STEP 2018 Solutions

Contents

Author's notes:

1. *At time of writing, I am not affiliated with Cambridge Assessment Admissions Testing. I did an undergraduate maths degree at Cambridge, so I sat the STEP II and III papers as an A-level student (in 2015), and I have also been one of a team of markers for the STEP exams (in 2019 and 2020). Any opinions given here are entirely my own, based on my own experiences of STEP.*

2. *These 'solutions' are not intended to be used as any sort of mark scheme. In terms of method, often there will be more than one correct way to answer a STEP question, and it is certainly not the case that the answers presented here are the only correct approaches to these questions. The worked solutions here were typed up after attempting the questions myself, and I have checked them against the official mark schemes published online. However, there is no guarantee that the solutions typed up here would achieve full marks. In particular, I have not provided diagrams for all questions due to the difficulties of typesetting them neatly. Many questions may ask the student to draw a diagram, and in these instances marks are often awarded for this. Another point of consideration is explanation: sometimes marks are awarded for explicitly justifying an assumption used. I have tried to justify these as I think necessary, but there is no guarantee that these solutions justify all assumptions to the standards of the mark schemes.*

3. *If you are preparing to sit the STEP exams, I hope these can be of some help.*

STEP I

Section A: Pure Mathematics

Question 1

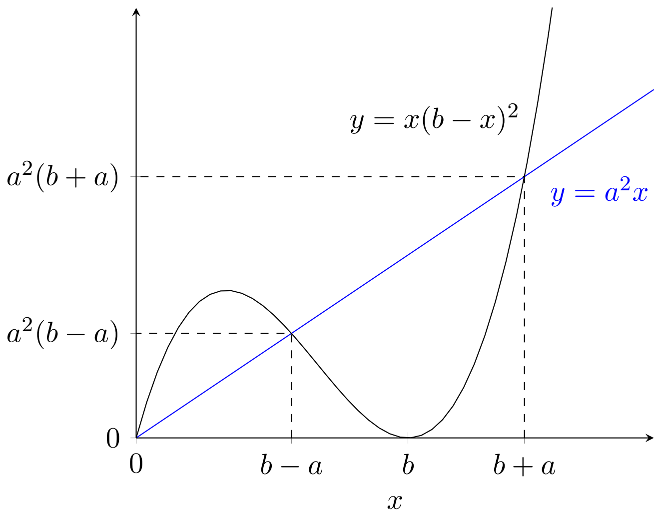
At intersection, we have:

$$\begin{aligned} a^2x &= x(b-x)^2 \\ a^2x &= b^2x - 2bx^2 + x^3 \\ x^3 - 2bx^2 + (b^2 - a^2)x &= 0 \\ x(x^2 - 2bx + b^2 - a^2) &= 0 \\ x(x-(b+a))(x-(b-a)) &= 0 \quad . \end{aligned}$$

Hence (given $a > 0$) the x-coordinates of P and Q are $x = b - a$ and $x = b + a$ respectively. As both points lie on $y = a^2x$ then, we find that the coordinates are

$$\begin{aligned} P: &\quad (x_p, y_p) = (b-a, a^2(b-a)) \\ Q: &\quad (x_q, y_q) = (b+a, a^2(b+a)) \quad . \end{aligned}$$

Our sketch is as follows.



Differentiating the equation for the curve gives:

$$\frac{dy}{dx} = (b-x)^2 - 2x(b-x) \quad ,$$

hence the gradient at P is

$$\frac{dy}{dx} = a^2 - 2a(b-a) = 3a^2 - 2ab \quad ,$$

and the equation of the tangent at P is

$$
\begin{aligned}
y - a^2(b-a) &= (3a^2 - 2ab)(x - (b-a)) \\
y &= a(3a-2b)x + a^2(b-a) - (3a^2 - 2ab)(b-a) \\
y &= a(3a-2b)x + (2ab - 2a^2)(b-a) \\
y &= a(3a-2b)x + 2a(b-a)^2 \quad ,
\end{aligned}
$$

as required. To find the area of the given region, we integrate the area below the curve and subtract off the area below the line:

$$
\begin{aligned}
S &= \int_0^{b-a} (b^2x - 2bx^2 + x^3)dx - \int_0^{b-a} a^2 x dx \\
&= \int_0^{b-a} (x^3 - 2bx^2 + (b^2 - a^2)x)dx \\
&= \left[\frac{1}{4}x^4 - \frac{2}{3}bx^3 + \frac{1}{2}(b+a)(b-a)x^2 \right]_0^{b-a} \\
&= \left(\frac{1}{4}(b-a) - \frac{2}{3}b + \frac{1}{2}(b+a) \right)(b-a)^3 \\
&= \frac{1}{12}\left(3b - 3a - 8b + 6b + 6a\right)(b-a)^3 \\
&= \frac{1}{12}(3a+b)(b-a)^3 \quad ,
\end{aligned}
$$

as given. By the equation of the tangent, the y-coordinate of R is $2a(b-a)^2$, hence we find the area of triangle OPR:

$$
T = \frac{1}{2} \cdot 2a(b-a)^2 \cdot (b-a) = a(b-a)^3 \quad ,
$$

so we can compute

$$
\begin{aligned}
S - \tfrac{1}{3}T &= \left(\frac{1}{4}a + \frac{1}{12}b \right)(b-a)^3 - \frac{1}{3}a(b-a)^3 \\
&= \left(\frac{1}{12}b - \frac{1}{12}a \right)(b-a)^3 \\
&= \frac{1}{12}(b-a)^4 \quad .
\end{aligned}
$$

Since we are given $b > a$, we conclude that $S - \frac{1}{3}T > 0$, hence $S > \frac{1}{3}T$.

Question 2

If $x = \log_b(c)$ we have that

$$c = b^x \quad .$$

Taking logs base a, we have

$$\begin{aligned}\log_a(c) &= x \log_a(b)\\ \frac{\log_a(c)}{\log_a(b)} &= x\\ \frac{\log_a(c)}{\log_a(b)} &= \log_b(c) \quad .\end{aligned}$$

(i) Given $\pi^2 < 10$, we have

$$\begin{aligned} 2\log_{10}(\pi) &< 1\\ \iff \log_{10}(\pi) &< \frac{1}{2} \quad .\end{aligned}$$

Then using the result from before, we find

$$\begin{aligned}\frac{1}{\log_2(\pi)} + \frac{1}{\log_5(\pi)} &= \frac{\log_{10}(2)}{\log_{10}(\pi)} + \frac{\log_{10}(5)}{\log_{10}(\pi)}\\ &= \frac{\log_{10}(2) + \log_{10}(5)}{\log_{10}(\pi)}\\ &= \frac{\log_{10}(10)}{\log_{10}(\pi)}\\ &= \frac{1}{\log_{10}(\pi)}\\ \implies \frac{1}{\log_2(\pi)} + \frac{1}{\log_5(\pi)} &> 2 \qquad \text{by the above inequality.}\end{aligned}$$

(ii) Now we have

$$\log_2(\pi) - \log_2(e) > \frac{1}{5} \quad ,$$

and

$$\begin{aligned} e^2 < 8 \qquad \implies \qquad 2\log_2(e) &< 3\\ \log_2(e) &< \frac{3}{2} \quad .\end{aligned}$$

We can write

$$\begin{aligned}\ln(\pi) &= \frac{\log_2(\pi)}{\log_2(e)}\\ &= 1 + \frac{\log_2(\pi) - \log_2(e)}{\log_2(e)} \quad ,\end{aligned}$$

so by the above inequalities:

$$\ln(\pi) > 1 + \frac{1}{5} \Big/ \frac{3}{2}$$
$$\ln(\pi) > 1 + \frac{2}{15}$$
$$\ln(\pi) > \frac{17}{15} \quad .$$

(iii) Now we have

$$e^3 > 20 \quad \Longrightarrow \quad 3\log_{10}(e) > \log_{10}(20) = \log_{10}(10) + \log_{10}(2)$$
$$3\log_{10}(e) > 1 + \frac{3}{10} = \frac{13}{10}$$
$$\log_{10}(e) > \frac{13}{30} \quad ,$$

and

$$\pi^2 < 10 \quad \Longrightarrow 2\log_{10}(\pi) < 1$$
$$\log_{10}(\pi) < \frac{1}{2} \quad .$$

Writing

$$\ln(\pi) = \frac{\log_{10}(\pi)}{\log_{10}(e)} \quad ,$$

the above inequalities give

$$\ln(\pi) < \frac{1}{2} \Big/ \frac{13}{30}$$
$$\ln(\pi) < \frac{15}{13} \quad .$$

Question 3

We have that:

$$\tan(\alpha) = \frac{y}{x+a} \quad \text{and} \quad \tan(\beta) = \frac{y}{2a-x}$$

A diagram is very useful to see this.

(i) If $\beta = 2\alpha$, then the tan addition formula gives

$$\begin{aligned}
\tan(\beta) = \tan(2\alpha) &= \frac{2\tan(\alpha)}{1-\tan^2(\alpha)} \\
\frac{y}{2a-x} &= \frac{2\frac{y}{x+a}}{1-\frac{y^2}{(x+a)^2}} \\
&= \frac{2y(x+a)}{(x+a)^2-y^2} \\
\implies (x+a)^2 - y^2 &= 2(x+a)(2a-x) \qquad \text{(since } y>0) \\
y^2 &= (x+a)(x+a-2(2a-x)) \\
&= (x+a)(3x-3a) \\
y^2 &= 3(x^2-a^2) \quad ,
\end{aligned}$$

that is, P lies on $y^2 = 3(x^2-a^2)$.

(ii) In general, we have

$$\begin{aligned}
\tan(\beta - 2\alpha) = \frac{\tan(\beta)-\tan(2\alpha)}{1+\tan(\beta)\tan(2\alpha)} &= \frac{\frac{y}{2a-x} - \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}}{1+\frac{y}{2a-x}\frac{2\tan(\alpha)}{1-\tan^2(\alpha)}} \\
&= \frac{\frac{y}{2a-x} - \frac{2\frac{y}{x+a}}{1-\frac{y^2}{(x+a)^2}}}{1+\frac{y}{2a-x}\frac{2\frac{y}{x+a}}{1-\frac{y^2}{(x+a)^2}}} = \frac{\frac{y}{2a-x}\left(1-\frac{y^2}{(x+a)^2}\right) - \frac{2y}{x+a}}{1-\frac{y^2}{(x+a)^2}+\frac{2y^2}{(2a-x)(x+a)}} \\
&= \frac{y\left((x+a)^2-y^2\right) - 2y(2a-x)(x+a)}{(2a-x)(x+a)^2 - y^2(2a-x) + 2y^2(x+a)} \\
&= \frac{y(x^2+2ax+a^2-y^2-(4a-2x)(x+a))}{(2a-x)(x+a)^2+3xy^2} \\
&= \frac{y(3x^2-3a^2-y^2)}{(2a-x)(x+a)^2+3xy^2} \quad .
\end{aligned}$$

If P lies on $y^2 = 3(x^2 - a^2)$, the numerator here vanishes, while the denominator gives

$$\begin{aligned}(2a - x)(x + a)^2 + 3xy^2 &= (2a - x)(x^2 + 2ax + a^2) + 9x(x^2 - a^2)\\ &= 8x^3 - 6a^2x + 2a^3\\ &= 2(x + a)(4x^2 - 4ax + a^2)\\ &= 2(x + a)(2x - a)^2 \quad .\end{aligned}$$

We are given that $y > 0$, hence $x^2 - a^2 > 0$; that is: $x < -a$ or $x > a$. Hence we cannot have $x = -a$ or $x = \frac{1}{2}a$, and so the denominator does not vanish. Thus when P lies on $y^2 = 3(x^2 - a^2)$, we must have

$$\tan(\beta - 2\alpha) = 0 \quad .$$

Since $y > 0$, we have $0 < \alpha < \pi$, and $0 < \beta < \pi$, giving $-2\pi < \beta - 2\alpha < \pi$. Hence the possible relations are:

$$\beta - 2\alpha = -\pi \quad \text{or} \quad \beta - 2\alpha = 0 \quad .$$

Question 4

We have

$$
\begin{aligned}
f(x) &= \frac{1}{x\log(x)}(1-(\log(x))^2)^2 \\
&= \frac{1}{x}\left((\log(x))^{-1} - 2\log(x) + (\log(x))^3\right) \quad , \\
f'(x) &= -\frac{1}{x^2}\left((\log(x))^{-1} - 2\log(x) + (\log(x))^3\right) + \frac{1}{x}\left(-\frac{(\log(x))^{-2}}{x} - \frac{2}{x} + \frac{3(\log(x))^2}{x}\right) \\
&= -\frac{1}{x^2}\left((\log(x))^3 - 3(\log(x))^2 - 2\log(x) + 2 + (\log(x))^{-1} + (\log(x))^{-2}\right) \quad .
\end{aligned}
$$

Hence when $(\log(x))^2 = 1$, we have

$$
\begin{aligned}
f(x) &= \frac{1}{x\log(x)}(1-1)^2 = 0 \qquad \text{and} \\
f'(x) &= -\frac{1}{x^2}\left(\log(x) - 3 - 2\log(x) + 2 + \frac{\log(x)}{(\log(x))^2} + 1\right) \\
&= -\frac{1}{x^2}\left(-\log(x) + \log(x)\right) = 0 \quad .
\end{aligned}
$$

(i) Using the form

$$
\begin{aligned}
f(x) &= \frac{1}{x}\left((\log(x))^{-1} - 2\log(x) + (\log(x))^3\right) \\
&= \left(\frac{d}{dx}(\log(x))\right)\left((\log(x))^{-1} - 2\log(x) + (\log(x))^3\right) \quad ,
\end{aligned}
$$

we can integrate

$$
\begin{aligned}
\int_{\frac{1}{e}}^{x} f(t)dt &= \left[\log(\log(t)) - (\log(t))^2 + \frac{1}{4}(\log(t))^4\right]_{\frac{1}{e}}^{x} \\
&= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 \\
&\qquad - \left(\log(|-1|) - (-1)^2 + \frac{1}{4}(-1)^4\right) \\
&= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 + \frac{3}{4} \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{e}^{x} f(t)dt &= \left[\log(\log(t)) - (\log(t))^2 + \frac{1}{4}(\log(t))^4\right]_{e}^{x} \\
&= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 \\
&\qquad - \left(\log(|1|) - (1)^2 + \frac{1}{4}(1)^4\right) \\
&= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 + \frac{3}{4} \quad ,
\end{aligned}
$$

hence

$$F(x) = \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 + \frac{3}{4} \qquad \text{for all } x > 0,\, x \neq 1 \quad .$$

Using the result that for all $x > 0$ $\log\left(x^{-1}\right) = -\log(x)$, we have that

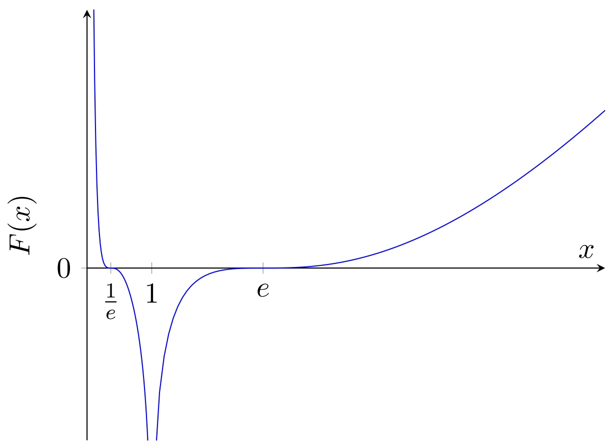
$$\begin{aligned} F(x^{-1}) &= \log(|-\log(x)|) - (-\log(x))^2 + \frac{1}{4}(-\log(x))^4 + \frac{3}{4} \\ &= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 + \frac{3}{4} \\ &= F(x) \quad . \end{aligned}$$

(ii) We know (for $x > 0$, $x \neq 1$)

$$\begin{aligned} F(x) &= \log(|\log(x)|) - (\log(x))^2 + \frac{1}{4}(\log(x))^4 + \frac{3}{4} \\ F'(x) = f(x) &= \frac{1}{x\log(x)}(1 - (\log(x))^2)^2 \quad , \end{aligned}$$

further, from the integral definition of $F(x)$, we can see that at $x = \frac{1}{e}$ and $x = e$, $F(x) = 0$. At these points we have $(\log(x))^2 = 1$, hence we also have $F'(x) = F''(x) = 0$ at $x = \frac{1}{e}$ and $x = e$.

As $x \to 0^+$, $F(x) \to \infty$, and $F'(x) \to -\infty$, so we have a vertical asymptote $x = 0$. We see that $F'(x)$ is negative but increasing until $x = \frac{1}{e}$, so F is positive, decreasing, and convex over this domain. For $\frac{1}{e} < x < 1$, $F'(x)$ is negative and decreasing; so $F(x)$ is negative, decreasing, and concave over this domain. As $x \to 1^-$, $F(x) \to -\infty$ and $F'(x) \to -\infty$, so we have another vertical asymptote $x = 1$. Finally, from the property $F(x^{-1}) = F(x)$ we can then deduce the shape of F for $x > 1$:



Question 5

(i) The polynomial must be 1 more than a polynomial with roots $x = 1, 2, 3, 4$. Since the polynomial is degree 4, none of these roots may be repeated and our polynomial is therefore

$$f(x) = k(x-1)(x-2)(x-3)(x-4) + 1 \quad ,$$

for some constant $k \neq 0$.

(ii) Analogously to part (i), we have

$$P(x) = k(x-1)(x-2)\cdots(x-N) + 1$$

for some constant $k \neq 0$. Thus

$$\begin{aligned} P(N+1) &= k \cdot N! + 1 \\ P(N+1) - 1 &= k \cdot N! \end{aligned}$$

and since $k \neq 0$, we must therefore have $P(N+1) \neq 1$.

If we are given that $P(N+1) = 2$, then

$$\begin{aligned} k \cdot N! + 1 &= 2 \\ k &= \frac{1}{N!} \quad . \end{aligned}$$

Hence for positive integer r, we have

$$\begin{aligned} P(N+r) &= \frac{1}{N!}(N+r-1)(N+r-2)\cdots r + 1 \\ &= \frac{(N+r-1)!}{N!(r-1)!} + 1 = \binom{N+r-1}{N} + 1 \quad . \end{aligned}$$

Fixing $P(N+r) = N+r$, for positive integer r we have

$$\begin{aligned} \binom{N+r-1}{N} + 1 &= N + r \\ \binom{N+r-1}{N} &= N + r - 1 \quad , \end{aligned}$$

this is satisfied if and only if $N + r - 1 = N + 1$ or $N = 1$, that is: if and only if $r = 2$ or $N = 1$, so our answer is $r = 2$.

