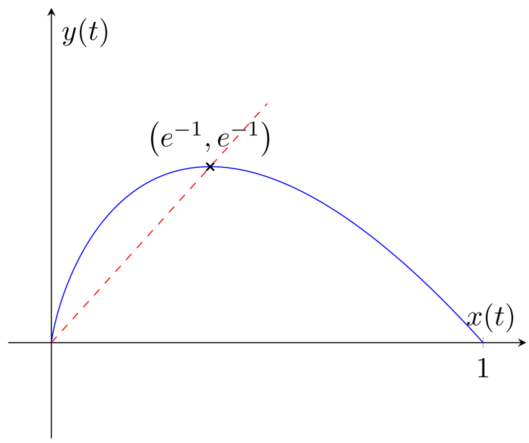
From these, we see that $x \to 0$ and $y \to 0$ as $t \to \infty$. x is monotonically decreasing for all $t \geqslant 0$, while

$$\dot{y} = 0 \iff x = y \iff t = 1 \quad ,$$

hence y achieves a maximum at $x = y = e^{-1}$. Further we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x - y}{-x} = \frac{e^{-t} - te^{-t}}{-e^{-t}} = t - 1 \quad ,$$

hence at $(x, y) = (1, 0)$ the trajectory has gradient -1, and as $t \to \infty$ the trajectory tends to the origin, where it is vertical ($\frac{dy}{dx} \to \infty$). Our sketch is



Question 2

(i) Setting $y = 0$, we have that for all x

$$f(x) = f(x)f(0) \quad ,$$

thus $f(0) = 1$ or $f(x) = 0$ for all x. If $f(x)$ is identically 0, then $f'(x)$ is also identically 0, but we are given that $k = f'(0) \neq 0$, hence we must have $f(0) = 1$.

We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f(x)f'(0) \\ f'(x) &= kf(x) \quad , \end{aligned}$$

which we can then solve:

$$\begin{aligned} f'(x) = kf(x) \implies f(x) &= f(0)e^{kx} \\ f(x) &= e^{kx} \quad . \end{aligned}$$

(ii) Setting $x = y = 0$, we have

$$\begin{aligned} g(0) &= \frac{2g(0)}{1 + g(0)^2} \\ g(0)^3 + g(0) &= 2g(0) \\ g(0)^3 - g(0) &= 0 \\ g(0)(g(0)^2 - 1) &= 0 \quad . \end{aligned}$$

Since we are given that $|g(x)| < 1$ for all x, we must have $g(0) = 0$. Then

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x) + g(h) - g(x) - g(x)^2 g(h)}{h(1 + g(x)g(h))} \\ &= (1 - g(x)^2) \lim_{h \to 0} \frac{g(h)}{h(1 + g(x)g(h))} \quad . \end{aligned}$$

We have that $\lim_{h\to 0}(1+g(x)g(h)) = 1$, and

$$\lim_{h\to 0}\frac{g(h)}{h} = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = g'(0) = k \quad ,$$

thus we can split the limit into two parts as follows:

$$g'(x) = (1-g(x)^2)\frac{\lim_{h\to 0}\left(\frac{g(h)}{h}\right)}{\lim_{h\to 0}(1+g(x)g(h))} = (1-g(x)^2)\frac{k}{1} = k(1-g(x)^2) \quad .$$

And now we can solve this:

$$\begin{aligned}
\frac{1}{1-g(x)^2}g'(x) &= k \\
\operatorname{arctanh}(g(x)) - \operatorname{arctanh}(g(0)) &= k(x-0) \\
\operatorname{arctanh}(g(x)) &= kx \\
g(x) &= \tanh(kx) \quad .
\end{aligned}$$

Question 3

(i) We have

$$\begin{cases} ax + by = x \\ cx + dy = y \end{cases} \implies \begin{cases} (a-1)x = -by \\ (d-1)y = -cx \end{cases} ,$$

thus

$$\begin{aligned} (a-1)x \cdot (d-1)y &= (-by) \cdot (-cx) = bcxy \\ ((a-1)(d-1) - bc)xy &= 0 \ . \end{aligned}$$

This gives three options: either $(a-1)(d-1) = bc$, or $x = 0$ for all points on L_1, or $y = 0$ for all points on L_1. If $x = 0$ for all points on L_1, then L_1 is the y-axis and

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ y \end{pmatrix} \quad \forall y \\ \implies by = 0 \ &, \ dy = y \quad \forall y \\ \implies b = 0 \ &, \ d = 1 \\ \implies (a-1)(d-1) - bc &= 0 \ . \end{aligned}$$

Similarly, if $y = 0$ for all points on L_1, then L_1 is the x-axis and

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} &= \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \forall x \\ \implies ax = x \ &, \ cx = 0 \quad \forall x \\ \implies a = 1 \ &, \ c = 0 \\ \implies (a-1)(d-1) - bc &= 0 \ . \end{aligned}$$

Thus in all cases we have $(a-1)(d-1) = bc$.

If L_1 does not pass through the origin, then L_1 is either of the form $y = mx + k$ for some $k \neq 0$, or of the form $x = k$ for some $k \neq 0$. If it is $y = mx + k$, $k \neq 0$, then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} &= \begin{pmatrix} x \\ mx + k \end{pmatrix} \quad \forall x \\ \implies ax + bmx + bk = x \ &, \ cx + dmx + dk = mx + k \quad \forall x \\ \implies a + bm = 1 \ , \ bk = 0 \ &, \ c + dm = m \ , \ dk = k \\ \implies a = 1 \ , \ b = 0 \ &, \ c = 0 \ , \ d = 1 \ , \end{aligned}$$

so then $\boldsymbol{A} = \boldsymbol{I}$, the 2×2 identity matrix.

Similarly if L_1 is $x = k$, $k \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k \\ y \end{pmatrix} = \begin{pmatrix} k \\ y \end{pmatrix} \quad \forall y$$

$$\implies \quad ak + by = k \quad , \quad ck + dy = y \quad \forall y$$

$$\implies \quad ak = k \quad , \quad b = 0 \quad , \quad ck = 0 \quad , \quad d = 1$$

$$\implies \quad a = 1 \quad , \quad b = 0 \quad , \quad c = 0 \quad , \quad d = 1 \quad ,$$

so again $\boldsymbol{A} = \boldsymbol{I}$.

(ii) If $b \neq 0$, consider the line $(a-1)x + by = 0$: we have

$$\begin{aligned} & (a-1)x + by = 0 \\ \implies & (a-1)(d-1)x + b(d-1)y = 0 \\ \implies & bcx + b(d-1)y = 0 \\ \implies & cx + (d-1)y = 0 \quad . \end{aligned}$$

so for all points on this line, we have

$$\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ,$$

that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad .$$

Thus in this case $\boldsymbol{A}$ has a line of invariant points, given by $(a-1)x + by = 0$ (or equivalently $cx + (d-1)y = 0$).

If $b = 0$, then $(a-1)(d-1) = bc$ requires $a = 1$, or $a \neq 1$ and $d = 1$. If $a = 1$, consider the line $cx + (d-1)y = 0$: we have

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \quad ,$$

thus for all points on the given line

$$\boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ cx + dy \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad .$$

Thus in this case $\boldsymbol{A}$ has a line of invariant points given by $cx + (d-1)y = 0$.
Finally, in the case $b = 0$, $a \neq 1$, $d = 1$, consider the line $x = 0$: we have

$$\boldsymbol{A} = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \quad ,$$

thus for all points with $x = 0$

$$\boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad .$$

Thus in this case $\boldsymbol{A}$ has a line of invariant points given by $x = 0$.

(iii) Since L_2 does not pass through the origin, we must have $k \neq 0$. By the fact that $\boldsymbol{A}$ transforms any given point on L_2 to some point on L_2, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ m\tilde{x} + k \end{pmatrix} \quad ,$$

where the x-coordinate of the new point, $\tilde{x}$, is some function of x, the starting x-coordinate. This gives

$$\begin{aligned} ax + bmx + bk &= \tilde{x} \\ cx + dmx + dk &= m\tilde{x} + k \quad , \end{aligned}$$

so after eliminating $\tilde{x}$, we must have

$$\begin{aligned} cx + dmx + dk &= m(ax + bmx + bk) + k \quad \forall x \\ \implies \quad c + dm = m(a + bm) \quad &, \quad dk = bmk + k \quad . \end{aligned}$$

Since $k \neq 0$, the second equation here gives $d = bm + 1$, so then we can eliminate m from the first equation as follows:

$$\begin{aligned} m(a + bm) &= c + dm \\ m(a + d - 1) &= c + dm \\ m(a - 1) &= c \\ \implies bm(a - 1) &= bc \\ (a - 1)(d - 1) &= bc \quad . \end{aligned}$$

Question 4

(i) We find the reflexive polynomials for $n = 1, 2, 3$ in turn.

$n = 1$: Degree one polynomials $P(x)$ of the given form are

$$P(x) = x - a_1 \quad ,$$

which has a_1 as a root, thus this is reflexive. Hence the reflexive polynomials of degree one are $P(x) = x - a_1$, where a_1 is any real number.

$n = 2$: Degree two polynomials of the given form are

$$P(x) = x^2 - a_1 x + a_2 \quad ,$$

and this is reflexive if and only if

$$x^2 - a_1 x + a_2 \equiv (x - a_1)(x - a_2) \quad .$$

By Vieta's formulae, this occurs if and only if

$$\begin{aligned} a_1 = a_1 + a_2 \quad &, \quad a_2 = a_1 a_2 \\ \iff \quad a_2 = 0 \quad &, \end{aligned}$$

hence the reflexive polynomials of degree two are $P(x) = x^2 - a_1 x$, where a_1 is any real number.

$n = 3$: Degree three polynomials of the given form are

$$P(x) = x^3 - a_1 x^2 + a_2 x - a_3 \quad ,$$

and this is reflexive if and only if

$$x^3 - a_1 x^2 + a_2 x - a^3 \equiv (x - a_1)(x - a_2)(x - a_3) \quad .$$

By Vieta's formulae, this occurs if and only if

$$\begin{aligned} & a_1 = a_1 + a_2 + a_3 \quad , \quad a_2 = a_1 a_2 + a_2 a_3 + a_3 a_1 \quad , \quad a_3 = a_1 a_2 a_3 \\ \iff & \quad a_3 = -a_2 \quad , \quad a_2 = a_1 a_2 - a_2^2 - a_1 a_2 \quad , \quad a_2 = a_1 a_2^2 \\ \iff & \quad a_3 = -a_2 \quad , \quad a_2^2 + a_2 = 0 \quad , \quad a_2(1 - a_1 a_2) = 0 \quad . \end{aligned}$$

From the second equation here we have two cases: $a_2 = 0$ and $a_2 = -1$. If $a_2 = 0$, then the third equation is already satisfied, and $a_3 = -a_2 = 0$, giving

$$P(x) = x^3 - a_1 x^2 \quad ,$$

where a_1 can be any real number. If $a_2 = -1$, then $a_3 = -1$ and the third equation gives $a_1 = -1$, so

$$P(x) = x^3 + x^2 - x - 1 \quad .$$

Thus the reflexive polynomials of degree less than or equal to three are

$$\begin{aligned} n=1: &\quad P(x) = x - a_1 \quad , \\ n=2: &\quad P(x) = x^2 - a_1 x \quad , \\ n=3: &\quad P(x) = x^3 - a_1 x^2 \quad , \\ \text{and} &\quad P(x) = x^3 + x^2 - x - 1 \quad , \end{aligned}$$

where a_1 can be any real number.

(ii) For a reflexive polynomial with $n > 3$:

$$\begin{aligned} P(x) &= x^n - a_1 x^{n-1} + a_2 x^{n-2} + \cdots + (-1)^n a_n \\ &\equiv (x - a_1)(x - a_2)(x - a_3)\cdots(x - a_n) \quad , \end{aligned}$$

Vieta's formulae for the coefficients of x^{n-1} and x^{n-2} give

$$\begin{aligned} a_1 &= a_1 + a_2 + a_3 + \cdots + a_n \quad , \\ a_2 &= a_1 a_2 + a_1 a_3 + \cdots + a_1 a_n + a_2 a_3 + \cdots + a_{n-1} a_n \quad , \end{aligned}$$

and we can rewrite the second equation here as

$$a_2 = \frac{1}{2}\left((a_1 + a_2 + \cdots + a_n)^2 - (a_1^2 + a_2^2 + \cdots + a_n^2)\right) \quad .$$

Thus

$$\begin{aligned} a_2 &= \frac{1}{2}\left(a_1^2 - (a_1^2 + a_2^2 + \cdots + a_n^2)\right) \\ 2a_2 &= -a_2^2 - a_3^2 - \cdots - a_n^2 \quad . \end{aligned}$$

Completing the square in this,

$$\begin{aligned} a_2^2 + 2a_2 &= -a_3^2 - \cdots - a_n^2 \\ (a_2 + 1)^2 &= 1 - a_3^2 - \cdots - a_n^2 \quad , \end{aligned}$$

hence if all the coefficients are integers and $a_n \neq 0$, we will have $a_n^2 \geqslant 1$ and the only way for the above equation to hold is if $a_n^2 = 1$ (so $a_n = \pm 1$) $a_2 = -1$ and $a_3 = \cdots = a_{n-1} = 0$, giving

$$P(x) = x^n - a_1 x^{n-1} - x^{n-2} \pm 1 \quad ,$$

but then $a_3 = 0$ is not a root of this $P(x)$, and so it is not reflexive. Thus if all the coefficients of a reflexive polynomial of degree n are integers and $a_n \neq 0$, then $n \leqslant 3$.

(iii) If all the coefficients of a reflexive polynomial of degree n are integers and $a_n = 0$,

$$\begin{aligned} P(x) &= x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^{n-1} a_{n-1} x \\ &= x \left(x^{n-1} - a_1 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1} \right) \quad , \end{aligned}$$

and Vieta's formulae for the coefficients $a_1, \ldots, a_{n-1}$ become exactly those for the polynomial of degree $n-1$

$$P(x) = x^{n-1} - a_1 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}$$

to be reflexive (since all terms involving a_n will vanish, and the final equation $a_n = a_1 a_2 \cdots a_n$ is consistent). Hence the degree n polynomial was reflexive if and only if this degree $n-1$ polynomial is reflexive. Repeating this logic, with the result from part (ii), any reflexive polynomial of degree $n > 3$ with integer coefficients must be of the form $x^r Q(x)$ where $r = 0, 1, 2, \ldots$ and $Q(x)$ is a reflexive polynomial of degree less than or equal to three with integer coefficients.