(iii) We know that S is of the form

$$S(x) = k(x-a)(x-b)(x-c)(x-d) + 2001 \quad ,$$

for some constant k, and since we are given that the coefficient of x^4 is 1, we must have $k = 1$, thus

$$S(x) = (x-a)(x-b)(x-c)(x-d) + 2001 \quad .$$

(a) Supposing there exists some integer e such that $S(e) = 2018$:

$$\begin{aligned} 2018 &= (e-a)(e-b)(e-c)(e-d) + 2001 \\ \iff (e-a)(e-b)(e-c)(e-d) &= 17 \quad . \end{aligned}$$

Each bracket here must be an integer (since a, b, c, d, e are all integers), hence (as factors of 17) each one must be one of $\{-17, -1, 1, 17\}$. Further we cannot have factors of both 17 and -17, but all four brackets must be distinct (since a, b, c, d are all distinct): contradiction.

(b) Requiring $S(0) = 2017$ gives

$$\begin{aligned} (-a)(-b)(-c)(-d) &= 16 \\ abcd &= 16 \quad . \end{aligned}$$

hence a, b, c, d are each one of $\{\pm 16, \pm 8, \pm 4, \pm 2, \pm 1\}$. Since they must all be distinct, we cannot use ± 16 (because we cannot write ± 1 as a product of three distinct integers). We note further that we cannot have a, b, c, d all positive or all negative since $|1 \cdot 2 \cdot 4 \cdot 8| = 64 > 16$. Hence we must have two negative (necessarily a and b, by the ordering) and two positive (c and d).

We can use $a = -8$, requiring $bcd = -2$, which can only be done one way:

$$b = -1 \qquad c = 1 \qquad d = 2 \quad .$$

We can use $a = -4$, requiring $bcd = -4$, and this gives two options:

$$\begin{aligned} b &= -2 \qquad c = 1 \qquad d = 2 \\ b &= -1 \qquad c = 1 \qquad d = 4 \quad . \end{aligned}$$

Finally, we can set $a = -2$, requiring $b = -1$, $cd = 8$, giving a further two options:

$$\begin{aligned} c &= 1 \qquad d = 8 \\ c &= 2 \qquad d = 4 \quad . \end{aligned}$$

In total then, there are five ways to choose a, b, c, d.

Question 6

Using the given identity:

$$\begin{aligned}
&2\sin(\theta)(\sin(\theta)+\sin(3\theta)+\cdots+\sin((2n-1)\theta))\\
&\qquad = 2\sin(\theta)\sin(\theta)+2\sin(\theta)\sin(3\theta)+\cdots+2\sin(\theta)\sin((2n-1)\theta)\\
&\qquad = \cos(0)-\cos(2\theta)+\cos(2\theta)-\cos(4\theta)+\cdots+\cos((2n-2)\theta)-\cos(2n\theta)\\
&\qquad = 1-\cos(2n\theta) \quad .
\end{aligned}$$

(i) We have

$$A_n = \frac{\pi}{n}\left(\sin\left(\frac{\pi}{2n}\right)+\sin\left(\frac{3\pi}{2n}\right)+\cdots+\sin\left(\frac{(2n-1)\pi}{2n}\right)\right) \quad .$$

Let $\frac{\pi}{2n}=\theta$, then we have

$$A_n = \frac{\pi}{n}(\sin(\theta)+\sin(3\theta)+\cdots+\sin((2n-1)\theta)) \quad ,$$

and so, by the first result:

$$\begin{aligned}
2\sin(\theta)\frac{n}{\pi}A_n &= 1-\cos(2n\theta)\\
\frac{2n}{\pi}\sin\left(\frac{\pi}{2n}\right)A_n &= 1-\cos(\pi)=2\\
A_n\sin\left(\frac{\pi}{2n}\right) &= \frac{\pi}{n} \quad .
\end{aligned}$$

(ii) We have

$$\begin{aligned}
B_n &= \frac{1}{2}\cdot\frac{\pi}{n}\left(\sin(0)+2\sin\left(\frac{\pi}{n}\right)+2\sin\left(\frac{2\pi}{n}\right)+\cdots+2\sin\left(\frac{(n-1)\pi}{n}\right)+\sin(\pi)\right)\\
&= \frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{(n-1)\pi}{n}\right)\right) \quad ,
\end{aligned}$$

hence, using the given identity, we have

$$\begin{aligned}
B_n\sin\left(\frac{\pi}{2n}\right) &= \frac{\pi}{n}\sin\left(\frac{\pi}{2n}\right)\left(\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{(n-1)\pi}{n}\right)\right)\\
&= \frac{\pi}{2n}\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\frac{3\pi}{2n}\right)+\cos\left(\frac{3\pi}{2n}\right)-\cos\left(\frac{5\pi}{2n}\right)\right.\\
&\qquad\qquad \left.+\cdots+\cos\left(\frac{(2n-3)\pi}{2n}\right)-\cos\left(\frac{(2n-1)\pi}{2n}\right)\right)\\
&= \frac{\pi}{2n}\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\frac{(2n-1)\pi}{2n}\right)\right)\\
&= \frac{\pi}{2n}\left(\cos\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)\right)\\
&= \frac{\pi}{n}\cos\left(\frac{\pi}{2n}\right) \quad .
\end{aligned}$$

(iii) Using the derived results for A_n and B_n, we can write

$$A_n = \frac{\pi}{n \sin\left(\frac{\pi}{2n}\right)} \qquad \text{and} \qquad B_n = \frac{\pi \cos\left(\frac{\pi}{2n}\right)}{n \sin\left(\frac{\pi}{2n}\right)} \quad ,$$

thus

$$\begin{aligned}
\frac{1}{2}(A_n + B_n) &= \frac{1}{2}\left(\frac{\pi}{n \sin\left(\frac{\pi}{2n}\right)} + \frac{\pi \cos\left(\frac{\pi}{2n}\right)}{n \sin\left(\frac{\pi}{2n}\right)}\right) \\
&= \frac{\pi}{2n} \frac{1 + \cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \\
&= \frac{\pi}{2n} \frac{1 + \cos^2\left(\frac{\pi}{4n}\right) - \sin^2\left(\frac{\pi}{4n}\right)}{2 \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)} \\
&= \frac{\pi}{2n} \cdot \frac{2 \cos^2\left(\frac{\pi}{4n}\right)}{2 \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)} \\
&= \frac{\pi \cos\left(\frac{\pi}{4n}\right)}{2n \sin\left(\frac{\pi}{4n}\right)} \\
&= B_{2n} \quad .
\end{aligned}$$

Further, we have

$$\begin{aligned}
A_n B_{2n} &= \frac{\pi}{n \sin\left(\frac{\pi}{2n}\right)} \cdot \frac{\pi \cos\left(\frac{\pi}{4n}\right)}{2n \sin\left(\frac{\pi}{4n}\right)} \\
&= \frac{\pi}{2n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)} \cdot \frac{\pi \cos\left(\frac{\pi}{4n}\right)}{2n \sin\left(\frac{\pi}{4n}\right)} \\
&= \frac{\pi^2}{\left(2n \sin\left(\frac{\pi}{4n}\right)\right)^2} \\
&= A_{2n}^2 \quad .
\end{aligned}$$

Question 7

(i) Using the given substitution, we have

$$\left(\frac{pz+q}{z+1}\right)^3 - 3pq\frac{pz+q}{z+1} + pq(p+q) = 0$$
$$\implies (pz+q)^3 - 3pq(pz+q)(z+1)^2 + pq(p+q)(z+1)^3 = 0 \quad .$$

This gives a cubic equation in z where the coefficient of z^3 is

$$\begin{aligned} p^3 - 3p^2q + pq(p+q) &= p^3 - 2p^2q + pq^2 \\ &= p(p^2 - 2pq + q^2) \\ &= p(p-q)^2 \quad , \end{aligned}$$

the coefficient of z^2 is

$$\begin{aligned} 3p^2q - 3pq(2p+q) + 3pq(p+q) &= 3p^2q - 6p^2q - 3pq^2 + 3p^2q + 3pq^2 \\ &= 0 \quad , \end{aligned}$$

the coefficient of z is

$$\begin{aligned} 3pq^2 - 3pq(p+2q) + 3pq(p+q) &= 3pq^2 - 3p^2q - 6pq^2 + 3p^2q + 3pq^2 \\ &= 0 \quad , \end{aligned}$$

and the constant coefficient is

$$\begin{aligned} q^3 - 3pq^2 + pq(p+q) &= q^3 - 3pq^2 + p^2q + pq^2 \\ &= q(q^2 - 2pq + p^2) \\ &= q(p-q)^2 \quad , \end{aligned}$$

hence our equation is

$$p(p-q)^2z^3 + q(p-q)^2 = 0 \quad ,$$

that is, since p and q are distinct:

$$pz^3 + q = 0 \quad .$$

(So $a = p$, $b = q$.)

(ii) To write $x^3 - 3cx + d = 0$ in the given form, we need $pq = c$, and $p + q = \frac{d}{c}$, so p and q are the roots of the quadratic

$$y^2 - \frac{d}{c}y + c = 0 \quad ,$$

so the condition that p and q are distinct and real is exactly the condition that the discriminant of the quadratic is strictly positive, *id est*

$$\frac{d^2}{c^2} - 4c > 0$$
$$d^2 > 4c^3 \quad .$$

(iii) Here we have $c = -2$, $d = -2$, so the condition $d^2 > 4c^3$ is satisfied $(4 > -32)$, and our real root will be

$$x = \frac{pz + q}{z + 1} \quad ,$$

where $z = -\left(\frac{q}{p}\right)^{\frac{1}{3}}$, and p and q are the roots of

$$y^2 - y - 2 = 0 \quad ,$$

$$\implies \{p, q\} = \frac{1 \pm \sqrt{1 + 8}}{2} = \{2, -1\} \quad .$$

We choose $p = -1$, $q = 2$, to get

$$z = 2^{\frac{1}{3}}$$

$$x = \frac{2 - 2^{\frac{1}{3}}}{1 + 2^{\frac{1}{3}}} \quad .$$

(iv) Now we have

$$x^3 - 3p^2x + 2p^3 = 0$$
$$(x - p)(x^2 + px - 2p^2) = 0$$
$$(x - p)^2(x + 2p) = 0 \quad .$$

Hence in the case $d^2 = 4c^3$, the roots are $x = p$, $x = -2p$, where $p = \frac{d}{2c}$, *id est*: $x = \frac{d}{2c}$, $x = -\frac{d}{c}$.

Question 8

(i) Differentiation gives

$$\begin{aligned}\frac{d}{dx}\left(s(x)^3+c(x)^3\right) &= 3s(x)^2\frac{d}{dx}(s(x)) + 3c(x)^2\frac{d}{dx}(c(x)) \\ &= 3s(x)^2c(x)^2 - 3s(x)^2c(x)^2 \\ &= 0 \quad ,\end{aligned}$$

hence we have that $s(x)^3 + c(x)^3$ is constant, and so

$$s(x)^3 + c(x)^3 = s(0)^3 + c(0)^3 = 1 \quad .$$

(ii) We have

$$\begin{aligned}\frac{d}{dx}(s(x)c(x)) &= s'(x)c(x) + s(x)c'(x) \\ &= c(x)^3 - s(x)^3 \\ &= 2c(x)^3 - (s(x)^3 + c(x)^3) \\ &= 2c(x)^3 - 1 \qquad \text{using the result from (i).}\end{aligned}$$

Also

$$\begin{aligned}\frac{d}{dx}\left(\frac{s(x)}{c(x)}\right) &= \frac{c(x)s'(x) - s(x)c'(x)}{c(x)^2} \\ &= \frac{c(x)^3 + s(x)^3}{c(x)^2} \\ &= \frac{1}{c(x)^2} \qquad \text{using the result from (i).}\end{aligned}$$

(iii) By the property that $c'(x) = -s(x)^2$, we have

$$\int s(x)^2 dx = -c(x) + k \quad ,$$

for some arbitrary constant k. Also, using the result from (i),

$$\begin{aligned}\int s(x)^5 dx &= \int s(x)^2(s(x)^3)dx = \int s(x)^2(1 - c(x)^3)dx \\ &= \int s(x)^2 dx - \int \left(-\frac{d}{dx}c(x)\right) c(x)^3 dx \\ &= -c(x) + \frac{1}{4}c(x)^4 + k \quad ,\end{aligned}$$

for some arbitrary constant k.

(iv) Using the suggested substitution:

$$\begin{aligned}\int \frac{1}{(1-u^3)^{\frac{2}{3}}}du &= \int \frac{1}{(1-s(x)^3)^{\frac{2}{3}}}s'(x)dx \\ &= \int \frac{1}{c(x)^2}c(x)^2 dx \\ &= \int dx = x + k \\ &= s^{-1}(u) + k \quad ,\end{aligned}$$

for some arbitrary constant k.

(v) Using the same substitution $u = s(x)$, we have

$$\begin{aligned}\int \frac{1}{(1-u^3)^{\frac{4}{3}}}du &= \int \frac{1}{c(x)^4}c(x)^2 dx \\ &= \int \frac{1}{c(x)^2}dx \\ &= \frac{s(x)}{c(x)} + k \qquad \text{using the result of (ii)} \\ &= \frac{s(x)}{(1-s(x)^3)^{\frac{1}{3}}} + k \qquad \text{using the result of (i)} \\ &= \frac{u}{(1-u^3)^{\frac{1}{3}}} + k\end{aligned}$$

for some arbitrary constant k. Also

$$\begin{aligned}\int (1-u^3)^{\frac{1}{3}}du &= \int c(x)(c(x)^2)dx \\ &= \int \frac{1}{2}\left(1 + \frac{d}{dx}\left(s(x)c(x)\right)\right)dx \qquad \text{using the result of (ii)} \\ &= \frac{1}{2}x + \frac{1}{2}s(x)c(x) + k \\ &= \frac{1}{2}s^{-1}(u) + \frac{1}{2}u(1-u^3)^{\frac{1}{3}} + k \quad ,\end{aligned}$$

for some arbitrary constant k.

Section B: Mechanics

Question 9

The gravitational potential energy of the go-kart is initially $mgx\sin(\alpha)$, where m is the mass of the go-kart, and its kinetic energy is initially 0. At the house, its gravitational potential energy is $mgd\sin(\beta)$ and we are given that it has non-negative kinetic energy at this point. By conservation of energy

$$\begin{aligned}\text{gain in kinetic energy } &= mg(x\sin(\alpha) - d\sin(\beta)) \geqslant 0 \\ &\implies x\sin(\alpha) \geqslant d\sin(\beta) \quad .\end{aligned}$$

When going downhill, the go-kart accelerates with a constant acceleration $g\sin(\alpha)$ due to gravity. Its initial velocity is 0, and it travels the total distance x down the slope. Let t_1 be the time at which the go-kart passes the traffic lights and v be the velocity of the go-kart at the traffic lights. Using SUVAT equations we find v:

$$\begin{aligned}v^2 &= 0^2 + 2g\sin(\alpha)x \\ v &= \sqrt{2xg\sin(\alpha)} \quad ,\end{aligned}$$

where we take the positive root, since the kart is accelerating 'forwards'; and then we find t_1:

$$\begin{aligned}v &= 0 + g\sin(\alpha)t_1 \\ t_1 &= \frac{v}{g\sin(\alpha)} = \sqrt{\frac{2x}{g\sin(\alpha)}} \quad .\end{aligned}$$

Now the go-kart decelerates uphill with acceleration $-g\sin(\beta)$ for a distance d. Let the time taken be t_2. We use SUVAT equations to find t_2:

$$\begin{aligned}d &= vt_2 - \frac{1}{2}g\sin(\beta)t^2 \\ \frac{g\sin(\beta)}{2}t_2^2 - vt_2 + d &= 0 \\ t_2 &= \frac{v \pm \sqrt{v^2 - 2dg\sin(\beta)}}{g\sin(\beta)} \quad ,\end{aligned}$$

we take the negative sign for the square-root, since this gives the first time the go-kart will reach the house (given that it does reach the house, any excess energy will take it further uphill before it eventually stops and rolls back to the house again – this will be the other, greater value for t_2):

$$t_2 = \frac{v - \sqrt{v^2 - 2dg\sin(\beta)}}{g\sin(\beta)} \quad .$$

The total time T thus satisfies

$$\begin{aligned}
\sqrt{\frac{g\sin(\alpha)}{2}}T &= \sqrt{\frac{g\sin(\alpha)}{2}}t_1 + \sqrt{\frac{g\sin(\alpha)}{2}}t_2 \\
&= \sqrt{x} + \sqrt{\frac{g\sin(\alpha)}{2}}\frac{v-\sqrt{v^2-2dg\sin(\beta)}}{g\sin(\beta)} \\
&= \sqrt{x} + \sqrt{\frac{g\sin(\alpha)}{2}}\frac{\sqrt{2xg\sin(\alpha)}-\sqrt{2xg\sin(\alpha)-2dg\sin(\beta)}}{g\sin(\beta)} \\
&= \sqrt{x} + \sqrt{\sin(\alpha)}\frac{\sqrt{x\sin(\alpha)}-\sqrt{x\sin(\alpha)-d\sin(\beta)}}{\sin(\beta)} \\
&= \left(1+\frac{\sin(\alpha)}{\sin(\beta)}\right)\sqrt{x} - \sqrt{\frac{\sin^2(\alpha)}{\sin^2(\beta)}x - \frac{\sin(\alpha)}{\sin(\beta)}d} \\
&= (1+k)\sqrt{x} - \sqrt{k^2x-kd} \quad .
\end{aligned}$$

The left-hand side here is just a positive multiple of T, hence the value of x that minimises T will minimise the right-hand side. Differentiating we have

$$\begin{aligned}
\sqrt{\frac{g\sin(\alpha)}{2}}\frac{dT}{dx} &= \frac{1}{2}(1+k)\frac{1}{\sqrt{x}} - \frac{1}{2}\frac{k^2}{\sqrt{k^2x-kd}} \\
&= \frac{1}{2}\frac{(1+k)\sqrt{k^2x-kd}-k^2\sqrt{x}}{\sqrt{k^2x^2-kdx}} \quad .
\end{aligned}$$

Hence we want

$$\begin{aligned}
(1+k)\sqrt{k^2x-kd} &= k^2\sqrt{x} \\
\implies (1+k)^2(k^2x-kd) &= k^4x \\
(1+2k+k^2)k^2x-(1+k)^2kd &= k^4x \\
(1+2k)k^2x &= (1+k)^2kd \\
x &= \frac{(1+k)^2}{k(1+2k)}d \quad .
\end{aligned}$$

Question 10

(i) By Newtons second law, if a is the acceleration of the train,

$$\begin{aligned} 2D - nR &= (2M + nm)a \\ a &= \frac{2D - nR}{2M + nm} \quad . \end{aligned}$$

Now using Newton's second law on the front engine, we have

$$\begin{aligned} D - T = Ma &= \frac{M(2D - nR)}{2M + nm} \\ T &= D - \frac{M(2D - nR)}{2M + nm} \\ &= \frac{D(2M + nm) + M(nR - 2D)}{2M + nm} \\ &= \frac{n(mD + MR)}{2M + nm} \quad . \end{aligned}$$

(ii) Let T_i be the tension in the coupling after the i-th carriage for $1 \leqslant i \leqslant k$. Using Newton's second law on the j-th carriage ($1 \leqslant j \leqslant k$)

$$\begin{aligned} T_{j-1} - T_j - R &= ma \\ T_j &= T_{j-1} - (R + ma) \\ \implies T_j &= T - j(R + ma) \quad . \end{aligned}$$

Hence $T > T_2 > T_3 > \cdots > T_k$. Similarly, the tensions in the couplings after the second engine will decrease getting further from the engine. Suppose U is the tension in the coupling just after the second engine. Using Newton's second law on all the carriages after the second engine gives

$$\begin{aligned} U - (n-k)R &= (n-k)ma \\ U &= (n-k)(R + ma) \end{aligned}$$

Hence we have

$$T - U = \frac{n(mD + MR)}{2M + nm} - (n-k)(R + ma) \quad .$$

We have

$$\begin{aligned} 2D - nR &= (2M + nm)a \\ D &= \tfrac{1}{2}nR + Ma + \tfrac{1}{2}nma \\ mD + MR &= \tfrac{1}{2}nmR + mMa + \tfrac{1}{2}nm^2a + MR \\ &= \tfrac{1}{2}(2MR + 2mMa + nmR + nm^2a) \\ &= \tfrac{1}{2}(R + ma)(2M + nm) \quad , \end{aligned}$$

so upon substitution

$$\begin{aligned} T - U &= \tfrac{1}{2}n(R + ma) - (n - k)(R + ma) \\ &= (k - \tfrac{1}{2}n)(R + ma) \quad . \end{aligned}$$

Thus if $k > \frac{1}{2}n$, then $T > U$, and so T is greater than the tension in any other coupling.

(iii) Using Newton's second law on the second engine

$$\begin{aligned} T_k + D - U &= Ma \\ T_k &= U - (D - Ma) \quad , \end{aligned}$$

but from Newton's second law on the first engine $D - Ma = T$:

$$T_k = U - T \quad .$$

By the result in (ii), if $k > \frac{1}{2}n$ then $T > U$; thus if $k > \frac{1}{2}n$ then $T_k < 0$.

Question 11

For this question, a diagram is particularly useful.

(i) As P approaches A, the angle θ approaches α from above, and as P approaches B, the angle θ approaches 2α from below; thus $\alpha \leqslant \theta \leqslant 2\alpha$, as required.

(ii) Consider the forces on the particle P resolved perpendicularly to L. Since P is at equilibrium, the normal reaction force R combined with a resolved component of the tension in the string T, balances a resolved component of the particles weight mg. Specifically:

$$R + T\sin\theta = mg\cos\alpha \quad .$$

T must be equal to λmg (the weight of the freely hanging particle), hence

$$\begin{aligned} R + \lambda mg\sin\theta &= mg\cos\alpha \\ R &= mg(\cos\alpha - \lambda\sin\theta) \quad . \end{aligned}$$

Since P can rest at equilibrium anywhere between O and A, we know that $R \geqslant 0$ for all $\theta \in [\alpha, 2\alpha]$. Hence

$$\cos\alpha \geqslant \lambda\sin\theta \quad \text{for all } \theta \in [\alpha, 2\alpha] \quad .$$

The right-hand side here achieves a maximum when $\theta = 2\alpha$ (since $2\alpha \leqslant \frac{\pi}{2}$). Hence we must have

$$\begin{aligned} & \cos\alpha \geqslant \lambda\sin(2\theta) \\ \iff & \cos\alpha \geqslant 2\lambda\sin\alpha\cos\alpha \\ \implies & 2\lambda\sin\alpha \leqslant 1 \quad , \end{aligned}$$

since $\cos\alpha > 0$.

(iii) Now consider the forces on P resolved parallel to L. The friction F, will be directed up the slope, with resolved components of the tension and weight acting down the slope. If P is at equilibrium then:

$$\begin{aligned} F &= mg\sin\alpha + T\cos\theta \\ &= mg(\sin\alpha + \lambda\cos\theta) \quad . \end{aligned}$$

From the coefficient of friction, we know that $F \leqslant \mu R$, giving

$$\begin{aligned} mg(\sin\alpha + \lambda\cos\theta) &\leqslant mg\tan\beta(\cos\alpha - \lambda\sin\theta) \\ \sin\alpha\cos\beta + \lambda\cos\beta\cos\theta &\leqslant \sin\beta\cos\alpha - \lambda\sin\beta\sin\theta \\ \lambda(\cos\beta\cos\theta + \sin\beta\sin\theta) &\leqslant \sin\beta\cos\alpha - \sin\alpha\cos\beta \\ \lambda\cos(\beta - \theta) &\leqslant \sin(\beta - \alpha) \\ \lambda &\leqslant \frac{\sin(\beta - \alpha)}{\cos(\beta - \theta)} \quad . \end{aligned}$$

Since $\beta \geqslant 2\alpha$, we have that $\beta - \theta \geqslant 0$. Thus the right-hand side is minimised when θ is maximised (so that $|\beta - \theta|$ is minimised), giving the necessary condition

$$\lambda \leqslant \frac{\sin(\beta - \alpha)}{\cos(\beta - 2\alpha)} \quad .$$

If $\alpha \leqslant \beta \leqslant 2\alpha$, then the right-hand side is minimised when $\theta = \beta$, giving the necessary condition

$$\lambda \leqslant \sin(\beta - \alpha) \quad .$$

Section C: Probability and Statistics

Question 12

(i) Since each coin of the three coins is chosen with probability $\frac{1}{3}$, the probability of the toss showing a head is

$$\frac{1}{3}(p_1 + p_2 + p_3) \quad .$$

(ii) Each of the two tosses contributing to N_1 is equivalent to the toss in part (i), which as a probability $p = \frac{1}{3}(p_1 + p_2 + p_3)$ of showing a head. Thus

$$\begin{aligned} \mathbb{P}\{N_1 = 1\} &= p(1-p) + (1-p)p = 2p(1-p) \\ \text{and} \quad \mathbb{P}\{N_1 = 2\} &= p^2 \quad . \end{aligned}$$

Hence we find the expectation

$$\mathbb{E}(N_1) = 1 \cdot 2p(1-p) + 2 \cdot p^2 = 2p \quad ,$$

and the variance

$$\begin{aligned} \text{Var}(N_1) &= \mathbb{E}(N_1^2) - \mathbb{E}(N_1)^2 \\ &= 1 \cdot 2p(1-p) + 4 \cdot p^2 - (2p)^2 = 2p(1-p) \quad . \end{aligned}$$

(iii) Now each possible pair of coins (1 and 2, 2 and 3, 1 and 3) is drawn with probability $\frac{1}{3}$, giving

$$\begin{aligned} \mathbb{P}\{N_2 = 1\} &= \frac{1}{3}(p_1(1-p_2) + (1-p_1)p_2 + p_2(1-p_3) + (1-p_2)p_3 \\ &\qquad + p_1(1-p_3) + (1-p_1)p_3) \\ &= \frac{2}{3}(p_1 + p_2 + p_3 - (p_1p_2 + p_2p_3 + p_1p_3)) \\ \text{and} \quad \mathbb{P}\{N_2 = 2\} &= \frac{1}{3}(p_1p_2 + p_2p_3 + p_1p_3) \quad . \end{aligned}$$

Hence we find the expectation

$$\begin{aligned} \mathbb{E}(N_2) &= \frac{2}{3}(p_1 + p_2 + p_3 - (p_1p_2 + p_2p_3 + p_1p_3)) + \frac{2}{3}(p_1p_2 + p_2p_3 + p_1p_3) \\ &= \frac{2}{3}(p_1 + p_2 + p_3) = 2p \quad , \end{aligned}$$

and the variance

$$\begin{aligned} \text{Var}(N_2) &= \mathbb{E}(N_2^2) - \mathbb{E}(N_2)^2 \\ &= \frac{2}{3}(p_1 + p_2 + p_3 - (p_1p_2 + p_2p_3 + p_1p_3)) + \frac{4}{3}(p_1p_2 + p_2p_3 + p_1p_3) - 4p^2 \\ &= 2p - 4p^2 + \frac{2}{3}(p_1p_2 + p_2p_3 + p_1p_3) \quad . \end{aligned}$$

(iv) We have

$$\begin{aligned}
\mathrm{Var}(N_1) - \mathrm{Var}(N_2) &= 2p(1-p) - 2p + 4p^2 - \frac{2}{3}(p_1p_2 + p_2p_3 + p_1p_3) \\
&= 2p^2 - \frac{2}{3}(p_1p_2 + p_2p_3 + p_1p_3) \\
&= \frac{2}{9}(p_1^2 + p_2^2 + p_3^2 + 2(p_1p_2 + p_2p_3 + p_1p_3)) \\
&\qquad\qquad - \frac{2}{3}(p_1p_2 + p_2p_3 + p_1p_3) \\
&= \frac{1}{9}(2p_1^2 + 2p_2^2 + 2p_3^2 - 2p_1p_2 - 2p_2p_3 - 2p_1p_3) \\
&= \frac{1}{9}((p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_1 - p_3)^2) \geqslant 0 \quad .
\end{aligned}$$

This right-hand side is zero if and only if $p_1 = p_2$, $p_2 = p_3$, and $p_1 = p_3$. In other words:

$$\mathrm{Var}(N_1) \geqslant \mathrm{Var}(N_2) \quad ,$$

with equality if and only if $p_1 = p_2 = p_3$.

Question 13

(i) Candidates cannot score 5 points or more without answering at least 3 questions correctly, thus Candidate A must choose $k \geqslant 3$ if she wishes to pass. If she chooses $k = 3$, then to pass she must answer all 3 questions correctly (scoring 6); thus the probability of passing given a candidate answers exactly 3 questions is

$$P_3 = \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^3} \quad .$$

If Candidate A chooses $k = 4$, then to pass she must answer either 3 of 4 questions correctly (scoring 5), or all 4 questions correctly (scoring 8); thus the probability of passing given a candidate answers exactly 4 questions is

$$P_4 = \binom{4}{3} \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{n-1}{n} + \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{4}{n^3} - \frac{3}{n^4} \quad .$$

Lastly, if Candidate A chooses $k = 5$, then to pass she must answer either 4 of 5 questions correctly (scoring 7), or all 5 questions correctly (scoring 10); thus the probability of passing given a candidate answers exactly 5 questions is

$$P_5 = \binom{5}{4} \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{n-1}{n} + \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{5}{n^4} - \frac{4}{n^5} \quad .$$

We find that

$$P_4 - P_3 = \frac{3}{n^3} - \frac{3}{n^4} = \frac{3}{n^4}(n-1) > 0 \quad ,$$

thus for any $n \geqslant 2$, Candidate A has a greater probability of passing if she answers 4 questions than if she answers 3. Similarly

$$P_4 - P_5 = \frac{4}{n^3} - \frac{8}{n^4} + \frac{4}{n^5} = \frac{4}{n^5}(n^2 - 2n + 1) = \frac{4}{n^5}(n-1)^2 > 0 \quad ,$$

thus for any $n \geqslant 2$, Candidate A has a greater probability of passing if she answers 4 questions than if she answers 5. Hence Candidate A maximises her probability of passing by choosing $k = 4$.

(ii) For Candidate B, we have

$$\mathbb{P}\{\text{Exactly 4 questions} \mid \text{Passed}\} = \frac{\mathbb{P}\{\text{Exactly 4 questions and passed}\}}{\mathbb{P}\{\text{Passed}\}} \quad .$$

We are given that the probability Candidate B attempts any given number of questions is $\frac{1}{6}$, hence using P_3, P_4, and P_5 as calculated in part (i):

$$\begin{aligned} \mathbb{P}\{\text{Exactly 4 questions} \mid \text{Passed}\} &= \frac{\frac{1}{6}P_4}{\frac{1}{6}P_3 + \frac{1}{6}P_4 + \frac{1}{6}P_5} \\ &= \frac{\frac{4}{n^3} - \frac{3}{n^4}}{\frac{5}{n^3} + \frac{2}{n^4} - \frac{4}{n^5}} \\ &= \frac{4n^2 - 3n}{5n^2 + 2n - 4} \quad . \end{aligned}$$

(iii) For Candidate C, the probability of passing is

$$\begin{aligned}
\mathbb{P}\{\text{Pass}\} &= \mathbb{P}\{\text{Toss 3 heads}\} \cdot P_3 + \mathbb{P}\{\text{Toss 4 heads}\} \cdot P_4 + \mathbb{P}\{\text{Toss 5 heads}\} \cdot P_5 \\
&= \binom{5}{3}\frac{n^3}{(n+1)^5}P_3 + \binom{5}{4}\frac{n^4}{(n+1)^5}P_4 + \binom{5}{5}\frac{n^5}{(n+1)^5}P_5 \\
&= \frac{10n^3}{(n+1)^5}\frac{1}{n^3} + \frac{5n^4}{(n+1)^5}(\frac{4}{n^3} - \frac{3}{n^4}) + \frac{n^5}{(n+1)^5}(\frac{5}{n^4} - \frac{4}{n^5}) \\
&= \frac{10}{(n+1)^5} + \frac{5}{(n+1)^5}(4n-3) + \frac{1}{(n+1)^5}(5n-4) \\
&= \frac{25n-9}{(n+1)^5} \quad .
\end{aligned}$$

STEP II

Section A: Pure Mathematics

Question 1

First we note (by inspection, or explicit substitution), that $x = 0$ cannot be a solution to $(*)$; hence if k is a root, we know that k^{-1} will be well-defined.

Suppose that k is a root of $(*)$, that is:

$$k^4 + ak^3 + bk^2 + ak + 1 = 0 \quad .$$

Then, dividing through by k^4 (which, as noted above, is okay to do),

$$1 + ak^{-1} + bk^{-2} + ak^{-3} + k^{-4} = 0 \quad ,$$

that is:

$$(k^{-1})^4 + a(k^{-1})^3 + b(k^{-1})^2 + ak^{-1} + 1 = 0 \quad ,$$

so k^{-1} is a root of $(*)$.

(i) Since any real number k being a root of $(*)$ implies that k^{-1} is a root, we can have one distinct root k if and only if $k = k^{-1}$. Thus we have exactly two possibilities: $k = 1$ or $k = -1$, giving

$$\begin{aligned} & x^4 + ax^3 + bx^2 + ax + 1 \equiv (x-1)^4 \quad \iff \quad (a, b) = (-4, 6) \\ \text{or} \quad & x^4 + ax^3 + bx^2 + ax + 1 \equiv (x+1)^4 \quad \iff \quad (a, b) = (4, 6) \end{aligned}$$

respectively.

(ii) For exactly three distinct roots, we must have a pair of distinct reciprocal roots $x = k$, $x = k^{-1}$ for some $k \neq 1$, $k \neq -1$, and a pair of identical reciprocal roots; so either a repeated root of $x = 1$, or $x = -1$. If $x = 1$ is a root, then substitution gives

$$\begin{aligned} 1 + a + b + a + 1 &= 0 \\ b &= -2a - 2 \quad , \end{aligned}$$

whereas if $x = -1$ is a root, then substitution gives

$$\begin{aligned} 1 - a + b - a + 1 &= 0 \\ b &= 2a - 2 \quad , \end{aligned}$$

and we derive the desired result.

(iii) In the case $b = 2a - 2$, we have

$$x^4 + ax^3 + (2a-2)x^2 + ax + 1 = (x^2 + 2x + 1)(x^2 + (a-2)x + 1) \quad ,$$

hence our roots of $(*)$ are

$$x = -1 \quad , \quad x = \frac{2 - a + \sqrt{(a-2)^2 - 4}}{2} \quad , \quad x = \frac{2 - a - \sqrt{(a-2)^2 - 4}}{2} \quad .$$

Similarly solving the case $b = -2a - 2$ we have

$$x^4 + ax^3 - (2a+2)x^2 + ax + 1 = (x^2 - 2x + 1)(x^2 + (a+2)x + 1) \quad ,$$

giving roots

$$x = 1 \quad , \quad x = \frac{-2 - a + \sqrt{(a+2)^2 - 4}}{2} \quad , \quad x = \frac{-2 - a - \sqrt{(a+2)^2 - 4}}{2} \quad .$$

Hence the necessary and sufficient conditions for exactly three distinct real roots are

$$(b+2)^2 = 4a^2$$

with

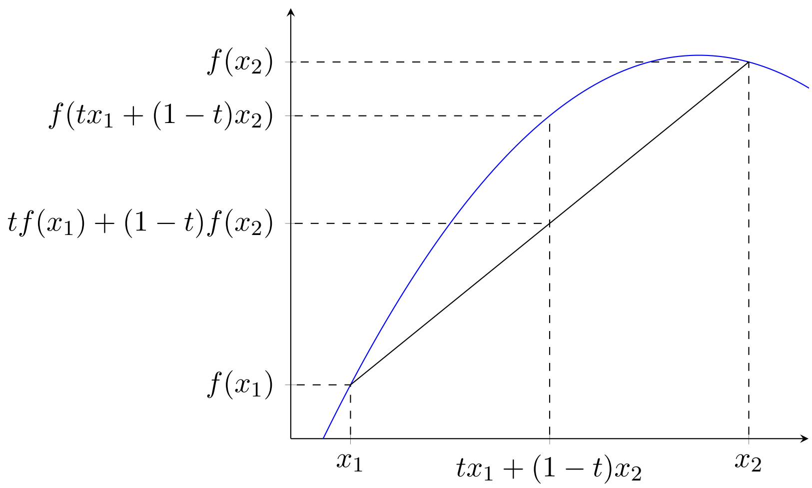
$$\begin{aligned} (a-2)^2 > 4 \quad \text{if} \quad b + 2 = 2a \quad (&\iff a^2 > 4a \ , \ b + 2 = 2a) \\ \text{and} \quad (a+2)^2 > 4 \quad \text{if} \quad b + 2 = -2a \quad (&\iff a^2 > -4a \ , \ b + 2 = -2a) \quad , \end{aligned}$$

where the inequalities are strict to guarantee that the roots of $x^2 + (a \mp 2)x + 1$ are distinct (and thus not repeated roots of $x = 1$ or $x = -1$). In fact, from the equivalent expressions in brackets, we see that the necessary and sufficient conditions are:

$$(b+2)^2 = 4a^2 \quad \text{and} \quad a^2 > 2b + 4 \quad .$$

Question 2

We illustrate the definition of concavity with the following sketch:



If $f''(x) < 0$ for $a < x < b$, then the gradient of $f(x)$ decreases as x increases in this domain. Suppose that f is not concave, then there must exist x_1 and x_2 in (a, b), and $t \in (0, 1)$ such that $f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$. The gradient of $f(x)$ at $x = tx_1 + (1-t)x_2$ must be less than the gradient of the chord connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so the curve must continue to have a gradient of this value or less. The curve therefore cannot pass through $(x_2, f(x_2))$, giving a contradiction.