From our results in part (i) then, we see that the reflexive polynomials with integer coefficients are

$$\begin{aligned} P(x) &= x^{r+1} - a_1 x^r = x^r (x - a_1) \\ \text{and} \quad P(x) &= x^{r+3} + x^{r+2} - x^{r+1} - x^r = x^r (x+1)^2 (x-1) \quad , \end{aligned}$$

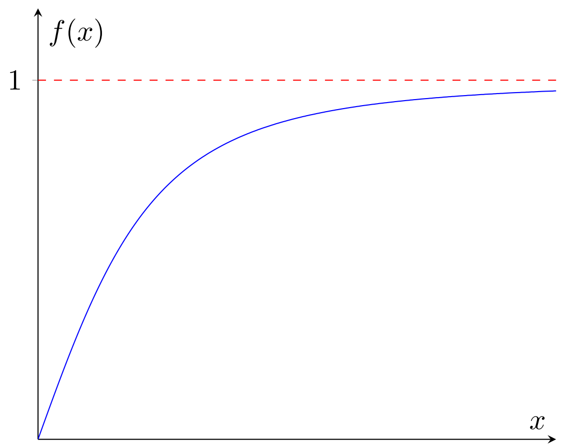
where $r = 0, 1, 2, \ldots$ and a_1 can be any integer.

Question 5

(i) We have

$$f(x) = \frac{x}{\sqrt{x^2+p}} \implies f'(x) = \frac{1}{\sqrt{x^2+p}} - \frac{x \cdot \frac{1}{2} \cdot 2x}{(x^2+p)^{\frac{3}{2}}} = \frac{x^2+p-x^2}{(x^2+p)^{\frac{3}{2}}}$$
$$= \frac{p}{(x^2+p)^{\frac{3}{2}}} \quad .$$

We can see that $0 \leqslant f(x) < 1$ for $x \geqslant 0$, with $f(0) = 0$, and $f(x) \to 1$ as $x \to \infty$; further $f'(0) = \frac{1}{\sqrt{p}}$, $f'(x) > 0$ for all $x \geqslant 0$, and $f'(x) \to 0$ as $x \to \infty$:



(ii) *Using the correct substitution, as noted in the erratum:*
We have

$$y = \frac{cx}{\sqrt{x^2+p}} \implies \frac{dy}{dx} = \frac{cp}{(x^2+p)^{\frac{3}{2}}} \quad ,$$

hence

$$I = \int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}} dy = \int \frac{1}{\left(b^2 - \frac{c^2x^2}{x^2+p}\right)\sqrt{c^2 - \frac{c^2x^2}{x^2+p}}} \frac{cp}{(x^2+p)^{\frac{3}{2}}} dx$$
$$= \int \frac{cp}{(b^2x^2+b^2p-c^2x^2)\sqrt{c^2x^2+c^2p-c^2x^2}} dx$$
$$= \int \frac{\sqrt{p}}{(b^2-c^2)x^2+b^2p} dx \quad ,$$

thus if we take $p = 1$, we have

$$I = \int \frac{1}{(b^2-c^2)x^2+b^2} dx \quad .$$

Setting $b = \sqrt{3}$, $c = \sqrt{2}$, the above result gives

$$\int \frac{1}{(3-y^2)\sqrt{2-y^2}}dy = \int \frac{1}{x^2+3}dx \quad .$$

To find the limits of the transformed integral, we invert the substitution:

$$\begin{aligned} y = \frac{cx}{\sqrt{x^2+p}} \quad \Longrightarrow \quad y^2 &= \frac{c^2x^2}{x^2+p} \\ x^2y^2 + py^2 &= c^2x^2 \\ x^2(c^2-y^2) &= py^2 \\ x &= y\sqrt{\frac{p}{c^2-y^2}} \quad , \end{aligned}$$

thus in this case we have

$$x = \frac{y}{\sqrt{2-y^2}} \quad ,$$

and so when $y = 1$ we have $x = 1$, and as $y \to \sqrt{2}$, we have $x \to \infty$. Hence

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}dy &= \int_1^{\infty} \frac{1}{x^2+3}dx \\ &= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_1^{\infty} \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \\ &= \frac{\pi}{3\sqrt{3}} \quad . \end{aligned}$$

Letting $y = \frac{1}{u}$, we have

$$\begin{aligned} \int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}dy &= \int_1^{\sqrt{2}} \frac{\frac{1}{u}}{\left(\frac{3}{u^2}-1\right)\sqrt{\frac{2}{u^2}-1}}\frac{1}{u^2}du \\ &= \int_1^{\sqrt{2}} \frac{1}{(3-u^2)\sqrt{2-u^2}}du \\ &= \frac{\pi}{3\sqrt{3}} \quad , \end{aligned}$$

since it is the same integral as before.

(iii) Using the substitution $y = \frac{cx}{\sqrt{x^2+p}}$, we have

$$\begin{aligned}
\int \frac{1}{(3y^2-1)\sqrt{2y^2-1}}dy &= \int \frac{1}{\left(\frac{3c^2x^2}{x^2+p}-1\right)\sqrt{\frac{2c^2x^2}{x^2+p}-1}}\frac{cp}{(x^2+p)^{\frac{3}{2}}}dx \\
&= \int \frac{cp}{(3c^2x^2-x^2-p)\sqrt{2c^2x^2-x^2-p}}dx \\
&= \int \frac{cp}{((3c^2-1)x^2-p)\sqrt{(2c^2-1)x^2-p}}dx \quad ,
\end{aligned}$$

hence if we take $c = \frac{1}{\sqrt{2}}$ and $p = -1$, we get

$$\int \frac{1}{(3y^2-1)\sqrt{2y^2-1}}dy = \int \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{2}x^2+1}dx = \sqrt{2}\int \frac{-1}{x^2+2}dx \quad .$$

From when we inverted the substitution before, we have

$$x = y\sqrt{\frac{p}{c^2-y^2}} = \frac{y}{\sqrt{y^2-\frac{1}{2}}} \quad ,$$

and so as $y \to \frac{1}{\sqrt{2}}$ we have $x \to \infty$, and at $y = 1$, we have $x = \sqrt{2}$. Hence

$$\begin{aligned}
\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2-1)\sqrt{2y^2-1}}dy &= \sqrt{2}\int_{\sqrt{2}}^{\infty} \frac{1}{2+x^2}dx \\
&= \left[\arctan\left(\frac{x}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\infty} \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4} \quad .
\end{aligned}$$

Question 6

We have

$$\begin{aligned} zz^* - az^* - a^*z + aa^* - r^2 &= 0 \\ (z-a)(z^*-a^*) &= r^2 \\ |z-a|^2 &= r^2 \quad , \end{aligned}$$

thus C (the locus of P) is a circle of radius r, centred at a.