(i) Let $x_1 = \frac{1}{3}(2u+v)$, $x_2 = \frac{1}{3}(v+2w)$, and $t = \frac{1}{2}$, then we have

$$\begin{aligned}
&f(tx_1 + (1-t)x_2) \geqslant tf(x_1) + (1-t)f(x_2) \\
\iff &f\left(\frac{1}{3}\left(u+\frac{1}{2}v\right)+\frac{1}{3}\left(\frac{1}{2}v+w\right)\right) \geqslant \frac{1}{2}f\left(\frac{1}{3}(2u+v)\right)+\frac{1}{2}f\left(\frac{1}{3}(v+2w)\right) \\
\iff &f\left(\frac{u+v+w}{3}\right) \geqslant \frac{1}{2}\left(f\left(\frac{2}{3}u+\frac{1}{3}v\right)+f\left(\frac{1}{3}v+\frac{2}{3}w\right)\right) \quad ,
\end{aligned}$$

using the concavity condition on both fs on the right, we get

$$\begin{aligned}
\iff f\left(\frac{u+v+w}{3}\right) &\geqslant \frac{1}{2}\left(f\left(\frac{2}{3}u+\frac{1}{3}v\right)+f\left(\frac{1}{3}v+\frac{2}{3}w\right)\right) \\
&\geqslant \frac{1}{2}\left(\frac{2}{3}f(u)+\frac{1}{3}f(v)+\frac{1}{3}f(v)+\frac{2}{3}f(w)\right) \\
\implies f\left(\frac{u+v+w}{3}\right) &\geqslant \frac{f(u)+f(v)+f(w)}{3} \quad .
\end{aligned}$$

(ii) First we establish that $f(x) = \sin(x)$ is concave for $0 < x < \pi$ (the possible interior angles of a triangle). We have that $f''(x) = -\sin(x)$, and so since $\sin(x) > 0$ for $0 < x < \pi$, we do indeed have $f''(x) < 0$ over this domain. Now by the result of (i), we have

$$\begin{aligned} \frac{1}{3}\left(\sin(A) + \sin(B) + \sin(C)\right) &\leqslant \sin\left(\frac{A+B+C}{3}\right) \\ \iff \sin(A) + \sin(B) + \sin(C) &\leqslant 3\sin\left(\frac{\pi}{3}\right) \\ \iff \sin(A) + \sin(B) + \sin(C) &\leqslant \frac{3\sqrt{3}}{2} \quad . \end{aligned}$$

(iii) Now let $f(x) = \ln(\sin(x))$; we have

$$\begin{aligned} f'(x) &= \frac{\cos(x)}{\sin(x)} \\ f''(x) &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)} \quad , \end{aligned}$$

hence this f also satisfies $f''(x) < 0$ for $0 < x < \pi$, hence this f is concave. By the result of (i) then, we have

$$\begin{aligned} \frac{1}{3}\left(\ln(\sin(A)) + \ln(\sin(B)) + \ln(\sin(C))\right) &\leqslant \ln\left(\sin\left(\frac{A+B+C}{3}\right)\right) \\ \iff \ln(\sin(A)\cdot\sin(B)\cdot\sin(C)) &\leqslant 3\ln\left(\sin\left(\frac{\pi}{3}\right)\right) \\ \iff \ln(\sin(A)\cdot\sin(B)\cdot\sin(C)) &\leqslant 3\ln\left(\frac{\sqrt{3}}{2}\right) \quad , \end{aligned}$$

and since log is an increasing function, we can exponentiate both sides and preserve the inequality, to get

$$\begin{aligned} \sin(A)\cdot\sin(B)\cdot\sin(C) &\leqslant \left(\frac{\sqrt{3}}{2}\right)^3 \\ \iff \sin(A)\cdot\sin(B)\cdot\sin(C) &\leqslant \frac{3\sqrt{3}}{8} \quad . \end{aligned}$$

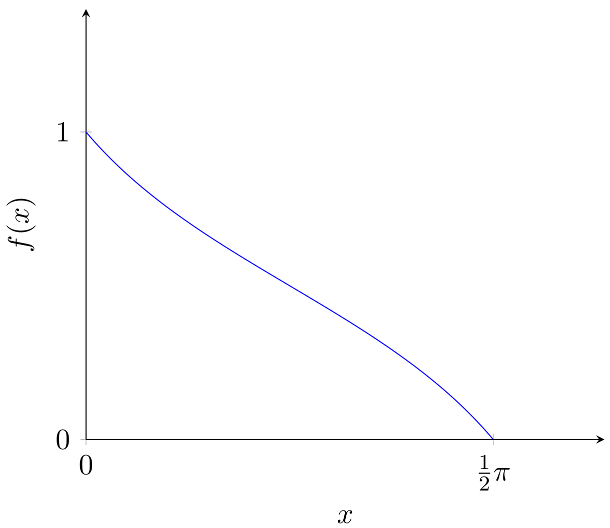
Question 3

(i) By the chain rule, we have

$$\begin{aligned} f'(x) &= -\frac{1}{(1+\tan(x))^2} \cdot \sec^2(x) \\ &= -\frac{1}{\cos^2(x)(1+\tan(x))^2} \\ &= -\frac{1}{(\cos(x)+\sin(x))^2} \\ &= -\frac{1}{\cos^2(x)+\sin^2(x)+2\cos(x)\sin(x)} \\ &= -\frac{1}{1+\sin(2x)} \quad . \end{aligned}$$

For $0 \leqslant x < \frac{1}{2}\pi$, we have $1 \leqslant 1+\sin(2x) \leqslant 2$, so the range of $f'(x)$ is $[-1, -\frac{1}{2}]$.

To sketch the curve, we note that $f(0) = 1$, $f'(0) = -1$, and as $x \to \frac{1}{2}\pi$ we have $f(x) \to 0$, $f'(x) \to -1$. Further, by the range of $f'(x)$, $f(x)$ is strictly decreasing for $0 \leqslant x < \frac{1}{2}\pi$; and we see that $f'(x)$ achieves its maximum value of $-\frac{1}{2}$ at $x = \frac{1}{4}\pi$, decreasing to $f'(x) = -1$ at $x = 0$ and $x \to \frac{1}{2}\pi$. Our sketch thus looks like



(ii) $y = g(x)$ will have rotational symmetry of order two about (a, b) if and only if $g(a) = b$ and the line joining (a, b) to $(a + x, g(a + x))$ is a 180-degree rotation of the line joining (a, b) to $(a - x, g(a - x))$ for all $a - x$ or $a + x$ in $[-1, 1]$. Since these lines will have equal horizontal component (namely x) they must have equal vertical component: that is, if and only if for all x

$$\begin{aligned} & g(a+x) - g(a) = g(a) - g(a-x) \\ \iff & g(a+x) + g(a-x) = 2g(a) \\ \iff & g(a+x) + g(a-x) = 2b \quad . \end{aligned}$$

Since this condition must hold for all relevant x, we can substitute x for $x - a$ to get that $y = g(x)$ has rotational symmetry of order two about (a, b) if and only if for all x

$$g(x) + g(2a - x) = 2b \quad .$$

Given the symmetry conditions on $y = g(x)$, we see that $g(x)$ must be an odd function over $-1 \leqslant x \leqslant 1$:

$$\begin{aligned} & g(x) + g(-x) = 2g(0) = 0 \\ \iff & g(x) = -g(-x) \quad , \end{aligned}$$

hence the integral is of an odd function over a symmetric domain, giving

$$\int_{-1}^{1} g(x) dx = 0 \quad .$$

(iii) Note that $\cot(x) = \tan(\frac{1}{2}\pi - x)$, hence

$$\begin{aligned} \frac{1}{1 + \tan^k(\frac{1}{2}\pi - x)} &= \frac{1}{1 + \cot^k(x)} \\ &= \frac{\tan^k(x)}{\tan^k(x) + 1} \\ &= 1 - \frac{1}{1 + \tan^k(x)} \quad , \end{aligned}$$

hence $g(x) = \frac{1}{1+\tan^k(x)}$ satisfies

$$\begin{aligned} g(\tfrac{1}{2}\pi - x) &= 1 - g(x) \\ g(x) + g(\tfrac{1}{2}\pi - x) &= 1 \quad . \end{aligned}$$

Hence the curve has rotational symmetry of order two about $(\frac{1}{4}\pi, \frac{1}{2})$.

With this we can evaluate

$$\begin{aligned}
\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k(x)} dx &= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1+\tan^k(x)} dx + \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k(x)} dx \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1+\tan^k(x)} dx - \int_{\frac{1}{4}\pi}^{\frac{1}{6}\pi} \frac{1}{1+\tan^k(\frac{1}{2}\pi - x)} dx \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1+\tan^k(x)} dx - \int_{\frac{1}{4}\pi}^{\frac{1}{6}\pi} \left(1 - \frac{1}{1+\tan^k(x)}\right) dx \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1+\tan^k(x)} dx + \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \left(1 - \frac{1}{1+\tan^k(x)}\right) dx \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} dx \\
&= \frac{1}{4}\pi - \frac{1}{6}\pi = \frac{1}{12}\pi \quad .
\end{aligned}$$

Question 4

(i) Using the given result, we have

$$\begin{aligned}
&\cos(x) + \cos(4x) + 3\cos(2x) + 3\cos(3x) \\
&\qquad = 2\cos\left(\tfrac{1}{2}(4x+x)\right)\cos\left(\tfrac{1}{2}(4x-x)\right) + 6\cos\left(\tfrac{1}{2}(3x+2x)\right)\cos\left(\tfrac{1}{2}(3x-2x)\right) \\
&\qquad = 2\cos\left(\tfrac{5}{2}x\right)\left(\cos\left(\tfrac{3}{2}x\right) + 3\cos\left(\tfrac{1}{2}x\right)\right) \\
&\qquad = 2\cos\left(\tfrac{5}{2}x\right)\left(2\cos\left(\tfrac{1}{2}\left(\tfrac{3}{2}x+\tfrac{1}{2}x\right)\right)\cos\left(\tfrac{1}{2}\left(\tfrac{3}{2}x-\tfrac{1}{2}x\right)\right) + 2\cos\left(\tfrac{1}{2}x\right)\right) \\
&\qquad = 4\cos\left(\tfrac{5}{2}x\right)\left(\cos(x)\cos\left(\tfrac{1}{2}x\right) + \cos\left(\tfrac{1}{2}x\right)\right) \\
&\qquad = 4\cos\left(\tfrac{5}{2}x\right)\cos\left(\tfrac{1}{2}x\right)(\cos(x)+1) \quad .
\end{aligned}$$

Hence we have $\cos(x) + \cos(2x) + \cos(3x) + \cos(4x) = 0$ if and only if

$$\cos\left(\tfrac{5}{2}x\right) = 0 \quad \text{or} \quad \cos\left(\tfrac{1}{2}x\right) = 0 \quad \text{or} \quad \cos(x) = -1 \quad ,$$

which (since $0 \leqslant \frac{5}{2}x \leqslant 5\pi$, $0 \leqslant \frac{1}{2}x \leqslant \pi$) holds if and only if

$$\tfrac{5}{2}x \in \left\{\tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \tfrac{7}{2}\pi, \tfrac{9}{2}\pi\right\} \quad \text{or} \quad \tfrac{1}{2}x = \tfrac{1}{2}\pi \quad \text{or} \quad x = \pi \quad .$$

That is,

$$x \in \left\{\tfrac{1}{5}\pi, \tfrac{3}{5}\pi, \pi, \tfrac{7}{5}\pi, \tfrac{9}{5}\pi\right\} \quad .$$

(ii) Again using the given result, we find that

$$\begin{aligned}
& \cos(x+y) + \cos(x-y) - \cos(2x) = 1 \\
\iff\ & 2\cos(x)\cos(y) - \cos^2(x) + \sin^2(x) = 1 \\
\iff\ & 2\cos(x)\cos(y) - 2\cos^2(x) + 1 = 1 \\
\iff\ & 2\cos(x)(\cos(y) - \cos(x)) = 0 \quad .
\end{aligned}$$

Hence either

$$\cos(x) = \cos(y) \quad ,$$

which holds if and only if $x = y$ (since $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$, and over this domain, cos is a one-to-one function); or

$$\cos(x) = 0 \quad ,$$

which (since $0 \leqslant x \leqslant \pi$) holds if and only if $x = \frac{1}{2}\pi$.

(iii) Lastly, we have

$$\begin{aligned}
\tfrac{3}{2} &= \cos(x) + \cos(y) - \cos(x+y) \\
&= 2\cos\left(\tfrac{1}{2}(x+y)\right)\cos\left(\tfrac{1}{2}(x-y)\right) - \cos^2\left(\tfrac{1}{2}(x+y)\right) + \sin^2\left(\tfrac{1}{2}(x+y)\right) \\
&= 2\cos\left(\tfrac{1}{2}(x+y)\right)\cos\left(\tfrac{1}{2}(x-y)\right) - 2\cos^2\left(\tfrac{1}{2}(x+y)\right) + 1 \quad ,
\end{aligned}$$

this gives

$$\begin{aligned}
0 &= \cos^2\left(\tfrac{1}{2}(x+y)\right) - \cos\left(\tfrac{1}{2}(x+y)\right)\cos\left(\tfrac{1}{2}(x-y)\right) + \tfrac{1}{4} \\
&= \left(\cos\left(\tfrac{1}{2}(x+y)\right) - \tfrac{1}{2}\cos\left(\tfrac{1}{2}(x-y)\right)\right)^2 + \tfrac{1}{4} - \tfrac{1}{4}\cos^2\left(\tfrac{1}{2}(x-y)\right)^2 \\
&= \left(\cos\left(\tfrac{1}{2}(x+y)\right) - \tfrac{1}{2}\cos\left(\tfrac{1}{2}(x-y)\right)\right)^2 + \tfrac{1}{4}\sin^2\left(\tfrac{1}{2}(x-y)\right)^2 \quad .
\end{aligned}$$

Thus for this to hold, we must have both

$$\cos\left(\tfrac{1}{2}(x+y)\right) = \tfrac{1}{2}\cos\left(\tfrac{1}{2}(x-y)\right) \quad \text{and} \quad \sin\left(\tfrac{1}{2}(x-y)\right) = 0 \quad .$$

Since $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$, we have that $-\frac{1}{2}\pi \leqslant \frac{1}{2}(x-y) \leqslant \frac{1}{2}\pi$, hence the second of these equations is satisfied if and only if $x = y$. Substituting into the first equation we have

$$\begin{aligned}
&\cos(x) = \tfrac{1}{2}\cos(0) = \tfrac{1}{2} \\
\iff\ & x = \tfrac{1}{3}\pi \quad ,
\end{aligned}$$

that is: the only values of x and y satisfying the given equation are $x = y = \frac{1}{3}\pi$.

Question 5

(i) For $|x| < 1$, we have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n \quad ,$$

so (*ignoring issues of convergence*), integrating this term-by-term gives

$$\begin{aligned}
\ln(1+x) = \int_0^x \frac{1}{1+t}dt &= \int_0^x \left(\sum_{n=0}^{\infty}(-t)^n\right) dt \\
&= \sum_{n=0}^{\infty}\left[-\frac{(-t)^{n+1}}{n+1}\right]_{t=0}^{t=x} \\
&= -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}
\end{aligned}$$

for $|x| < 1$.