(i) $\omega = \frac{1}{z}$ gives

$$\begin{aligned} \frac{1}{\omega\omega^*} - \frac{a}{\omega^*} - \frac{a^*}{\omega} + aa^* - r^2 &= 0 \\ (aa^* - r^2)\omega\omega^* - a\omega - a^*\omega^* + 1 &= 0 \\ \omega\omega^* - \frac{a}{|a|^2 - r^2}\omega - \frac{a^*}{|a|^2 - r^2}\omega^* &= -\frac{1}{|a|^2 - r^2} \\ \left(\omega - \frac{a^*}{|a|^2 - r^2}\right)\left(\omega^* - \frac{a}{|a|^2 - r^2}\right) &= \frac{|a|^2}{(|a|^2 - r^2)^2} - \frac{1}{|a|^2 - r^2} \\ \left|\omega - \frac{a^*}{|a|^2 - r^2}\right|^2 &= \frac{|a|^2 - (|a|^2 - r^2)}{(|a|^2 - r^2)^2} \\ \left|\omega - \frac{a^*}{|a|^2 - r^2}\right|^2 &= \frac{r^2}{(|a|^2 - r^2)^2} \quad , \end{aligned}$$

thus C' (the locus of Q) is a circle of radius $\frac{r}{||a|^2 - r^2|}$, centred at $\frac{a^*}{|a|^2 - r^2}$. If C and C' are the same circle, then their radii must be the same, so

$$\begin{aligned} r &= \frac{r}{||a|^2 - r^2|} \\ \iff ||a|^2 - r^2| &= 1 \\ \iff (|a|^2 - r^2)^2 &= 1 \quad , \end{aligned}$$

(hence $|a|^2 - r^2 = \pm 1$); and their centres must be coincident, so

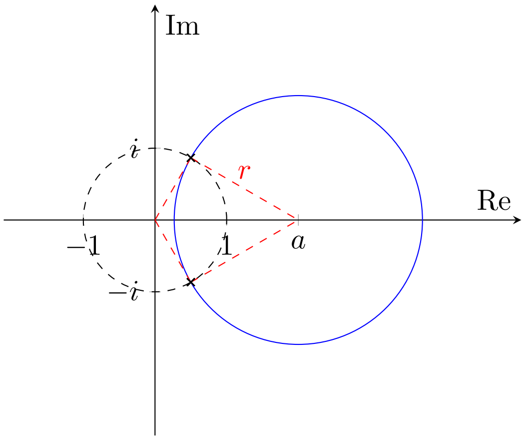
$$a = \frac{a^*}{|a|^2 - r^2} = \pm a^* \quad .$$

Letting $a = u + iv$, where u, v are real, we see that

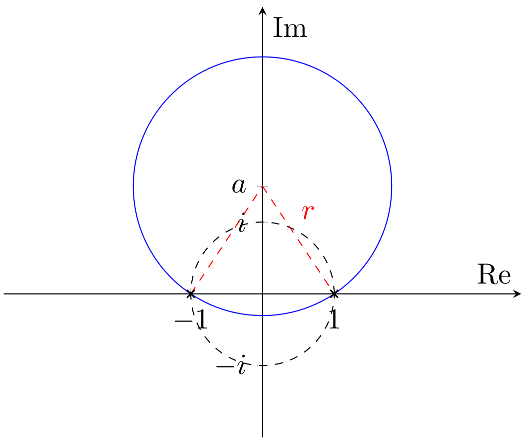
$$\begin{aligned} a = a^* &\implies u + iv = u - iv \iff v = 0 \iff a \text{ is real} \\ a = -a^* &\implies u + iv = -u + iv \iff u = 0 \iff a \text{ is imaginary} \quad , \end{aligned}$$

thus either a is real or a is imaginary.

In the case a is real, we have $|a|^2 - r^2 = 1$, hence $|a|^2 = r^2 + 1$. Thus the circle does not contain the origin, but intersects the unit circle centred at the origin, and where these circles intersect their radii are perpendicular:



In the case a is imaginary, we have $|a|^2 - r^2 = -1$, hence $r^2 = |a|^2 + 1$ thus the circle contains the origin, and intersects the unit circle centred at the origin at the points $z = \pm 1$:



(ii) If instead $\omega = \frac{1}{z^*}$, then we have

$$\begin{aligned} \left|\omega^* - \frac{a^*}{|a|^2 - r^2}\right|^2 &= \frac{r^2}{(|a|^2 - r^2)^2} \\ \left|\omega - \frac{a}{|a|^2 - r^2}\right|^2 &= \frac{r^2}{(|a|^2 - r^2)^2} \quad , \end{aligned}$$

thus now C' has the same radius as before, but is centred at $\frac{a}{|a|^2 - r^2}$. Now if C and C' are coincident, comparing radii gives $|a|^2 - r^2 = \pm 1$ as before, but comparing centres gives

$$\begin{aligned} a &= \frac{a}{|a|^2 - r^2} \\ \iff \quad a(1 - (|a|^2 - r^2)) &= 0 \quad , \end{aligned}$$

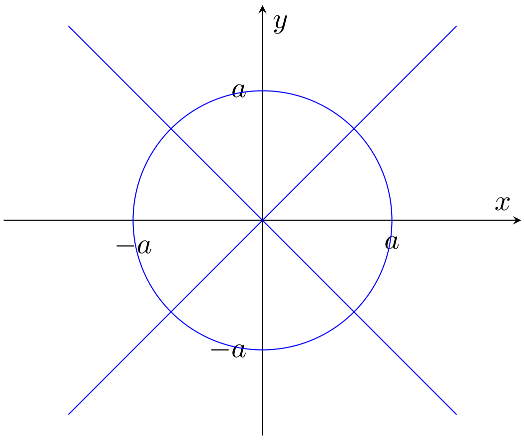
so if $|a|^2 - r^2 = -1$, then we must have $a = 0$, and if $|a|^2 - r^2 = 1$, then a can be any complex number with $|a| = \sqrt{1 + r^2}$. Thus it is not the case that either a is real or a is imaginary.

Question 7

(i) In the case $a = b$, we have

$$\begin{aligned} y^2(y^2 - a^2) &= x^2(x^2 - a^2) \\ y^4 - x^4 - a^2(y^2 - x^2) &= 0 \\ (y^2 - x^2)(x^2 + y^2 - a^2) &= 0 \\ (y - x)(y + x)(x^2 + y^2 - a^2) &= 0 \quad , \end{aligned}$$

thus the curve is comprised of the lines $y = x$, $y = -x$, and the circle $x^2 + y^2 = a^2$:



(ii) Now we have

$$y^2(y^2 - 5) = x^2(x^2 - 4) \quad .$$

(a) We rearrange the equation as follows:

$$x^4 - 4x^2 - y^2(y^2 - 5) = 0 \quad .$$

This is a quadratic equation in x^2. For there to be a point on the curve with a given y value, the discriminant of the quadratic must be nonnegative:

$$\begin{aligned} 16 + 4y^2(y^2 - 5) &\geqslant 0 \\ 4y^4 - 20y^2 + 16 &\geqslant 0 \\ y^4 - 5y^2 + 4 &\geqslant 0 \\ (y^2 - 4)(y^2 - 1) &\geqslant 0 \quad , \end{aligned}$$

thus

$$\begin{aligned} & y^2 \geqslant 4 \quad \Longrightarrow \quad y \geqslant 2 \\ \text{or} \quad & y^2 \leqslant 1 \quad \Longrightarrow \quad 0 \leqslant y \leqslant 1 \quad , \end{aligned}$$

since in each case we are only concerned with $y \geqslant 0$.

(b) For small x and y, $y^2(y^2-5) \approx -5y^2$ and $x^2(x^2-4) \approx -4x^2$, thus very close to the origin the curve can be approximated by

$$-5y^2 = -4x^2 \implies y = \pm\frac{2}{\sqrt{5}}x \quad .$$

For $x \geqslant 0$ and $y \geqslant 0$, only the line with positive gradient is relevant. For large x and y, $y^2(y^2-5) \approx y^4$ and $x^2(x^2-4) \approx x^4$, thus very far away from the origin the curve can be approximated by

$$y^4 = x^4 \implies y = \pm x \quad .$$

Again for $x \geqslant 0$ and $y \geqslant 0$, only the line with positive gradient is relevant.

(c) Differentiating, we have

$$(4y^3 - 10y)\frac{dy}{dx} = 4x^3 - 8x \implies \frac{dy}{dx} = \frac{2x^3-4x}{2y^3-5y}$$
$$\text{or equivalently} \quad \frac{dx}{dy} = \frac{2y^3-5y}{2x^3-4x} \quad .$$

Where the tangent to the curve is parallel to the x-axis, we have $\frac{dy}{dx} = 0$, hence we want

$$\begin{aligned} & 2x^3 - 4x = 0 \text{ and } 2y^3 - 5y \neq 0 \\ \implies & x(x-\sqrt{2})(x+\sqrt{2}) = 0 \text{ and } y(\sqrt{2}y-\sqrt{5})(\sqrt{2}y+\sqrt{5}) \neq 0 \\ \implies & x(x-\sqrt{2}) = 0 \text{ and } y(\sqrt{2}y-\sqrt{5}) \neq 0 \quad , \end{aligned}$$

since we are only concerned with $x \geqslant 0$, $y \geqslant 0$. Substituting $x = 0$ gives

$$y^2(y^2-5) = 0 \implies y = 0 \text{ or } y = \sqrt{5} \quad ,$$

and by the above we reject $y = 0$. Substituting $x = \sqrt{2}$ gives

$$y^4 - 5y^2 = -4 \implies (y^2-1)(y^2-4) \implies y = 1 \text{ or } y = 2 \quad .$$

Hence the tangent to the curve is parallel to the x-axis at $(0, \sqrt{5})$, $(\sqrt{2}, 1)$, and $(\sqrt{2}, 2)$.

Where the tangent to the curve is parallel to the y-axis, we have $\frac{dx}{dy} = 0$, hence we want

$$\begin{aligned} & 2y^3 - 5y = 0 \text{ and } 2x^3 - 4x \neq 0 \\ \implies & y(\sqrt{2}y-\sqrt{5}) = 0 \text{ and } x(x-\sqrt{2}) \neq 0 \quad , \end{aligned}$$

since we are only concerned with $x \geqslant 0$, $y \geqslant 0$. Substituting $y = 0$ gives

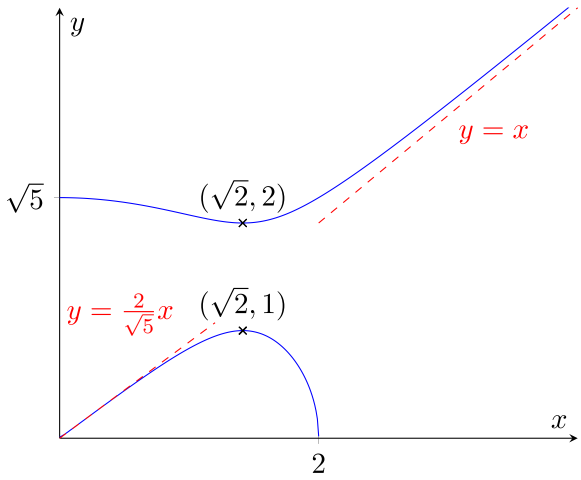
$$x^2(x^2-4) = 0 \implies x = 0 \text{ or } x = 2 \quad ,$$

and by the above we reject $x = 0$.

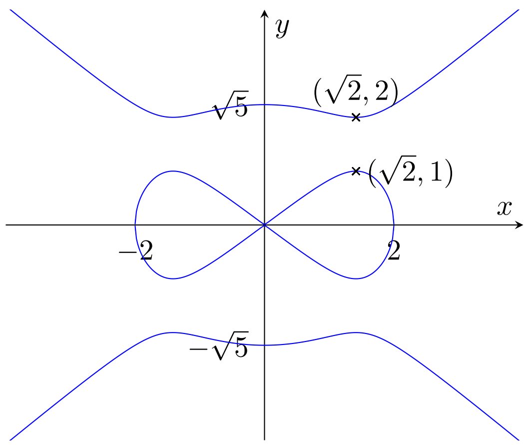
Substituting $y = \sqrt{\frac{5}{2}}$ gives

$$x^4 - 4x^2 = -\frac{25}{4} \implies 4x^4 - 16x^2 + 25 = 0 \quad ,$$

and the discriminant of this quadratic (in x^2) is $16^2 - 16 \cdot 25 < 0$, thus this offers no solutions. Hence the tangent to the curve is parallel to the y-axis only at $(2, 0)$. Our sketch for $x \geqslant 0$, $y \geqslant 0$ is as follows.



(iii) Since the equation of the curve is in x^2 and y^2 only, the full graph is gained from the above sketch by using the fact that the graph is symmetrical in both axes:



Question 8

(i) Let M be the midpoint of AB and N be the midpoint of BC. We have that vectors $\overrightarrow{MV}$ and $\overrightarrow{AB}$ lie in the plane AVB, with

$$\overrightarrow{MV} = |MV|(\cos\alpha\boldsymbol{j} + \sin\alpha\boldsymbol{k}) \quad , \quad \overrightarrow{AB} = |AB|\boldsymbol{i} \quad ;$$

hence a unit vector that is perpendicular to the face AVB is

$$\boldsymbol{i} \times (\cos\alpha\boldsymbol{j} + \sin\alpha\boldsymbol{k}) = -\sin\alpha\boldsymbol{j} + \cos\alpha\boldsymbol{k} \quad .$$

Similarly, vectors $\overrightarrow{NV}$ and $\overrightarrow{BC}$ lie in the plane BVC, with

$$\overrightarrow{NV} = |NV|(-\cos\beta\boldsymbol{i} + \sin\beta\boldsymbol{k}) \quad , \quad \overrightarrow{BC} = |BC|\boldsymbol{j} \quad ;$$

hence a unit vector that is perpendicular to the face BVC is

$$\boldsymbol{j} \times (-\cos\beta\boldsymbol{i} + \sin\beta\boldsymbol{k}) = \sin\beta\boldsymbol{i} + \cos\beta\boldsymbol{k} \quad .$$

These unit vectors point out of the pyramid from the faces AVB and BVC respectively. We have that the acute angle between these faces is θ, and so

$$\cos\theta = (-\sin\alpha\boldsymbol{j} + \cos\alpha\boldsymbol{k}) \cdot (\sin\beta\boldsymbol{i} + \cos\beta\boldsymbol{k}) = \cos\alpha\cos\beta \quad .$$

(ii) Let P be the centre of the base. We have

$$|MP| = |PV|\cot\alpha \quad , \quad |BM| = |NP| = |PV|\cot\beta \quad , \quad |BP| = |PV|\cot\phi \quad ,$$

thus using Pythagoras' theorem on triangle BMP we have

$$\begin{aligned} |BP|^2 &= |MP|^2 + |BM|^2 \\ \implies |PV|^2\cot^2\phi &= |PV|^2\cot^2\alpha + |PV|^2\cot^2\beta \\ \cot^2\phi &= \cot^2\alpha + \cot^2\beta \quad . \end{aligned}$$