(ii) We have

$$e^{-ax} = \sum_{n=0}^{\infty}\frac{(-ax)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-a)^n}{n!}x^n \quad ,$$

hence (*ignoring issues of convergence*)

$$\begin{aligned}
\int_0^{\infty}\frac{(1-e^{-ax})e^{-x}}{x}dx &= \int_0^{\infty}\frac{1}{x}\left(1-\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}x^n\right)e^{-x}dx \\
&= -\sum_{n=1}^{\infty}\left(\frac{(-a)^n}{n!}\int_0^{\infty}x^{n-1}e^{-x}dx\right) \quad .
\end{aligned}$$

Now let

$$I_n = \int_0^{\infty}x^{n-1}e^{-x}dx \quad :$$

we have

$$I_1 = \int_0^{\infty}e^{-x}dx = 1 \quad ,$$

and for $n > 1$:

$$\begin{aligned}
I_{n+1} &= \int_0^{\infty}x^n e^{-x}dx \\
&= \left[x^n\left(-e^{-x}\right)\right]_{x=0}^{x\to\infty} - \int_0^{\infty}nx^{n-1}\left(-e^{-x}\right)dx \\
&= (0-0) + nI_n = nI_n \quad .
\end{aligned}$$

Thus, by induction,

$$I_n = (n-1)! \quad .$$

Using this result, we have that for $|a| < 1$

$$\begin{aligned}
\int_0^\infty \frac{(1 - e^{-ax})e^{-x}}{x} dx &= -\sum_{n=1}^\infty \frac{(-a)^n}{n!} I_n \\
&= -\sum_{n=1}^\infty \frac{(-a)^n \cdot (n-1)!}{n!} \\
&= -\sum_{n=1}^\infty \frac{(-a)^n}{n} \\
&= \ln(1 + a) \qquad\qquad \text{by part (i).}
\end{aligned}$$

(iii) Substituting $x = e^{-u}$, we find

$$\begin{aligned}
\int_0^1 \frac{x^p - x^q}{\ln x} dx &= \int_\infty^0 \frac{e^{-pu} - e^{-qu}}{-u} \left(\frac{d}{du} e^{-u} \right) du \\
&= \int_0^\infty \frac{e^{-pu} - e^{-qu}}{u} \left(-e^{-u} \right) du \\
&= \int_0^\infty \frac{-e^{-pu}e^{-u} - (-e^{-qu})e^{-u}}{u} du \\
&= \int_0^\infty \left(\frac{(1 - e^{-pu})e^{-u}}{u} - \frac{(1 - e^{-qu})e^{-u}}{u} \right) du \\
&= \ln(1 + p) - \ln(1 + q) \\
&= \ln \left(\frac{1 + p}{1 + q} \right) \quad .
\end{aligned}$$

Question 6

(i) If $n \geqslant 5$, then $n!$ (as the product of each positive integer up to and including n) will be a multiple of 5. Thus for any $n \geqslant 5$, $n! + 5$ is divisible by 5, and so cannot be prime (we can discount the possibility that it could be equal to 5 – the only prime number that is a multiple of 5 – since $n! + 5$ is a strictly increasing function of n and we have that $5! + 5 = 125 \gg 5$).

Evaluating for $1 \leqslant n \leqslant 4$ then, we have:

$$\begin{aligned} n = 1 &: \quad n! + 5 = 1 + 5 = 6 \quad \text{(not prime)} \\ n = 2 &: \quad n! + 5 = 2 + 5 = 7 \quad \text{(prime)} \\ n = 3 &: \quad n! + 5 = 6 + 5 = 11 \quad \text{(prime)} \\ n = 4 &: \quad n! + 5 = 24 + 5 = 29 \quad \text{(prime)} \quad , \end{aligned}$$

and so our answers are

$$(n,p) = (2,7) \quad , \quad (n,p) = (3,11) \quad , \quad (n,p) = (4,29) \quad .$$

(ii) Suppose positive integers (n, m) satisfy $1! \times 3! \times \cdots \times (2n-1)! = m!$ (call this equation $(*)$) and $n \geqslant 7$. By theorem 1, we must have $m > 4n$. By theorem 2 then, there is some prime number p satisfying $2n < p < m$.

Since the left hand side of $(*)$ is a product of integers less than $2n$, which is less than p, and p is a prime number, the left hand side of $(*)$ is not a multiple of p. Since the right hand side of $(*)$ is the product of all integers up to and including m, which is greater than $4n$, which is greater than p, the right hand side of $(*)$ must be a multiple of p. The equality $(*)$ thus gives a contradiction, and we cannot have any solution pair (n, m) with $n \geqslant 7$.

Evaluating for $1 \leqslant n \leqslant 6$ then, we have:

$$\begin{aligned} n = 1 &: \quad 1! = 1! \\ n = 2 &: \quad 1! \times 3! = 3! \\ n = 3 &: \quad 3! \times 5! = 6 \times 5! = 6! \\ n = 4 &: \quad 6! \times 7! = 2^4 \times 3^2 \times 5 \times 7! = 2 \times 3^2 \times 5 \times 8! = 2 \times 5 \times 9! = 10! \end{aligned}$$

We can see that $n = 5$ cannot gives us a solution, since (using the result for $n = 4$) the left hand of $(*)$ will equal $10! \times 9!$, which contains no factor of 11 (which is prime), while if m is to satisfy $(*)$ with $n = 5$ it must be at least 11 (since for $n = 4$ we had $m = 10$), so the right hand side of $(*)$ must be a multiple of 11 (contradiction).

Finally then, we check $n = 6$, and find

$$10! \times 9! \times 11! > 2 \times 3 \times 4 \times 7 \times 11! = 12 \times 14 \times 11! > 13! \quad .$$

So for $n = 6$ to be a solution, we must have $m > 13$, and the right hand side of $(*)$ will be a multiple of 13, but the left hand side of $(*)$ is a product of integers less than or equal to 11, so cannot be a multiple of 13, and so by contradiction, $n = 6$ cannot be a solution.

Hence our answers are

$$(n, m) = (1, 1) \quad , \quad (n, m) = (2, 3) \quad , \quad (n, m) = (3, 6) \quad , \quad (n, m) = (4, 10) \quad .$$

Question 7

For this question a diagram is particularly useful.

Let $\boldsymbol{m} = \alpha\boldsymbol{a}$ and $\boldsymbol{n} = \beta\boldsymbol{b}$ (where α and β are in $[0,1]$). We have that $|MQ| = \mu|QB|$, hence

$$\begin{aligned}\boldsymbol{q} - \boldsymbol{m} &= \mu(\boldsymbol{b} - \boldsymbol{q})\\ \boldsymbol{q} - \alpha\boldsymbol{a} &= \mu\boldsymbol{b} - \mu\boldsymbol{q}\\ (1+\mu)\boldsymbol{q} &= \alpha\boldsymbol{a} + \mu\boldsymbol{b}\\ \boldsymbol{q} &= \frac{\alpha}{1+\mu}\boldsymbol{a} + \frac{\mu}{1+\mu}\boldsymbol{b} \quad .\end{aligned}$$

Similarly $|NQ| = \nu|QA|$ gives

$$\boldsymbol{q} = \frac{\nu}{1+\nu}\boldsymbol{a} + \frac{\beta}{1+\nu}\boldsymbol{b} \quad .$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, the coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$ must be the same in each of these expressions for $\boldsymbol{q}$: that is

$$\begin{aligned}\frac{\alpha}{1+\mu} &= \frac{\nu}{1+\nu}\\ \alpha &= \frac{(1+\mu)\nu}{1+\nu} \quad .\end{aligned}$$

Hence

$$\boldsymbol{m} = \frac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} \quad ,$$

as required. By symmetry, equating the coefficients of $\boldsymbol{b}$ to find β gives

$$\boldsymbol{n} = \frac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} \quad .$$

Given $\boldsymbol{l} = \lambda\boldsymbol{b}$ (by the fact that L lies on OB and $|OL| = \lambda|OB|$), the vector $\vec{ML}$ can be written

$$\begin{aligned}\vec{ML} &= \boldsymbol{l} - \boldsymbol{m}\\ &= \lambda\boldsymbol{b} - \frac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} \quad .\end{aligned}$$

The vector $\vec{AN}$ can be written

$$\begin{aligned}\vec{AN} &= \boldsymbol{n} - \boldsymbol{a}\\ &= \frac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a} \quad .\end{aligned}$$

Given that ML is parallel to AN, we know that the vector equation for $\vec{ML}$ is simply a scalar multiple of that for $\vec{AN}$; hence comparing coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\begin{aligned}\lambda &= \frac{(1+\nu)\mu}{1+\mu} \cdot \frac{(1+\mu)\nu}{1+\nu} \\ &= \mu\nu \quad .\end{aligned}$$

Since $\boldsymbol{l} = \mu\nu\boldsymbol{b}$, the significance of $\mu\nu < 1$ is that L lies on the side OB, as opposed to on the ray OB extended past B.

Question 8

(i) With the given substitution, we have $y = v^2$, giving $\frac{dy}{dt} = 2v\frac{dv}{dt}$, so our differential equation becomes

$$\begin{aligned} 2v\frac{dv}{dt} &= \alpha v - \beta v^2 \\ \frac{dv}{dt} &= \frac{\alpha}{2} - \frac{\beta}{2}v \\ \frac{dv}{dt} + \frac{\beta}{2}v &= \frac{\alpha}{2} \quad . \end{aligned}$$

We can then solve this using an integrating factor:

$$\begin{aligned} e^{\frac{\beta}{2}t}\frac{dv}{dt} + \frac{\beta}{2}e^{\frac{\beta}{2}t}v &= \frac{\alpha}{2}e^{\frac{\beta}{2}t} \\ \frac{d}{dt}\left(e^{\frac{\beta}{2}t}v\right) &= \frac{\alpha}{2}e^{\frac{\beta}{2}t} \\ e^{\frac{\beta}{2}t}v &= \frac{\alpha}{\beta}e^{\frac{\beta}{2}t} + c \quad , \end{aligned}$$

where c is an arbitrary constant, giving

$$\begin{aligned} v(t) &= \frac{\alpha}{\beta} + ce^{-\frac{\beta}{2}t} \\ y(t) &= \left(\frac{\alpha}{\beta} + ce^{-\frac{\beta}{2}t}\right)^2 \quad . \end{aligned}$$

Requiring $y_1(0) = 0$, we need

$$\begin{aligned} \left(\frac{\alpha}{\beta} + c\right)^2 &= 0 \\ \iff c &= -\frac{\alpha}{\beta} \quad , \end{aligned}$$

that is:

$$y_1(t) = \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{-\frac{\beta}{2}t}\right)^2 = \frac{\alpha^2}{\beta^2}\left(1 - e^{-\frac{\beta}{2}t}\right)^2 \quad .$$

(ii) We try the substitution $v = y^{\frac{1}{3}}$, so $y = v^3$ and $\frac{dy}{dt} = 3v^2\frac{dv}{dt}$:

$$\begin{aligned} 3v^2\frac{dv}{dt} &= \alpha v^2 - \beta v^3 \\ \frac{dv}{dt} &= \frac{\alpha}{3} - \frac{\beta}{3}v \\ \frac{dv}{dt} + \frac{\beta}{3}v &= \frac{\alpha}{3} \quad . \end{aligned}$$

Again using an integrating factor, we have

$$\begin{aligned}
e^{\frac{\beta}{3}t}\frac{dv}{dt} + \frac{\beta}{3}e^{\frac{\beta}{3}t}v &= \frac{\alpha}{3}e^{\frac{\beta}{3}t} \\
\frac{d}{dt}\left(e^{\frac{\beta}{3}t}v\right) &= \frac{\alpha}{3}e^{\frac{\beta}{3}t} \\
e^{\frac{\beta}{3}t}v &= \frac{\alpha}{\beta}e^{\frac{\beta}{3}t} + c \quad ,
\end{aligned}$$

where c is an arbitrary constant, giving

$$\begin{aligned}
v(t) &= \frac{\alpha}{\beta} + ce^{-\frac{\beta}{3}t} \\
y(t) &= \left(\frac{\alpha}{\beta} + ce^{-\frac{\beta}{3}t}\right)^3 \quad .
\end{aligned}$$

Requiring $y_2(0) = 0$, we need

$$\begin{aligned}
\left(\frac{\alpha}{\beta} + c\right)^3 &= 0 \\
\iff c &= -\frac{\alpha}{\beta} \quad ,
\end{aligned}$$

that is:

$$y_2(t) = \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{-\frac{\beta}{3}t}\right)^3 = \frac{\alpha^3}{\beta^3}\left(1 - e^{-\frac{\beta}{3}t}\right)^3 \quad .$$

(iii) With $\alpha = \beta$, we have

$$\begin{aligned}
y_1(t) &= \left(1 - e^{-\frac{\beta}{2}t}\right)^2 \quad , \\
y_2(t) &= \left(1 - e^{-\frac{\beta}{3}t}\right)^3 \quad .
\end{aligned}$$

At $t = 0$, we have $y_1(0) = y_2(0) = 0$, and looking at the defining differential equations, we must also have $y_1'(0) = y_2'(0) = 0$. As $t \to \infty$, we have $y_1 \to 1$ and $y_2 \to 1$. By considering the decay in $e^{-\frac{\beta}{2}t}$ and $e^{-\frac{\beta}{3}t}$, we know that both $y_1(t)$ and $y_2(t)$ are increasing functions of t, thus for all $t > 0$ we have $0 < y_1(t) < 1$ and $0 < y_2(t) < 1$. Lastly, we note that since β is positive we have that for all $t > 0$

$$\begin{aligned}
-\frac{\beta}{2}t < -\frac{\beta}{3}t &< 0 \\
\implies 0 < e^{-\frac{\beta}{2}t} < e^{-\frac{\beta}{3}t} &< 1 \quad ,
\end{aligned}$$

so we have

$$0 < \left(1 - e^{-\frac{\beta}{3}t}\right) < \left(1 - e^{-\frac{\beta}{2}t}\right) < 1$$

and so also

$$\left(1 - e^{-\frac{\beta}{3}t}\right)^2 < \left(1 - e^{-\frac{\beta}{2}t}\right)^2 \quad .$$

Combining this last inequality with

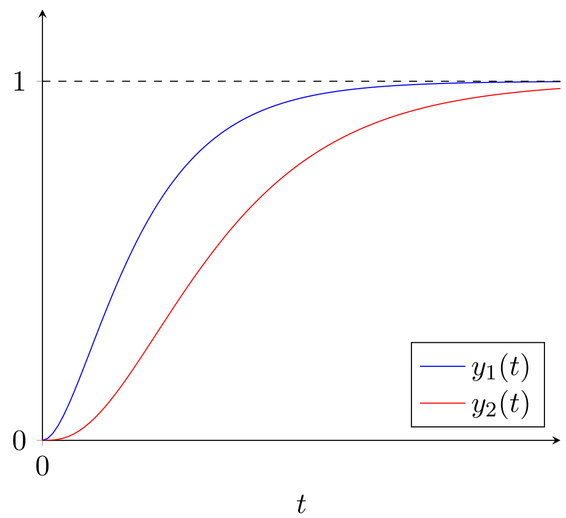
$$\left(1 - e^{-\frac{\beta}{3}t}\right) < 1 \quad ,$$

we get

$$\left(1 - e^{-\frac{\beta}{3}t}\right)^3 < \left(1 - e^{-\frac{\beta}{2}t}\right)^2 \quad ,$$

that is: $y_2(t) < y_1(t)$ for all t.

Our sketch is as follows:



Section B: Mechanics

Question 9

We split this question into parts according to the different motions of the beads. Throughout positive velocities and displacements will correspond to downward motion, and negative velocities and displacements to upward motion:

(i) Both A and B descend until A collides with the ground. A has initial vertical displacement $8h$, starts at rest, and accelerates under gravity for a time t_1. We use SUVAT equations to determine the final speed V just before the collision:

$$\begin{aligned} V^2 &= 0^2 + 2g \cdot 8h \\ \implies V &= 4\sqrt{gh} \quad . \end{aligned}$$

During this time B identically descends through a vertical distance of $8h$ to a final vertical displacement of $9h$ from the ground, and when A collides with the ground, B is descending with a velocity of $V = 4\sqrt{gh}$.

(ii) A rebounds from the floor, with coefficient of restitution $\frac{1}{2}$, hence its velocity changes instantaneously to $-2\sqrt{gh}$ (now travelling upwards).

(iii) A starts ascending with B still descending for some further time t_2 until they collide at height H at some time $t_1 + t_2$. Considering the motion of A, we have

$$-H = -2\sqrt{gh}t_2 + \frac{1}{2}gt_2^2 \quad .$$

Considering the motion of B, we have

$$9h - H = 4\sqrt{gh}t_2 + \frac{1}{2}gt_2^2 \quad .$$

Subtracting the first of these two equations from the second gives

$$\begin{aligned} 9h &= 6\sqrt{gh}t_2 \\ \implies t_2 &= \frac{3}{2}\sqrt{\frac{h}{g}} \quad , \end{aligned}$$

and substituting this into the first of the two equations, we have

$$\begin{aligned} -H &= -2\sqrt{gh} \cdot \frac{3}{2}\sqrt{\frac{h}{g}} + \tfrac{1}{2}g \cdot \frac{9}{4}\frac{h}{g} \\ \implies H &= 3h - \frac{9}{8}h \\ &= \frac{15}{8}h \quad , \end{aligned}$$

as we were told to show.

We also use SUVAT equations to find the velocities u_A, u_B before the impact:

$$u_A = -2\sqrt{gh} + \frac{3}{2}\sqrt{gh} = -\frac{1}{2}\sqrt{gh} \quad ,$$
$$\text{and} \quad u_B = 4\sqrt{gh} + \frac{3}{2}\sqrt{gh} = \frac{11}{2}\sqrt{gh} \quad .$$

(iv) A descends again until it collides with the ground a second time, again at speed $V = 4\sqrt{gh}$. Using SUVAT equations we determine the velocity v_A of A immediately after the collision with B:

$$\begin{aligned} V^2 &= v_A^2 + 2gH \\ v_A^2 &= 16gh - \frac{15}{4}gh = \frac{49}{4}gh \\ \implies v_A &= \frac{7}{2}\sqrt{gh} \quad . \end{aligned}$$

We know we must take the positive root here because the total momentum of A and B in the collision is directed downwards, thus both beads cannot be ascending immediately after the collision, and so A (being the lower bead) must be descending immediately after the collision.

By conservation of momentum the velocity v_B of B immediately after the collision must satisfy

$$u_A + u_B = v_A + v_B$$

(where we have cancelled factors of mass, since the beads have equal masses). Thus we find

$$v_B = \frac{11}{2}\sqrt{gh} + \frac{1}{2}\sqrt{gh} - \frac{7}{2}\sqrt{gh} = \frac{3}{2}\sqrt{gh} \quad ,$$

and hence the coefficient of restitution between the two beads is

$$e = \frac{\frac{7}{2}\sqrt{gh} - \frac{3}{2}\sqrt{gh}}{\frac{11}{2}\sqrt{gh} + \frac{1}{2}\sqrt{gh}} = \frac{2\sqrt{gh}}{6\sqrt{gh}} = \frac{1}{3} \quad .$$

Question 10

The speed at which the string moves relative to the table surface varies linearly along its length. At time t the length of the string is $a + ut$, thus at time t the speed of the given point on the string is

$$\frac{x}{a+ut} \cdot u \quad .$$

Combining the speed v of the ant relative to the string, and the speed of a point on the string relative to the table surface, we have

$$\frac{dx}{dt} = v + \frac{xu}{a+ut} \quad .$$

From this we can verify

$$\begin{aligned} \frac{d}{dt}\left(\frac{x}{a+ut}\right) &= \frac{(a+ut)\frac{dx}{dt} - xu}{(a+ut)^2} \\ &= \frac{(a+ut)v + xu - xu}{(a+ut)^2} \\ &= \frac{v}{a+ut} \quad . \end{aligned}$$

By the relation just verified, we have

$$\begin{aligned} \frac{x(t)}{a+ut} &= \int_0^t \frac{v}{a+u\tau} d\tau \\ &= \frac{v}{u} \int_0^t \frac{1}{\frac{a}{u} + \tau} d\tau \\ &= \frac{v}{u} \frac{\log(\frac{a}{u} + t)}{\log(\frac{a}{u})} \quad . \end{aligned}$$

We can now evaluate this at time T, using the fact that at this time the ant has reached the end of the string, and so $x(T) = a + uT$:

$$\begin{aligned} \frac{a+uT}{a+uT} &= \frac{v}{u} \log\left(1 + \frac{u}{a}T\right) \\ \frac{u}{v} &= \log\left(1 + \frac{u}{a}T\right) \\ e^{\frac{u}{v}} &= 1 + \frac{u}{a}T \quad , \end{aligned}$$

giving

$$uT = a(e^k - 1) \quad ,$$

where $k = \frac{u}{v}$, as required.