For any angle ψ we have

$$\cot^2\psi = \frac{\cos^2\psi}{\sin^2\psi} = \frac{\cos^2\psi}{1-\cos^2\psi} \quad \text{and} \quad \cos^2\psi = \frac{\cot^2\psi}{\csc^2\psi} = \frac{\cot^2\psi}{1+\cot^2\psi} \quad ,$$

hence

$$\begin{aligned} \cos^2\phi = \frac{\cot^2\phi}{1+\cot^2\phi} &= \frac{\cot^2\alpha + \cot^2\beta}{1+\cot^2\alpha+\cot^2\beta} = \frac{\frac{\cos^2\alpha}{1-\cos^2\alpha} + \frac{\cos^2\beta}{1-\cos^2\beta}}{1 + \frac{\cos^2\alpha}{1-\cos^2\alpha} + \frac{\cos^2\beta}{1-\cos^2\beta}} \\ &= \frac{\cos^2\alpha(1-\cos^2\beta) + \cos^2\beta(1-\cos^2\alpha)}{(1-\cos^2\alpha)(1-\cos^2\beta) + \cos^2\alpha(1-\cos^2\beta) + \cos^2\beta(1-\cos^2\alpha)} \quad . \end{aligned}$$

Simplifying, using the result from (i), we have

$$\begin{aligned}\cos^2\phi &= \frac{\cos^2\alpha - \cos^2\theta + \cos^2\beta - \cos^2\theta}{1 - \cos^2\alpha - \cos^2\beta + \cos^2\theta + \cos^2\alpha - \cos^2\theta + \cos^2\beta - \cos^2\theta} \\ &= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta} \quad .\end{aligned}$$

Rearranging the numerator, we have

$$\begin{aligned}\cos^2\phi &= \frac{(\cos\alpha - \cos\beta)^2 + 2\cos\alpha\cos\beta - 2\cos^2\theta}{1 - \cos^2\theta} \\ &= \frac{(\cos\alpha - \cos\beta)^2 + 2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} \\ &\geqslant \frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} \quad .\end{aligned}$$

We can simplify this further:

$$\cos^2\phi \geqslant \frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} = \frac{2\cos\theta(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)} = \frac{2\cos\theta}{1 + \cos\theta}$$

(since $\theta > 0$ and θ is acute, thus $\cos\theta \neq 1$). We have $0 < \theta < \frac{1}{2}\pi$, hence $0 < \cos\theta < 1$ and so

$$1 + \cos\theta < 2 \implies \frac{2}{1 + \cos\theta} > 1 \text{ and } 0 < \cos\theta < 1 \implies \cos\theta > \cos^2\theta \quad .$$

Using these we find

$$\begin{aligned}&\cos^2\phi \geqslant \frac{2}{1 + \cos\theta}\cos\theta > 1 \cdot \cos^2\theta \\ &\cos^2\phi > \cos^2\theta \quad .\end{aligned}$$

Since we also have $0 < \phi < \frac{\pi}{2}$ we can deduce

$$\begin{aligned}&\cos^2\phi > \cos^2\theta \\ \implies &\cos\phi > \cos\theta \\ \implies &\phi < \theta \quad .\end{aligned}$$

Section B: Mechanics

Question 9

(i) The position of the particle relative to O is $a\sin\theta\boldsymbol{i} + a\cos\theta\boldsymbol{j}$, and the position of O relative to its initial position is $-s\boldsymbol{i}$, hence

$$\boldsymbol{r} = (a\sin\theta - s)\boldsymbol{i} + a\cos\theta\boldsymbol{j} \quad ,$$

and differentiating with respect to time:

$$\dot{\boldsymbol{r}} = (a\dot{\theta}\cos\theta - \dot{s})\boldsymbol{i} - a\dot{\theta}\sin\theta\boldsymbol{j} \quad .$$

From this, we see that the horizontal velocity of the particle is $a\dot{\theta}\cos\theta - \dot{s}$ and the horizontal velocity of the hemisphere is $-\dot{s}$. The initial horizontal momentum of the particle and the hemisphere is zero, hence conservation of linear momentum gives

$$\begin{aligned} 0 &= m(a\dot{\theta}\cos\theta - \dot{s}) - M\dot{s} \\ \dot{s}(m+M) &= ma\dot{\theta}\cos\theta \\ \dot{s} &= \frac{m}{m+M}a\dot{\theta}\cos\theta \\ &= \left(1 - \frac{M}{m+M}\right)a\dot{\theta}\cos\theta \\ &= (1-k)a\dot{\theta}\cos\theta \quad . \end{aligned}$$

Substituting this back into our expression for $\dot{\boldsymbol{r}}$ gives

$$\begin{aligned} \dot{\boldsymbol{r}} &= (a\dot{\theta}\cos\theta - (1-k)a\dot{\theta}\cos\theta)\boldsymbol{i} - a\dot{\theta}\sin\theta\boldsymbol{j} \\ &= a\dot{\theta}(k\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j}) \quad . \end{aligned}$$

(ii) The loss in gravitational potential energy of the particle is

$$-\Delta E_{GP} = mga - mga\cos\theta = mga(1-\cos\theta) \quad .$$

The gain in kinetic energy of the particle and hemisphere combined is

$$\begin{aligned} \Delta E_K &= \frac{1}{2}m(a\dot{\theta})^2(k^2\cos^2\theta + \sin^2\theta) + \frac{1}{2}M\dot{s}^2 \\ &= \frac{1}{2}a^2\dot{\theta}^2(m(k^2\cos^2\theta + \sin^2\theta) + M(1-k)^2\cos^2\theta) \quad . \end{aligned}$$

Thus conservation of energy gives

$$\begin{aligned} mga(1-\cos\theta) &= \frac{1}{2}a^2\dot{\theta}^2(m(k^2\cos^2\theta + \sin^2\theta) + M(1-k)^2\cos^2\theta) \\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left(k^2\cos^2\theta + \sin^2\theta + \frac{M}{m}(1-k)^2\cos^2\theta\right) \quad , \end{aligned}$$

and we have

$$\frac{M}{m} = \frac{M/(m+M)}{m/(m+M)} = \frac{k}{1-k} \quad ,$$

thus

$$\begin{aligned}
2g(1-\cos\theta) &= a\dot{\theta}^2(k^2\cos^2\theta + \sin^2\theta + k(1-k)\cos^2\theta) \\
&= a\dot{\theta}^2(k^2\cos^2\theta + \sin^2\theta + k\cos^2\theta - k^2\cos^2\theta) \\
&= a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta) \quad .
\end{aligned}$$

(iii) Differentiating with respect to time again

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \frac{d}{dt}\Big(a\dot{\theta}(k\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j})\Big) \\
&= a\ddot{\theta}(k\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j}) + a\dot{\theta}^2(-k\sin\theta\boldsymbol{i} - \cos\theta\boldsymbol{j}) \quad .
\end{aligned}$$

The vector in the first term here is perpendicular to $\sin\theta\boldsymbol{i} + k\cos\theta\boldsymbol{j}$, thus taking a dot product with the suggested vector gives

$$\begin{aligned}
\ddot{\boldsymbol{r}}\cdot(\sin\theta\boldsymbol{i} + k\cos\theta\boldsymbol{j}) &= a\dot{\theta}^2(-k\sin\theta\boldsymbol{i} - \cos\theta\boldsymbol{j})\cdot(\sin\theta\boldsymbol{i} + k\cos\theta\boldsymbol{j}) \\
&= a\dot{\theta}^2(-k\sin^2\theta - k\cos^2\theta) \\
&= -ak\dot{\theta}^2 \quad .
\end{aligned}$$

At time T, the particle loses contact with the hemisphere and is accelerating only under gravity, hence at time T we have $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$. Hence, evaluating the above equation at time T gives

$$\begin{aligned}
-g\boldsymbol{j}\cdot(\sin\alpha\boldsymbol{i} + k\cos\alpha\boldsymbol{j}) &= -ak\dot{\theta}^2 \\
-gk\cos\alpha &= -ak\dot{\theta}^2 \\
a\dot{\theta}^2 &= g\cos\alpha \quad .
\end{aligned}$$

Using this with the result from part (ii) evaluated at $\theta = \alpha$ then, we have

$$\begin{aligned}
2g(1-\cos\alpha) &= a\dot{\theta}^2(k\cos^2\alpha + \sin^2\alpha) \\
2g(1-\cos\alpha) &= g\cos\alpha(k\cos^2\alpha + \sin^2\alpha) \\
2 - 2\cos\alpha &= k\cos^3\alpha + \cos\alpha(1-\cos^2\alpha) \\
(1-k)\cos^3\alpha - 3\cos\alpha + 2 &= 0 \quad .
\end{aligned}$$

Finally, we know that $k < 1$ and α is acute, thus $\cos\alpha > 0$ and

$$\begin{aligned}
& 3\cos\alpha - 2 = (1-k)\cos^3\alpha \\
\implies & 3\cos\alpha - 2 > 0 \quad ,
\end{aligned}$$

and so $\cos\alpha > \frac{2}{3}$.

Question 10

Let the mass of each sphere be m.

(i) Consider momentum in the direction perpendicular to the final motion of Q: the initial and final momenta of Q in this direction are both zero; the initial momentum of P in this direction is $mu\sin\alpha$, and the final momentum of P in this direction is $mv\sin(\alpha+\theta)$, hence by conservation of momentum

$$\begin{aligned} mu\sin\alpha &= mv\sin(\alpha+\theta) \\ u\sin\alpha &= v\sin(\alpha+\theta) \quad . \end{aligned}$$

Now consider momentum in the direction perpendicular to the initial motion of P: the initial momenta of P and Q in this direction are both zero; the final momentum of P in this direction is $-mv\sin\theta$, and the final momentum of Q in this direction is $mw\sin\alpha$, hence by conservation of momentum

$$\begin{aligned} 0 &= -mv\sin\theta + mw\sin\alpha \\ \implies \quad w &= v\frac{\sin\theta}{\sin\alpha} \quad . \end{aligned}$$

(ii) Using the restitution equation (so, in the direction of the final motion of Q) with the above relations

$$\begin{aligned} w - v\cos(\alpha+\theta) &= eu\cos\alpha \\ w\sin\alpha - v\cos(\alpha+\theta)\sin\alpha &= eu\sin\alpha\cos\alpha \\ v\sin\theta - v\cos(\alpha+\theta)\sin\alpha &= ev\sin(\alpha+\theta)\cos\alpha \\ \sin\theta &= \cos(\alpha+\theta)\sin\alpha + e\sin(\alpha+\theta)\cos\alpha \quad . \end{aligned}$$

Expanding this

$$\sin\theta = \sin\alpha\cos\alpha\cos\theta - \sin^2\alpha\sin\theta + e\sin\alpha\cos\alpha\cos\theta + e\cos^2\alpha\sin\theta \quad ,$$

thus

$$\begin{aligned} (1+\sin^2\alpha - e\cos^2\alpha)\sin\theta &= \sin\alpha\cos\alpha\cos\theta + e\sin\alpha\cos\alpha\cos\theta \\ (2\sin^2\alpha + (1-e)\cos^2\alpha)\sin\theta &= (1+e)\sin\alpha\cos\alpha\cos\theta \\ (2\tan^2\alpha + 1 - e)\tan\theta &= (1+e)\tan\alpha \\ \tan\theta &= \frac{(1+e)\tan\alpha}{2\tan^2\alpha + 1 - e} \quad . \end{aligned}$$

Let $\tan\alpha = t$, then

$$\tan\theta = \frac{(1+e)t}{2t^2+1-e} \quad ,$$

and we want to find the maximum value of this for $t > 0$. At $\alpha = 0$ we have $t = 0$ and $\tan\theta = 0$, and as $\alpha \to \frac{\pi}{2}$ we have $t \to \infty$ and $\tan\theta \to 0$. For $0 < \alpha < \frac{\pi}{2}$ we have $t > 0$ and so $\tan\theta > 0$. Hence the maximum value of $\tan\theta$ is achieved at a stationary point. Differentiating with respect to t gives

$$\begin{aligned}\frac{d}{dt}\tan\theta &= \frac{(2t^2+1-e)(1+e)-(1+e)t(4t)}{(2t^2+1-e)^2} \\ &= \frac{1+e}{(2t^2+1-e)^2}(1-e-2t^2) \quad ,\end{aligned}$$

hence

$$\frac{d}{dt}\tan\theta = 0 \iff t^2 = \frac{1}{2}(1-e) \quad ,$$

and the maximum value is at $t = \sqrt{\frac{1-e}{2}}$, giving

$$\tan\theta = \frac{(1+e)\sqrt{\frac{1-e}{2}}}{1-e+1-e} = \frac{1}{2}\frac{1+e}{1-e}\sqrt{\frac{1-e}{2}} = \frac{1}{2\sqrt{2}}\frac{1+e}{\sqrt{1-e}} \quad .$$

Section C: Probability and Statistics

Question 11

(i) Let Y be the number of customers arriving and let X be the number of customers that take sand. We have

$$\begin{aligned}
\mathbb{P}\{X=x\} &= \sum_{y=x}^{\infty} \mathbb{P}\{Y=y\}\mathbb{P}\{\text{out of } y \text{ customers, } x \text{ take sand}\} \\
&= \sum_{y=x}^{\infty} e^{-\lambda}\frac{\lambda^y}{y!}\binom{y}{x}p^x(1-p)^{y-x} \\
&= e^{-\lambda}p^x \sum_{y=x}^{\infty} \frac{\lambda^y}{y!}\frac{y!}{x!(y-x)!}(1-p)^{y-x} \\
&= e^{-\lambda}\frac{p^x}{x!}\sum_{y=x}^{\infty}\frac{\lambda^y}{(y-x)!}(1-p)^{y-x} \\
&= e^{-\lambda}\frac{p^x}{x!}\sum_{y=0}^{\infty}\frac{\lambda^{y+x}}{y!}(1-p)^y \\
&= e^{-\lambda}\frac{(p\lambda)^x}{x!}\sum_{y=0}^{\infty}\frac{((1-p)\lambda)^y}{y!} \\
&= e^{-\lambda}\frac{(p\lambda)^x}{x!}e^{(1-p)\lambda} \\
&= e^{-p\lambda}\frac{(p\lambda)^x}{x!} \quad ,
\end{aligned}$$

thus X follows a Poisson distribution with mean $p\lambda$.