For the last part, we can either go through a similar method to the above, by finding an ODE for the ant's motion, OR ... :

If we change our frame of reference to travel along with the end of the string being pulled, now it appears as if that end is fixed, and the end where the ant started is being pulled away at a constant speed u. In this new frame of reference, the ant will still move with speed v relative to the string, so the question is identical to what we have done above, except now the 'initial' length of the string is $a + uT$ instead of a. Hence the remaining time T_2 until the ant gets back to where it started satisfies

$$\begin{aligned} uT_2 &= (a + uT)(e^k - 1) \\ &= a(e^k - 1) + uT(e^k - 1) \\ &= uT + uT(e^k - 1) \\ &= uTe^k \\ \implies T_2 &= Te^k \quad . \end{aligned}$$

Question 11

Taking moments about the centre of mass, we have

$$R_f \cdot d + F \cdot h = R_r \cdot (b - d)$$
$$\implies F = \frac{1}{h}(R_r(b-d) - R_f d) \quad ,$$

where R_f and R_r are the normal reaction forces on the front and rear wheels respectively, and F is the (frictional) driving force on the rear wheel. By resolving forces vertically we have

$$R_f + R_r = mg \quad .$$

Supposing that the front wheel loses contact with the road (before the rear wheel slips), this would occur when

$$R_f = 0 \quad , \quad R_r = mg \quad , \quad F = \frac{mg}{h}(b-d) \quad .$$

By friction, the maximum possible F is

$$F_{\max} = \mu mg \quad .$$

Thus the rear wheel will necessarily slip before the front wheel loses contact with the road if

$$\mu mg < \frac{mg}{h}(b-d)$$
$$\iff \mu < \frac{b-d}{h} \quad .$$

Now assuming the inequality holds; just before the rear wheel slips F will achieve its maximum $F = \mu R_r$, giving

$$\frac{1}{h}(R_r(b-d) - (mg - R_r)d) = \mu R_r$$
$$R_r b - mgd = \mu R_r h$$
$$R_r = \frac{mgd}{b - \mu h} \quad .$$

This gives a maximum driving force of

$$F = \frac{\mu mgd}{b - \mu h} \quad ,$$

and hence (by $F = ma$) a maximum acceleration

$$a = \frac{\mu gd}{b - \mu h} \quad .$$

Now supposing the inequality does not hold (so the front wheel loses contact with the road before the rear wheel slips). The acceleration is given by

$$a = \frac{F}{m} = \frac{R_r(b-d) - R_f d}{mh} \quad ,$$

hence $R_f + R_r = mg$ gives

$$a = \frac{R_r b - mgd}{mh} \quad \text{and} \quad a = \frac{mg(b-d) - R_f b}{mh} \quad ,$$

thus the acceleration increases as R_r increases and R_f decreases. The maximum acceleration is thus achieved just before the front wheel loses contact with the road when $R_f = 0$ and $R_r = mg$:

$$a_{\max} = \frac{mg(b-d)}{mh} = \frac{g(b-d)}{h} \quad .$$

Section C: Probability and Statistics

Question 12

(i) In version 1, we either win £h, if we toss h consecutive Heads, with probability

$$P = p^h \quad ,$$

or we lose and win nothing. Thus the expected winnings are

$$E_h = hp^h \quad .$$

We have

$$\begin{aligned} \frac{E_{h+1}}{E_h} &= \frac{(h+1)p^{h+1}}{hp^h} \\ &= \frac{h+1}{h}p = \frac{(h+1)N}{h(N+1)} \quad , \end{aligned}$$

thus

$$\begin{aligned} E_{h+1} \geqslant E_h \iff & (h+1)N \geqslant h(N+1) \\ \iff & h \leqslant N \quad . \end{aligned}$$

Similarly

$$\begin{aligned} \frac{E_{h-1}}{E_h} &= \frac{(h-1)p^{h-1}}{hp^h} \\ &= \frac{h-1}{h}\frac{1}{p} = \frac{(h-1)(N+1)}{hN} \quad , \end{aligned}$$

thus

$$\begin{aligned} E_{h-1} \geqslant E_h \iff & (h-1)(N+1) \geqslant hN \\ \iff & h \geqslant N+1 \quad . \end{aligned}$$

Hence we have that E_h increases monotonically as h increases until $h = N$; we have $E_N = E_{N+1}$, and that as h increases past $h = N+1$, E_h monotonically decreases. Thus we can maximise our expected winnings by choosing $h = N$ (or $h = N+1$).

(ii) In version 2, we either win £h, if we toss h Heads without tossing 2 consecutive Tails, or we lose and win nothing. In tossing h Heads without tossing 2 consecutive Tails, there are h points in the sequence where we can toss a single Tail without losing.

The probability of winning despite tossing t Tails (each separated by at least one Head) is

$$P_t = \binom{h}{t} p^h (1-p)^t \quad ,$$

thus our probability of winning is

$$\begin{aligned} P &= \sum_{t=0}^{h} P_t = \sum_{t=0}^{h} \binom{h}{t} p^h (1-p)^t \\ &= p^h \sum_{t=0}^{h} \binom{h}{t} (1-p)^t = p^h (1 + (1-p))^h \\ &= \left(\frac{N}{N+1}\right)^h \left(2 - \frac{N}{N+1}\right)^h \\ &= \left(\frac{N}{N+1}\right)^h \left(\frac{N+2}{N+1}\right)^h = \frac{N^h (N+2)^h}{(N+1)^{2h}} \quad , \end{aligned}$$

and so our expected winnings are

$$E_h = \frac{h N^h (N+2)^h}{(N+1)^{2h}} \quad .$$

In the case $N = 2$, the expected winnings are

$$E_h = h \frac{2^h \cdot 4^h}{3^{2h}} = h \left(\frac{8}{9}\right)^h \quad .$$

Comparing this to part (i) with $N = 8$, we know that this is maximised by choosing $h = 8$ or $h = 9$. Using the given approximation we have

$$2 \approx 3^{0.63} = 3^{\frac{63}{100}} \quad ,$$

so we can approximate the maximum expected winnings:

$$\begin{aligned} E_8 &= \frac{8^9}{9^8} = \frac{2^{27}}{3^{16}} \\ &\approx \frac{3^{\frac{63*27}{100}}}{3^{16}} = 3^{\frac{1701}{100} - 16} = 3^{1.01} \\ &\approx 3 \quad . \end{aligned}$$

Question 13

(i) From the given probabilities, we can read off that

$$A_1 = \frac{1}{2} \quad , \quad B_1 = \frac{1}{4} \quad , \quad C_1 = 0 \quad , \quad D_1 = \frac{1}{4} \quad .$$

Considering how the parcel can be passed at the second whistle we compute

$$\begin{aligned} A_2 &= \frac{1}{2}A_1 + \frac{1}{4}(B_1 + D_1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \quad , \\ B_2 &= \frac{1}{2}B_1 + \frac{1}{4}(A_1 + C_1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \quad , \\ C_2 &= \frac{1}{2}C_1 + \frac{1}{4}(B_1 + D_1) = 0 + \frac{1}{8} = \frac{1}{8} \quad , \\ D_2 &= \frac{1}{2}D_1 + \frac{1}{4}(A_1 + C_1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \quad . \end{aligned}$$

(ii) We have

$$\begin{aligned} B_{n+1} + D_{n+1} &= \frac{1}{2}B_n + \frac{1}{4}(A_n + C_n) + \frac{1}{2}D_n + \frac{1}{4}(A_n + C_n) \\ &= \frac{1}{2}(A_n + B_n + C_n + D_n) = \frac{1}{2} \quad . \end{aligned}$$

We note also that by symmetry $B_n = D_n$ for each n. Hence for all n we have

$$B_n = D_n = \frac{1}{4} \quad .$$

Since $B_n + D_n = \frac{1}{2}$, we also have $A_n + C_n = \frac{1}{2}$. Consider

$$\begin{aligned} A_{n+1} - C_{n+1} &= \frac{1}{2}A_n + \frac{1}{4}(B_n + D_n) - \frac{1}{2}C_n - \frac{1}{4}(B_n + D_n) \\ &= \frac{1}{2}(A_n - C_n) \quad , \end{aligned}$$

hence, by induction

$$A_n - C_n = \frac{1}{2^{n-1}}(A_1 - C_1) = \frac{1}{2^n} \quad .$$

Now we can solve for A_n and C_n:

$$A_n = \frac{1}{2}\left((A_n + C_n) + (A_n - C_n)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2^n}\right) = \frac{1}{4} + \frac{1}{2^{n+1}} \quad ,$$

and

$$C_n = \frac{1}{2}\left((A_n + C_n) - (A_n - C_n)\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^n}\right) = \frac{1}{4} - \frac{1}{2^{n+1}} \quad .$$

STEP III

Section A: Pure Mathematics

Question 1

(i) Assuming that $\beta \neq 0$,

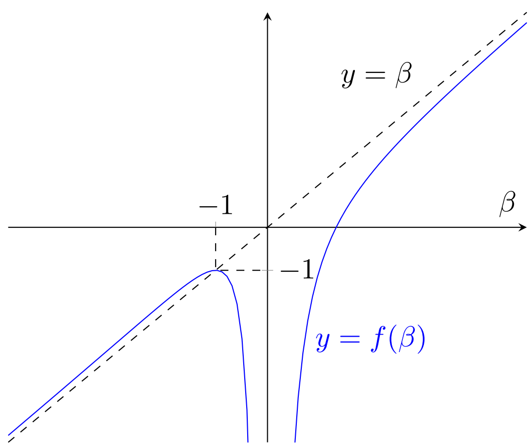
$$f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \quad ,$$

and we have

$$\begin{aligned} f'(\beta) &= 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} \\ &= \frac{1}{\beta^3}(\beta^3 + \beta + 2) \\ &= \frac{1}{\beta^3}(\beta + 1)(\beta^2 - \beta + 2) \\ &= \frac{1}{\beta^3}(\beta + 1)\left(\left(\beta - \frac{1}{2}\right)^2 + \frac{7}{4}\right) \quad , \end{aligned}$$

thus (given $\beta \neq 0$), the only turning point ($f'(\beta) = 0$) is $\beta = -1$, $f(-1) = -1$.

In order to draw the graph: for $\beta < -1$, $f'(\beta)$ is positive (so the function is increasing); for $-1 < \beta < 0$, $f'(\beta)$ is negative (so the function is decreasing); we have that $f'(\beta) \to -\infty$ as $\beta \to 0^-$, and $f'(\beta) \to \infty$ as $\beta \to 0^+$; lastly for $\beta > 0$, $f'(\beta)$ is again positive. As $|\beta| \to \infty$, $f(\beta) \sim \beta$ ($\frac{1}{\beta}$ and $\frac{1}{\beta^2}$ decay in modulus, while β grows in modulus). This leads to the following graph:

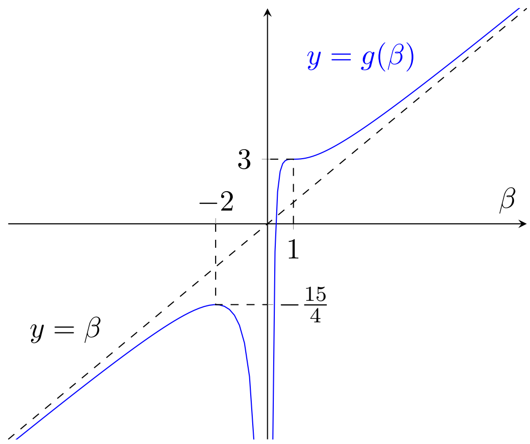


Similarly with $g(\beta)$, we find

$$g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}$$
$$= \frac{1}{\beta^3}(\beta^3 - 3\beta + 2)$$
$$= \frac{1}{\beta^3}(\beta - 1)(\beta^2 + \beta - 2)$$
$$= \frac{1}{\beta^3}(\beta - 1)^2(\beta + 2) \quad ,$$

hence (given $\beta \neq 0$), we have turning points $\beta = 1$, $g(\beta) = 3$, and $\beta = -2$, $g(\beta) = -\frac{15}{4}$.

In order to draw the graph: for $\beta < -2$, $g'(\beta)$ is positive (so the function is increasing); for $-2 < \beta < 0$, $g'(\beta)$ is negative (so the function is decreasing); we have that $g'(\beta) \to -\infty$ as $\beta \to 0^-$, and $g'(\beta) \to \infty$ as $\beta \to 0^+$; lastly for $0 < \beta < 1$ and $\beta > 1$, $g'(\beta)$ is again positive, with $g'(1) = 0$. As $|\beta| \to \infty$, $g(\beta) \sim \beta$. This leads to the following graph:



(ii) Vieta's formulae give that

$$u + v = -\alpha \qquad \text{and} \qquad uv = \beta \quad ,$$

thus we find

$$u + v + \frac{1}{uv} = -\alpha + \frac{1}{\beta} \quad ,$$

and

$$\frac{1}{u} + \frac{1}{v} + uv = \frac{u+v}{uv} + uv = -\frac{\alpha}{\beta} + \beta \quad .$$

(iii) We are given that

$$
\begin{aligned}
-\alpha + \frac{1}{\beta} &= -1 \\
\alpha &= 1 + \frac{1}{\beta} \quad ,
\end{aligned}
$$

thus

$$
\begin{aligned}
\frac{1}{u} + \frac{1}{v} + uv &= -\frac{\alpha}{\beta} + \beta \\
&= -\frac{1}{\beta} - \frac{1}{\beta^2} + \beta \\
&= f(\beta) \quad .
\end{aligned}
$$

Since u and v are real, the discriminant of the quadratic gives that

$$\alpha^2 - 4\beta \geqslant 0 \quad ,$$

so

$$
\begin{aligned}
\left(1 + \frac{1}{\beta}\right)^2 - 4\beta \geqslant 0 \\
\iff 4\beta - 1 - \frac{2}{\beta} - \frac{1}{\beta^2} \leqslant 0 \\
\iff 4\beta^3 - \beta^2 - 2\beta - 1 \leqslant 0 \\
(\beta - 1)(4\beta^2 + 3\beta + 1) \leqslant 0 \\
(\beta - 1)\left(\left(2\beta + \frac{3}{4}\right)^2 + \frac{7}{16}\right) \leqslant 0 \\
\iff \beta \leqslant 1 \quad .
\end{aligned}
$$

We have that $\beta \leqslant 1$ and find that $f(1) = -1$, so by the shape of the curve $y = f(\beta)$, the maximum possible value for $f(\beta)$ on this domain is -1, *id est*:

$$\frac{1}{u} + \frac{1}{v} + uv \leqslant -1 \quad .$$

(iv) Now we are instead given that

$$-\alpha + \frac{1}{\beta} = 3$$
$$\alpha = -3 + \frac{1}{\beta} \quad ,$$

thus

$$\begin{aligned} \frac{1}{u} + \frac{1}{v} + uv &= -\frac{\alpha}{\beta} + \beta \\ &= \frac{3}{\beta} - \frac{1}{\beta^2} + \beta \\ &= g(\beta) \quad . \end{aligned}$$

Again, since u and v are real, the discriminant of the quadratic gives that

$$\alpha^2 - 4\beta \geqslant 0 \quad ,$$

so

$$\begin{aligned} \left(-3 + \frac{1}{\beta}\right)^2 - 4\beta &\geqslant 0 \\ \iff 4\beta - 9 + \frac{6}{\beta} - \frac{1}{\beta^2} &\leqslant 0 \\ \iff 4\beta^3 - 9\beta^2 + 6\beta - 1 &\leqslant 0 \\ (\beta - 1)(4\beta^2 - 5\beta + 1) &\leqslant 0 \\ (\beta - 1)^2(4\beta - 1) &\leqslant 0 \\ \iff \beta = 1 \quad \text{or} \quad \beta &\leqslant \frac{1}{4} \quad . \end{aligned}$$

By the shape of the curve $y = g(\beta)$, the maximum possible value for $g(\beta)$ on this domain is $g(1) = 3$, *id est*:

$$\frac{1}{u} + \frac{1}{v} + uv \leqslant 3 \quad .$$

Question 2

(i) We have that

$$\frac{dz}{dx} = -2xe^{-x^2} = -2xz \quad ,$$

and that for all $n \geqslant 1$

$$zy_n = (-1)^n \frac{d^n z}{dx^n} \quad .$$

Hence for all $n \geqslant 1$

$$\begin{aligned} zy_{n+1} &= (-1)^{n+1} \frac{d^{n+1} z}{dx^{n+1}} \\ &= -\frac{d}{dx}\left((-1)^n \frac{d^n z}{dx^n}\right) \\ &= -\frac{d}{dx}(zy_n) = -\frac{dz}{dx} y_n - z\frac{dy_n}{dx} \\ &= 2xzy_n - z\frac{dy_n}{dx} \quad . \end{aligned}$$

Thus for all $n \geqslant 1$

$$y_{n+1} = 2xy_n - \frac{dy_n}{dx} \quad ,$$

that is:

$$\frac{dy_n}{dx} = 2xy_n - y_{n+1} \quad .$$

(ii) For $n = 1$ we have

$$\begin{aligned} y_2 &= \frac{1}{z}\frac{d^2 z}{dx^2} = \frac{1}{z}\frac{d}{dx}(-2xz) \\ &= -2 - \frac{2x}{z}\frac{dz}{dx} = -2y_0 + 2xy_1 \quad , \end{aligned}$$

hence the equation holds when $n = 1$. Now supposing the equation holds for a given value $n = k \geqslant 1$ and using the result from (i), we have

$$\begin{aligned} y_{k+1} &= 2xy_k - 2ky_{k-1} \\ \Longrightarrow \quad \frac{dy_{k+1}}{dx} &= 2y_k + 2x\frac{dy_k}{dx} - 2k\frac{dy_{k-1}}{dx} \\ \Longleftrightarrow \quad 2xy_{k+1} - y_{k+2} &= 2y_k + 2x(2xy_k - y_{k+1}) - 2k(2xy_{k-1} - y_k) \\ &= 2y_k + 4x^2 y_k - 2xy_{k+1} + 2x(y_{k+1} - 2xy_k) + 2ky_k \\ \Longleftrightarrow \quad y_{k+2} &= 2xy_{k+1} - (2k+2)y_k \quad , \end{aligned}$$

which is the required result for $n = k + 1$. Hence the equation holds for all $n \geqslant 1$.