(ii) If x customers take sand, then the mass of sand remaining at the end of the day is $(1-k)^x S$, and the mass of sand taken will be $(1-(1-k)^x)S$. Hence the expected mass of sand taken is

$$\begin{aligned}
\mathbb{E} &= \sum_{x=0}^{\infty}(1-(1-k)^x)S\cdot\mathbb{P}\{X=x\} \\
&= \sum_{x=0}^{\infty}(1-(1-k)^x)Se^{-p\lambda}\frac{(p\lambda)^x}{x!} \\
&= \left(\sum_{x=0}^{\infty}\frac{(p\lambda)^x}{x!} - \sum_{x=0}^{\infty}\frac{((1-k)p\lambda)^x}{x!}\right)e^{-p\lambda}S \\
&= \left(e^{p\lambda}-e^{(1-k)p\lambda}\right)e^{-p\lambda}S \\
&= (1-e^{-kp\lambda})S \quad .
\end{aligned}$$

(iii) If x customers take sand (so the mass of sand remaining after the customers have taken theirs is $(1-k)^x S$), the assistant will take a mass $k(1-k)^x S$ and the probability (conditional on there being x customers taking sand) that the assistant takes the gold grain is $k(1-k)^x$. Thus the total probability that the assistant takes the gold grain is

$$\begin{aligned}
P &= \sum_{x=0}^{\infty} k(1-k)^x \mathbb{P}\{X=x\} \\
&= \sum_{x=0}^{\infty} k(1-k)^x e^{-p\lambda} \frac{(p\lambda)^x}{x!} \\
&= k e^{-p\lambda} \sum_{x=0}^{\infty} \frac{((1-k)p\lambda)^x}{x!} \\
&= k e^{-p\lambda} e^{(1-k)p\lambda} \\
&= k e^{-kp\lambda} \quad .
\end{aligned}$$

In the case $k=0$ we have $P=0$: nobody takes any sand (including the assistant), hence the probability that the assistant takes the gold grain is zero. As $k \to 1$ we have $P \to e^{-p\lambda}$: any customer that takes sand will take all the sand (and hence will take the gold grain), thus the probability that the assistant takes the gold grain is exactly the probability that no customers take sand, that is $\mathbb{P}\{X=0\}$, which is indeed $e^{-p\lambda}$.

Differentiating with respect to k:

$$\begin{aligned}
\frac{dP}{dk} &= \frac{d}{dk}\left(k e^{-kp\lambda}\right) \\
&= e^{-kp\lambda} - kp\lambda e^{-kp\lambda} \\
&= (1-kp\lambda) e^{-kp\lambda} \quad .
\end{aligned}$$

Hence (given $\frac{1}{p\lambda} < 1$), $\frac{dP}{dk} = 0$ if and only if $k = \frac{1}{p\lambda}$, for $k < \frac{1}{p\lambda}$ we have $\frac{dP}{dk} > 0$, and for $k > \frac{1}{p\lambda}$ we have $\frac{dP}{dk} < 0$; thus $k = \frac{1}{p\lambda}$ gives the maximum value of P.

Question 12

For any given subset A of S, there are two possibilities for each integer: either it can be in A or in its complement $\bar{A}$. Hence there are 2^n possible distinct subsets of S; hence T contains exactly 2^n sets.

(i) The element 1 will be in exactly half of the sets in T. Hence

$$\mathbb{P}\{1 \in A_1\} = \frac{1}{2} \quad .$$

(ii) $A_1 \cap A_2 = \emptyset$ if and only if for each integer $x \in S$ we have $x \notin A_1 \cap A_2$. For each x we have

$$\begin{aligned}\mathbb{P}\{x \notin A_1 \cap A_2\} &= 1 - \mathbb{P}\{x \in A_1 \text{ and } x \in A_2\} \\ &= 1 - \mathbb{P}\{x \in A_1\} \cdot \mathbb{P}\{x \in A_2\} \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \quad ,\end{aligned}$$

hence

$$\mathbb{P}\{A_1 \cap A_2 = \emptyset\} = \left(\frac{3}{4}\right)^n \quad .$$

Similarly, $A_1 \cap A_2 \cap A_3 = \emptyset$ if and only if for each integer $x \in S$ we have $x \notin A_1 \cap A_2 \cap A_3$. For each x we have

$$\begin{aligned}\mathbb{P}\{x \notin A_1 \cap A_2 \cap A_3\} &= 1 - \mathbb{P}\{x \in A_1 \text{ and } x \in A_2 \text{ and } x \in A_3\} \\ &= 1 - \mathbb{P}\{x \in A_1\} \cdot \mathbb{P}\{x \in A_2\} \cdot \mathbb{P}\{x \in A_3\} \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{8} \quad ,\end{aligned}$$

hence

$$\mathbb{P}\{A_1 \cap A_2 \cap A_3 = \emptyset\} = \left(\frac{7}{8}\right)^n \quad .$$

Likewise, for each x we have

$$\mathbb{P}\{x \notin A_1 \cap A_2 \cap \cdots \cap A_m\} = 1 - \frac{1}{2^m} \quad ,$$

hence

$$\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_m = \emptyset\} = \left(1 - \frac{1}{2^m}\right)^n \quad .$$

(iii) $A_1 \subseteq A_2$ if and only if for each integer $x \in S$ we have $x \in A_1 \cap A_2$ or $x \in \bar{A}_1 \cap A_2$ or $x \in \bar{A}_1 \cap \bar{A}_2$. For each x we have

$$\mathbb{P}\{x \in A_1 \cap A_2\} = \frac{1}{4} \quad , \quad \mathbb{P}\{x \in \bar{A}_1 \cap A_2\} = \frac{1}{4} \quad , \quad \text{and} \quad \mathbb{P}\{x \in \bar{A}_1 \cap \bar{A}_2\} = \frac{1}{4}$$

$$\Longrightarrow \quad \mathbb{P}\{x \in A_1 \cap A_2 \text{ or } x \in \bar{A}_1 \cap A_2 \text{ or } x \in \bar{A}_1 \cap \bar{A}_2\} = \frac{3}{4}$$

(we can simply add the probabilities since these subsets are disjoint), hence

$$\mathbb{P}\{A_1 \subseteq A_2\} = \left(\frac{3}{4}\right)^n \quad .$$

Similarly, $A_1 \subseteq A_2 \subseteq A_3$ if and only if for each integer $x \in S$ we have $x \in A_1 \cap A_2 \cap A_3$ or $x \in \bar{A}_1 \cap A_2 \cap A_3$ or $x \in \bar{A}_1 \cap \bar{A}_2 \cap A_3$ or $x \in \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$. For each x we have

$$\mathbb{P}\{x \in A_1 \cap A_2 \cap A_3\} = \frac{1}{8} \quad , \quad \mathbb{P}\{x \in \bar{A}_1 \cap A_2 \cap A_3\} = \frac{1}{8} \quad ,$$

$$\mathbb{P}\{x \in \bar{A}_1 \cap \bar{A}_2 \cap A_3\} = \frac{1}{8} \quad , \quad \text{and} \quad \mathbb{P}\{x \in \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\} = \frac{1}{8} \quad ,$$

hence

$$\mathbb{P}\{A_1 \subseteq A_2 \subseteq A_3\} = \left(\frac{4}{8}\right)^n = \left(\frac{1}{2}\right)^n \quad .$$

For $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$, we need each $x \in S$ to be in one of the $m+1$ disjoint sets $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m$, $\bar{A}_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m$, $\bar{A}_1 \cap \bar{A}_2 \cap A_3 \cap \cdots \cap A_m$, ..., $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cdots \cap \bar{A}_m$. The probability that any given x is in any given one of these sets is $\frac{1}{2^m}$, hence the probability that any given x is in one of them is $\frac{m+1}{2^m}$, and so

$$\mathbb{P}\{A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_m\} = \left(\frac{m+1}{2^m}\right)^n \quad .$$