Thus we have, for all $n \geqslant 1$

$$\begin{aligned}
y_{n+1} &= 2xy_n - 2ny_{n-1} \\
\Longrightarrow \quad y_{n+1}^2 &= y_n(2xy_{n+1}) - 2ny_{n-1}y_{n+1} \\
&= y_n(y_{n+2} + 2(n+1)y_n) - 2ny_{n-1}y_{n+1} \\
\Longleftrightarrow \quad y_{n+1}^2 - y_n y_{n+2} &= (2n+2)y_n^2 - 2ny_{n-1}y_{n+1} \\
y_{n+1}^2 - y_n y_{n+2} &= 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2 \quad .
\end{aligned}$$

(iii) For all $n \geqslant 1$ let $q_n = y_n^2 - y_{n-1}y_{n+1}$. We have

$$\begin{aligned}
y_0 &= 1 \quad , \\
y_1 &= -\frac{1}{z}\frac{dz}{dx} = 2x \quad , \\
y_2 &= \frac{1}{z}\frac{d}{dx}(-2xz) \\
&= -2 - \frac{2x}{z}(-2xz) = 4x^2 - 2 \quad .
\end{aligned}$$

Hence

$$\begin{aligned}
q_1 &= 4x^2 - (4x^2 - 2) = 2 \\
\Longrightarrow \quad q_1 &> 0 \qquad \text{(for all } x\text{)} \quad .
\end{aligned}$$

By the result of part (ii), we have that for all $n \geqslant 1$

$$q_{n+1} = 2nq_n + 2y_n^2 \quad .$$

Thus for all $n \geqslant 1$ (and all x), we have $q_{n+1} > 2nq_n > q_n$. Hence by induction $q_n > 0$ (for all x) for all $n \geqslant 1$, which is the required result.

Question 3

By the product rule, we have:

$$\begin{aligned}x^a(x^b(x^cy)')' &= x^a(x^b(x^cy' + cx^{c-1}y))' \\ &= x^a(x^{b+c}y' + cx^{b+c-1}y)' \\ &= x^a(x^{b+c}y'' + (b+c)x^{b+c-1}y' + cx^{b+c-1}y' + c(b+c-1)x^{b+c-2}y) \\ &= x^{a+b+c}y'' + (b+2c)x^{a+b+c-1}y' + c(b+c-1)x^{a+b+c-2}y \quad ,\end{aligned}$$

hence if we can find a, b, c, such that:

$$\begin{aligned}a+b+c &= 2 \quad , \\ b+2c &= 1-2p \quad , \\ c(b+c-1) &= p^2-q^2 \quad ,\end{aligned}$$

then we can write the given differential equation in the form $(*)$. Attempting to solve for a, b, c in the general case:

$$\begin{aligned}a &= 2-b-c \quad , \\ b &= 1-2c-2p \quad ,\end{aligned}$$

giving

$$\begin{aligned}c(b+c-1) &= p^2-q^2 \\ \implies c(-c-2p) &= p^2-q^2 \\ c^2+2cp+p^2 &= q^2 \\ (c+p)^2 &= q^2 \\ c &= -p \pm q \quad , \\ \implies b &= 1 \mp 2q \quad , \\ \implies a &= 1+p \pm q \quad .\end{aligned}$$

So we can write the given differential equation in the form $(*)$ if we set

$$\begin{cases} a = 1+p+q \\ b = 1-2q \\ c = -p+q \end{cases} \qquad \text{or} \qquad \begin{cases} a = 1+p-q \\ b = 1+2q \\ c = -p-q \end{cases} \quad .$$

(i) With $f(x) \equiv 0$, $(*)$ gives

$$\begin{aligned}(x^b(x^cy)')' = 0 \iff x^b(x^cy)' &= A \\ (x^cy)' &= Ax^{-b} \quad ,\end{aligned}$$

where A is an arbitrary constant.

This now gives

$$\begin{cases} x^c y = -\dfrac{A}{b-1}x^{-b+1} + B & \text{if } b \neq 1 \\ x^c y = A\log(x) + B & \text{if } b = 1 \end{cases}$$
$$\iff \begin{cases} y = -\dfrac{A}{b-1}x^{-b-c+1} + Bx^{-c} & \text{if } b \neq 1 \\ y = Ax^{-c}\log(x) + Bx^{-c} & \text{if } b = 1 \end{cases} \quad ,$$

where A and B are arbitrary constants.

In terms of p and q, we have that

$$-b-c+1 = p \pm q \quad \text{and} \quad -c = p \mp q \quad ,$$

hence if $q \neq 0$ (so $b \neq 1$), our general solution is

$$y = Cx^{p-q} + Dx^{p+q} \quad ,$$

where C and D are arbitrary constants. Whereas if $q = 0$ (so $b = 1$), we have

$$-c = p \quad ,$$

and our general solution is

$$y = Ax^p \log(x) + Bx^p \quad ,$$

where A and B are arbitrary constants.

(ii) Now we have $q = 0$ and $f(x) = x^n$. By $(*)$ we have:

$$\begin{aligned} x^{1+p}(x(x^{-p}y)')' &= x^n \\ (x(x^{-p}y)')' &= x^{n-p-1} \quad , \end{aligned}$$

hence

$$\begin{cases} x(x^{-p}y)' = \dfrac{1}{n-p}x^{n-p} + A & \text{if } n \neq p \\ x(x^{-p}y)' = \log(x) + A & \text{if } n = p \end{cases}$$
$$\iff \begin{cases} (x^{-p}y)' = \dfrac{1}{n-p}x^{n-p-1} + Ax^{-1} & \text{if } n \neq p \\ (x^{-p}y)' = x^{-1}\log(x) + Ax^{-1} & \text{if } n = p \end{cases} \quad ,$$

where A is an arbitrary constant.

First consider $n \neq p$:

$$\begin{aligned}(x^{-p}y)' &= \frac{1}{n-p}x^{n-p-1} + Ax^{-1} \\ x^{-p}y &= \frac{1}{(n-p)^2}x^{n-p} + A\log(x) + B\end{aligned}$$

where B is an arbitrary constant. This gives the general solution

$$y = \frac{1}{(n-p)^2}x^n + Ax^p\log(x) + Bx^p \quad .$$

Now consider $n = p$:

$$\begin{aligned}(x^{-p}y)' &= x^{-1}\log(x) + Ax^{-1} \\ &= \frac{1}{2}\frac{d}{dx}\left(\log(x)^2\right) + Ax^{-1} \\ \implies x^{-p}y &= \frac{1}{2}\log(x)^2 + A\log(x) + B \quad ,\end{aligned}$$

where B is an arbitrary constant. This gives the general solution

$$y = \frac{1}{2}x^p\log(x)^2 + Ax^p\log(x) + Bx^p \quad .$$

Question 4

From the equation of the hyperbola, we have

$$\begin{aligned} \frac{2}{a^2}x - \frac{2}{b^2}y\frac{dy}{dx} &= 0 \\ \implies \frac{1}{b^2}y\frac{dy}{dx} &= \frac{1}{a^2}x \quad . \end{aligned}$$

Hence at P,

$$\begin{aligned} \frac{1}{b}\tan(\theta)\frac{dy}{dx} &= \frac{1}{a}\sec(\theta) \\ \implies \frac{dy}{dx} &= \frac{b}{a}\text{cosec}(\theta) \quad . \end{aligned}$$

Hence the equation of the tangent to the curve at P is

$$\begin{aligned} y - b\tan(\theta) &= \frac{b}{a}\text{cosec}(\theta)\,(x - a\sec(\theta)) \\ a\sin(\theta)y - ab\frac{\sin^2(\theta)}{\cos(\theta)} &= bx - ab\sec(\theta) \\ bx - ay\sin(\theta) &= ab\frac{1-\sin^2(\theta)}{\cos(\theta)} \\ bx - ay\sin(\theta) &= ab\cos(\theta) \quad . \end{aligned}$$

(i) Substituting $x = \frac{a}{b}y$, we find the coordinates (x_s, y_s) of S:

$$\begin{aligned} ay_s - ay_s\sin(\theta) &= ab\cos(\theta) \\ y_s &= \frac{b\cos(\theta)}{1-\sin(\theta)} \quad , \\ \implies x_s &= \frac{a\cos(\theta)}{1-\sin(\theta)} \quad . \end{aligned}$$

Similarly, we substitute $x = -\frac{a}{b}y$ to find the coordinates (x_t, y_t) of T:

$$\begin{aligned} -ay_t - ay_t\sin(\theta) &= ab\cos(\theta) \\ y_t &= -\frac{b\cos(\theta)}{1+\sin(\theta)} \quad , \\ \implies x_t &= \frac{a\cos(\theta)}{1+\sin(\theta)} \quad . \end{aligned}$$

Thus the coordinates of the midpoint of S and T (call it M) are:

$$\begin{aligned}(x_m, y_m) &= \left(\frac{1}{2}\left(\frac{a\cos(\theta)}{1-\sin(\theta)} + \frac{a\cos(\theta)}{1+\sin(\theta)}\right), \frac{1}{2}\left(\frac{b\cos(\theta)}{1-\sin(\theta)} - \frac{b\cos(\theta)}{1+\sin(\theta)}\right)\right) \\ &= \left(\frac{1}{2}a\cos(\theta)\cdot\frac{2}{1-\sin^2(\theta)}, \frac{1}{2}b\cos(\theta)\frac{2\sin(\theta)}{1-\sin^2(\theta)}\right) \\ &= \left(a\frac{\cos(\theta)}{\cos^2(\theta)}, b\frac{\sin(\theta)\cos(\theta)}{\cos^2(\theta)}\right) \\ &= (a\sec(\theta), b\tan(\theta)) \quad .\end{aligned}$$

Thus the midpoint of S and T is in fact the point P.

(ii) Similar to before, we have that the gradient at Q is

$$\frac{dy}{dx} = \frac{b}{a}\text{cosec}(\phi) \quad ,$$

and that the equation of the tangent to the curve at Q is

$$bx - ay\sin(\phi) = ab\cos(\phi) \quad .$$

Since the tangents at P and Q are perpendicular, we have

$$\begin{aligned}\frac{b^2}{a^2}\text{cosec}(\theta)\text{cosec}(\phi) &= -1 \\ \sin(\theta)\sin(\phi) &= -\frac{b^2}{a^2} \quad .\end{aligned}$$

Let the intersection of the two tangents be I at (x_i, y_i). Combining the two tangent equations, eliminating x_i, we have

$$\begin{aligned}ay_i\sin(\theta) + ab\cos(\theta) &= ay_i\sin(\phi) + ab\cos(\phi) \\ y_i(\sin(\theta) - \sin(\phi)) &= b(\cos(\phi) - \cos(\theta)) \\ y_i &= \frac{b(\cos(\phi) - \cos(\theta))}{\sin(\theta) - \sin(\phi)} \quad .\end{aligned}$$

(Note that since $\sin(\theta)\sin(\phi) < 0$ we must have $\sin(\theta) \neq \sin(\phi)$.)

Multiplying the two tangent equations by $\sin(\phi)$ and $\sin(\theta)$ respectively, we use the result that $\sin(\theta)\sin(\phi) = -\frac{b^2}{a^2}$ to get

$$\begin{cases} bx_i \sin(\phi) + \frac{b^2}{a} y_i = ab\cos(\theta)\sin(\phi) \\ bx_i \sin(\theta) + \frac{b^2}{a} y_i = ab\cos(\phi)\sin(\theta) \end{cases} \quad ,$$

and combining these equations, eliminating y_i, we have

$$\begin{aligned} bx_i(\sin(\theta) - \sin(\phi)) &= ab(\cos(\phi)\sin(\theta) - \cos(\theta)\sin(\phi)) \\ x_i &= \frac{a(\cos(\phi)\sin(\theta) - \cos(\theta)\sin(\phi))}{\sin(\theta) - \sin(\phi)} \quad . \end{aligned}$$

Thus we have

$$\begin{aligned} x_i^2 &= a^2 \left(\frac{\cos(\phi)\sin(\theta) - \cos(\theta)\sin(\phi)}{\sin(\theta) - \sin(\phi)} \right)^2 \quad , \\ y_i^2 &= b^2 \left(\frac{\cos(\phi) - \cos(\theta)}{\sin(\theta) - \sin(\phi)} \right)^2 \\ &= -a^2 \sin(\theta)\sin(\phi) \left(\frac{\cos(\phi) - \cos(\theta)}{\sin(\theta) - \sin(\phi)} \right)^2 \quad . \end{aligned}$$

Thus

$$\begin{aligned} x_i^2 + y_i^2 &= \frac{a^2 \left((\cos(\phi)\sin(\theta) - \cos(\theta)\sin(\phi))^2 - \sin(\theta)\sin(\phi)(\cos(\phi) - \cos(\theta))^2 \right)}{(\sin(\theta) - \sin(\phi))^2} \\ &= \frac{a^2 \left(\cos^2(\phi)\sin(\theta)(\sin(\theta) - \sin(\phi)) + \cos^2(\theta)\sin(\phi)(\sin(\phi) - \sin(\theta)) \right)}{(\sin(\theta) - \sin(\phi))^2} \\ &= \frac{a^2 \left(\cos^2(\phi)\sin(\theta) - \cos^2(\theta)\sin(\phi) \right)}{\sin(\theta) - \sin(\phi)} \\ &= \frac{a^2 \left(\sin(\theta) - \sin^2(\phi)\sin(\theta) - \sin(\phi) + \sin^2(\theta)\sin(\phi) \right)}{\sin(\theta) - \sin(\phi)} \\ &= a^2 - a^2 \sin(\theta)\sin(\phi) \frac{\sin(\phi) - \sin(\theta)}{\sin(\theta) - \sin(\phi)} \\ &= a^2 + a^2 \sin(\theta)\sin(\phi) \\ &= a^2 - b^2 \quad , \end{aligned}$$

as required.

Question 5

(i) Consider

$$\begin{aligned}(k+1)(A_{k+1}-G_{k+1})-k(A_k-G_k)&\\ &= a_1+a_2+\cdots+a_{k+1}-(k+1)G_{k+1}-(a_1+a_2+\cdots+a_k)+kG_k\\ &= a_{k+1}-(k+1)G_{k+1}+kG_k\\ &= G_k\left(\frac{a_{k+1}}{G_k}-(k+1)\frac{G_{k+1}}{G_k}+k\right)\\ &= G_k\left(\lambda_k^{k+1}-(k+1)\frac{\left(a_{k+1}G_k^k\right)^{\frac{1}{k+1}}}{G_k}+k\right)\\ &= G_k\left(\lambda_k^{k+1}-(k+1)\frac{a_{k+1}^{\frac{1}{k+1}}}{G_k^{\frac{1}{k+1}}}+k\right)\\ &= G_k\left(\lambda_k^{k+1}-(k+1)\lambda_k+k\right)\quad ,\end{aligned}$$

hence, since $G_k > 0$ for each k, we conclude that

$$(k+1)(A_{k+1}-G_{k+1}) \geqslant k(A_k-G_k)$$

if and only if

$$\lambda_k^{k+1}-(k+1)\lambda_k+k \geqslant 0 \quad .$$

(ii) We have

$$\begin{aligned}f'(x)&=(k+1)x^k-(k+1) \qquad \text{and}\\ f''(x)&=k(k+1)x^{k-1} \quad .\end{aligned}$$

We see that $f''(x) > 0$ for all $x > 0$, hence there is one stationary point at $x = 1$ (since k is positive and x is positive $(k+1)(x^{k+1}-1)=0 \iff x=1$), and it is a minimum. Evaluating:

$$f(1) = 1-(k+1)+k = 0 \quad ,$$

hence $f(x) \geqslant 0$ for all $x > 0$, with $f(x) = 0$ if and only if $x = 1$.

(iii) We know that $\lambda_k > 0$ for each k, hence by (ii), for each k we have

$$f(\lambda_k) = \lambda_k^{k+1} - (k+1)\lambda_k + k \geqslant 0 \quad .$$

By (i) then, for each k

$$(k+1)(A_{k+1} - G_{k+1}) \geqslant k(A_k - G_k) \quad .$$

For the basis case, consider $k = 1$:

$$A_1 = a_1 \quad \text{and} \quad G_1 = a_1 \quad \implies A_1 - G_1 = 0 \quad .$$

Now suppose that $A_n \geqslant G_n$ for some $n \geqslant 1$:

$$\begin{aligned} & (n+1)(A_{n+1} - G_{n+1}) \geqslant n(A_n - G_n) && \text{as concluded above} \\ \implies & (n+1)(A_{n+1} - G_{n+1}) \geqslant 0 && \text{by supposition (and } n \text{ being positive)} \\ \implies & A_{n+1} - G_{n+1} \geqslant 0 && \text{since } n+1 \text{ is positive} \\ \implies & A_{n+1} \geqslant G_{n+1} \quad . \end{aligned}$$

By induction then, we have that $A_n \geqslant G_n$ for all n.

Now suppose $A_n = G_n$ for some n. We have that $A_k \geqslant G_k$ for all k, hence $A_{n-1} \geqslant G_{n-1}$. By (i) and (ii), we must also have $(n-1)(A_{n-1} - G_{n-1}) \leqslant 0$, hence $A_{n-1} = G_{n-1}$, and by the above results we have

$$\lambda_{n-1} = \left(\frac{a_n}{G_{n-1}}\right)^{\frac{1}{n}} = 1 \quad ,$$

which occurs if and only if $G_{n-1} = a_n$. By iterating this reasoning, we conclude that for each $1 \leqslant k \leqslant n$ we must have $A_k = G_k$, and so $\lambda_{k-1} = 1$, and so for each $1 \leqslant k \leqslant n-1$ we must have $G_k = a_{k+1}$. This then implies

$$\begin{aligned} a_2 &= G_1 && \text{as concluded above} \\ &= a_1 && \text{by definition} \end{aligned}$$

then

$$\begin{aligned} a_3 &= G_2 && \text{as concluded above} \\ &= (a_1 \cdot a_2)^{\frac{1}{2}} && \text{by definition} \\ &= (a_1 \cdot a_1)^{\frac{1}{2}} && \text{since } a_2 = a_1 \\ &= a_1 \quad . \end{aligned}$$

Iterating this reasoning similarly, we conclude that $a_1 = a_2 = \cdots = a_n$.

Question 6

(i) Since the three points are collinear, we have that $\vec{AQ}$ is either parallel or anti-parallel to $\vec{AC}$, hence

$$\vec{AQ} = \lambda \vec{AC}$$

for some non-zero (since the points are distinct) real scalar λ. Hence:

$$\begin{aligned}(q-a) &= \lambda(c-a) \\ \frac{q-a}{c-a} &= \lambda \qquad \text{(which is real)} \quad .\end{aligned}$$

From this, we have

$$\begin{aligned} \left(\frac{q-a}{c-a}\right)^* &= \frac{q-a}{c-a} \\ \iff (c-a)(q-a)^* &= (c-a)^*(q-a) \\ (c-a)(q^*-a^*) &= (c^*-a^*)(q-a) \quad .\end{aligned}$$

Given $aa^* = cc^* = 1$, expanding and substituting $a^* = \frac{1}{a}$, $c^* = \frac{1}{c}$ gives

$$\begin{aligned}(c-a)(q^*-a^*) &= (c^*-a^*)(q-a) \\ (c-a)q^* - a^*c + 1 &= (c^*-a^*)q - ac^* + 1 \\ (c-a)q^* - a^*c &= (c^*-a^*)q - ac^* \\ (c-a)q^* - \frac{c}{a} &= \left(\frac{1}{c} - \frac{1}{a}\right) q - \frac{a}{c} \\ ac(c-a)q^* - c^2 &= (a-c)q - a^2 \\ (a-c)(q + acq^*) &= a^2 - c^2 \\ \implies q + acq^* &= a + c \quad ,\end{aligned}$$

where we have divided by $a-c$, since A and C are distinct, so $a \neq c$.

(ii) From the given information, the results from (i) hold for A, Q, and C and similarly for B, Q, and D, hence we have

$$\begin{aligned} q + acq^* &= a + c \quad \text{and} \\ q + bdq^* &= b + d \quad .\end{aligned}$$

Subtracting one of these equations from the other, we get

$$(ac - bd)q^* = (a+c) - (b+d) \quad .$$

Further, we can multiply the equations through by bd and ac respectively to get

$$\begin{aligned} bdq + abcdq^* &= (a+c)bd \quad \text{and} \\ acq + abcdq^* &= (b+d)ac \quad , \end{aligned}$$

where now taking one equation from the other gives

$$(ac - bd)q = (b+d)ac - (a+c)bd \quad .$$

Adding these two similar results, we have

$$\begin{aligned} (ac - bd)(q + q^*) &= (b+d)ac - (a+c)bd + a + c - b - d \\ &= a - b + acd - bcd + c - d + abc - abd \\ &= (a-b)(1+cd) + (c-d)(1+ab) \quad . \end{aligned}$$

(iii) Now we have the set up in (i) for points A, P, and B, hence by the above we know

$$p + abp^* = a + b \quad ,$$

and since p is real, we substitute $p^* = p$:

$$p(1+ab) = a + b \quad .$$

Identically, using points C, P, and D gives

$$p(1+cd) = c + d \quad .$$

Finally, multiplying through by p in the final result from (ii), the above equations give

$$\begin{aligned} (ac - bd)p(q + q^*) &= (a-b)p(1+cd) + (c-d)p(1+ab) \\ &= (a-b)(c+d) + (c-d)(a+b) \\ &= ac - bc + ad - bd + ac - ad + bc - bd \\ &= 2(ac - bd) \quad , \end{aligned}$$

so given $ac - bd \neq 0$, we divide through by $ac - bd$ to get

$$p(q + q^*) = 2 \quad .$$

Question 7

(i) For any positive integer n, if $\sin(\theta) \neq 0$, we have

$$\begin{aligned}
&\frac{(\cot(\theta)+i)^{2n+1}-(\cot(\theta)-i)^{2n+1}}{2i} \\
&\quad= \frac{(\cos(\theta)+i\sin(\theta))^{2n+1}-(\cos(\theta)-i\sin(\theta))^{2n+1}}{2i\sin(\theta)^{2n+1}} \\
&\quad= \frac{\cos((2n+1)\theta)+i\sin((2n+1)\theta)-(\cos((2n+1)\theta)-i\sin((2n+1)\theta)}{2i\sin(\theta)^{2n+1}} \\
&\quad= \frac{2i\sin((2n+1)\theta)}{2i\sin(\theta)^{2n+1}} = \frac{\sin((2n+1)\theta)}{\sin(\theta)^{2n+1}} \quad .
\end{aligned}$$

Consider

$$\begin{aligned}
&(y+i)^{2n+1}-(y-i)^{2n+1} \\
&\qquad= \sum_{k=0}^{2n+1}\binom{2n+1}{k} i^k y^{2n+1-k} - \sum_{k=0}^{2n+1}\binom{2n+1}{k}(-i)^k y^{2n+1-k} \\
&\qquad= \sum_{k=0}^{2n+1}\binom{2n+1}{k}(i^k-(-i)^k)y^{2n+1-k} \\
&\qquad= \sum_{j=0}^{n}\binom{2n+1}{2j+1} 2i^{2j+1}y^{2n+1-(2j+1)}
\end{aligned}$$

(that is, $i^k - (-i)^k = 0$ if k is even, and $i^k - (-i)^k = 2i^k$ if k is odd)

$$\frac{(y+i)^{2n+1}-(y-i)^{2n+1}}{2i} = \sum_{j=0}^{n}\binom{2n+1}{2j+1}(-1)^j y^{2n-2j} \quad .$$

We note that the right-hand side here is the polynomial we are asked to solve, but in y^2, as opposed to x. Supposing that x is positive, and setting $x = \cot^2(\theta)$ for some $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ (since over this domain $\cot^2(\theta)$ is one-to-one), then we have

$$\sum_{j=0}^{n}\binom{2n+1}{2j+1}(-1)^j x^{n-j} = \frac{(\cot(\theta)+i)^{2n+1}-(\cot(\theta)-i)^{2n+1}}{2i} = \frac{\sin((2n+1)\theta)}{\sin(\theta)^{2n+1}} \quad .$$

Hence, requiring $\sin((2n+1)\theta) = 0$, $\sin(\theta) \neq 0$, we find that n distinct solutions to the polynomial are given by

$$\theta = \frac{m\pi}{2n+1} \qquad m = 1, 2, \ldots, n \quad .$$

Since the polynomial is of degree n, these are all the solutions:

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right) \qquad m = 1, 2, \ldots, n \quad .$$

(ii) By Vieta's formulae, the sum of the roots of the polynomial is -1 times the ratio of the coefficient of x^{n-1} and the coefficient of x^n: that is

$$\begin{aligned}\sum_{m=1}^{n} \cot^2\left(\frac{m}{2n+1}\pi\right) &= \binom{2n+1}{3} \Big/ \binom{2n+1}{1} = \frac{(2n+1)!}{3!(2n-2)!}\frac{1!(2n)!}{(2n+1)!} \\ &= \frac{(2n)!}{3!(2n-2)!} = \frac{(2n)(2n-1)}{6} \\ &= \frac{n(2n-1)}{3} \quad .\end{aligned}$$

(iii) Using the given inequalities, we have

$$\begin{aligned}0 < \theta < \tan(\theta) &\implies \theta^2 < \tan^2(\theta) \\ &\implies \frac{1}{\theta^2} > \cot^2(\theta) \quad ,\end{aligned}$$

and

$$\begin{aligned}0 < \sin(\theta) < \theta &\implies \sin^2(\theta) < \theta^2 \\ &\implies \frac{1}{\theta^2} < \frac{1}{\sin^2(\theta)} \\ &\quad\ \ \frac{1}{\theta^2} < \frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)} \\ &\quad\ \ \frac{1}{\theta^2} < 1+\cot^2(\theta) \quad ,\end{aligned}$$

hence

$$\cot^2(\theta) < \frac{1}{\theta^2} < 1+\cot^2(\theta) \quad ,$$

as required.

Using this result, we can set $\theta = \frac{m\pi}{2n+1}$ for each $1 \leqslant m \leqslant n$ for some finite n, and sum the inequalities to get

$$\begin{aligned}\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n}\left(\frac{2n+1}{m\pi}\right)^2 &< \sum_{m=1}^{n}\left(1+\cot^2\left(\frac{m\pi}{2n+1}\right)\right) \\ \frac{n(2n-1)}{3} < \sum_{m=1}^{n}\left(\frac{2n+1}{m\pi}\right)^2 &< n + \frac{n(2n-1)}{3} \\ \frac{n(2n-1)}{3(2n+1)^2}\pi^2 < \sum_{m=1}^{n}\frac{1}{m^2} &< \frac{n(2n-1)+3n}{3(2n+1)^2}\pi^2 \\ \frac{2n^2-n}{12n^2+12n+3}\pi^2 < \sum_{m=1}^{n}\frac{1}{m^2} &< \frac{2n^2+2n}{12n^2+12n+3}\pi^2 \quad .\end{aligned}$$

Now, looking at the highest powers of n in the numerator and denominator, we can see that

$$\lim_{n\to\infty}\left(\frac{2n^2-n}{12n^2+12n+3}\pi^2\right)=\frac{1}{6}\pi^2 \qquad \text{and}$$
$$\lim_{n\to\infty}\left(\frac{2n^2+2n}{12n^2+12n+3}\pi^2\right)=\frac{1}{6}\pi^2 \quad ,$$

hence, by the squeeze theorem, we have

$$\lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m^2}\right)=\frac{1}{6}\pi^2 \quad ,$$

that is:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6} \quad ,$$

as required.

Question 8

(i) Using the change of variables $y = x^{-1}$, we have

$$\begin{aligned} I &= \int_{\infty}^{1} \frac{f(y)}{1+y^{-1}} \left(-\frac{1}{y^2}\right) dy \\ &= \int_{1}^{\infty} \frac{f(y)}{y^2(1+y^{-1})} dy \\ &= \int_{1}^{\infty} \frac{f(y)}{y(y+1)} dy \quad . \end{aligned}$$

Ignoring issues of convergence then, we can split the integral into an infinite sum of integrals:

$$I = \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{f(y)}{y(1+y)} dy \quad ,$$

as required. Using partial fractions, we can write

$$\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{y+1} \quad ,$$

hence, if f has the given periodicity, we find (again ignoring issues of convergence)

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left(\int_{n}^{n+1} \frac{f(y)}{y} dy - \int_{n}^{n+1} \frac{f(y)}{y+1} dy \right) \\ &= \sum_{n=1}^{\infty} \left(\int_{n}^{n+1} \frac{f(y)}{y} dy - \int_{n}^{n+1} \frac{f(y+1)}{y+1} dy \right) \qquad \text{(by periodicity of } f\text{)} \\ &= \sum_{n=1}^{\infty} \left(\int_{n}^{n+1} \frac{f(y)}{y} dy - \int_{n+1}^{n+2} \frac{f(y)}{y} dy \right) \\ &= \int_{1}^{2} \frac{f(y)}{y} dy \quad , \end{aligned}$$

where the last line follows since the sum telescopes. Now we have

$$\begin{aligned} I &= \int_{1}^{2} \frac{f(y)}{y} dy \\ &= \int_{0}^{1} \frac{f(y+1)}{y+1} dy \qquad \text{(changing variable)} \\ &= \int_{0}^{1} \frac{f(y)}{y+1} dy \qquad \text{(by periodicity)} \end{aligned}$$

as required.

(ii) Now we have $f(x) = \{x\}$, which we note satisfies the periodicity condition from (i) since

$$\{x+1\} = x + 1 - \lfloor x+1 \rfloor = x + 1 - \lfloor x \rfloor - 1 = x - \lfloor x \rfloor = \{x\} \quad .$$

From part (i) with this f then, we have

$$\int_0^1 \frac{\{x^{-1}\}}{1+x} dx = \int_0^1 \frac{\{y\}}{y+1} dy = \int_0^1 \frac{y}{y+1} dy \quad ,$$

since for $0 \leqslant y < 1$, we have $\{y\} = y$. Thus

$$\begin{aligned}
\int_0^1 \frac{\{x^{-1}\}}{1+x} dx &= \int_0^1 \frac{y}{y+1} dy \\
&= \int_0^1 \left(1 - \frac{1}{y+1}\right) dy \\
&= 1 - [\log(y+1)]_0^1 \\
&= 1 - \log(2) \quad .
\end{aligned}$$

Similarly $f(x) = \{2x\}$ also satisfies the periodicity condition:

$$\{2(x+1)\} = 2x + 2 - \lfloor 2x+2 \rfloor = 2x + 2 - \lfloor 2x \rfloor - 2 = 2x - \lfloor 2x \rfloor = \{2x\} \quad ,$$

so using part (i) again

$$\begin{aligned}
\int_0^1 \frac{\{2x^{-1}\}}{1+x} dx &= \int_0^1 \frac{\{2y\}}{y+1} dy \\
&= \int_0^{\frac{1}{2}} \frac{\{2y\}}{y+1} dy + \int_{\frac{1}{2}}^1 \frac{\{2y\}}{y+1} dy \\
&= \int_0^{\frac{1}{2}} \frac{2y}{y+1} dy + \int_{\frac{1}{2}}^1 \frac{2y-1}{y+1} dy \\
&= \int_0^{\frac{1}{2}} \left(2 - \frac{2}{y+1}\right) dy + \int_{\frac{1}{2}}^1 \left(2 - \frac{3}{y+1}\right) dy \\
&= 1 - [2\log(y+1)]_0^{\frac{1}{2}} + 1 - [3\log(y+1)]_{\frac{1}{2}}^1 \\
&= 2 - 2\log\left(\tfrac{3}{2}\right) - 3\log(2) + 3\log\left(\tfrac{3}{2}\right) \\
&= 2 + \log\left(\tfrac{3}{2}\right) - 3\log(2) \\
&= 2 - 4\log(2) + \log(3) \quad .
\end{aligned}$$

Section B: Mechanics

Question 9

(i) For the n-th collision, the restitution equation gives

$$v_n - u_n = e(v_{n-1} + u_{n-1}) \quad ,$$

and conservation of momentum states

$$u_n + kv_n = u_{n-1} - kv_{n-1} \quad ,$$

after cancelling factors of m. Using these we find

$$\begin{aligned} u_n + kv_n - k(v_n - u_n) &= u_{n-1} - kv_{n-1} - ke(v_{n-1} + u_{n-1}) \\ (1+k)u_n &= (1-ke)u_{n-1} - k(1+e)v_{n-1} \quad . \end{aligned}$$

For $n \geqslant 2$, we can use the same equations for the $n-1$-th collision:

$$\begin{aligned} v_{n-1} - u_{n-1} &= e(v_{n-2} + u_{n-2}) \\ u_{n-1} + kv_{n-1} &= u_{n-2} - kv_{n-2} \quad , \end{aligned}$$

to find v_{n-1} in terms of u_{n-1} and u_{n-2}:

$$\begin{aligned} k(v_{n-1} - u_{n-1}) + e(u_{n-1} + kv_{n-1}) &= ke(v_{n-2} + u_{n-2}) + e(u_{n-2} - kv_{n-2}) \\ k(1+e)v_{n-1} + (e-k)u_{n-1} &= e(1+k)u_{n-2} \\ k(1+e)v_{n-1} &= (k-e)u_{n-1} + e(1+k)u_{n-2} \quad . \end{aligned}$$

Combining these results, we get

$$\begin{aligned} (1+k)u_n &= (1-ke)u_{n-1} - ((k-e)u_{n-1} + e(1+k)u_{n-2}) \\ &= (1 - ke - k + e)u_{n-1} - e(1+k)u_{n-2} \\ &= (1-k)(1+e)u_{n-1} - e(1+k)u_{n-2} \quad , \end{aligned}$$

thus

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0 \quad .$$

(ii) With the given values for e and k, the first collision gives

$$\begin{aligned} v_1 - u_1 &= \tfrac{1}{2}(v_0 + u_0) \\ u_1 + \tfrac{1}{34}v_1 &= u_0 - \tfrac{1}{34}v_0 \quad , \end{aligned}$$

thus

$$\begin{aligned} u_1 + \tfrac{1}{34}v_1 - \tfrac{1}{34}(v_1 - u_1) &= u_0 - \tfrac{1}{34}v_0 - \tfrac{1}{68}(v_0 + u_0) \\ \tfrac{35}{34}u_1 &= \tfrac{67}{68}u_0 - \tfrac{3}{68}v_0 \\ u_1 &= \tfrac{67}{70}u_0 - \tfrac{3}{70}v_0 \quad . \end{aligned}$$

Then from the given form of the solution, we have

$$\begin{aligned} A + B &= u_0 \\ \tfrac{7}{10}A + \tfrac{5}{7}B &= \tfrac{67}{70}u_0 - \tfrac{3}{70}v_0 \quad . \end{aligned}$$

Thus

$$\begin{aligned} 50A + 50B &= 50u_0 \\ 49A + 50B &= 67u_0 - 3v_0 \quad , \end{aligned}$$

giving

$$A = -17u_0 + 3v_0 \quad ,$$

and

$$B = u_0 - A = 18u_0 - 3v_0 \quad .$$

If $0 < 6u_0 < v_0$, we have

$$\begin{aligned} u_n &= (-17u_0 + 3v_0)\left(\tfrac{7}{10}\right)^n + (18u_0 - 3v_0)\left(\tfrac{5}{7}\right)^n \\ &= \left(\tfrac{5}{7}\right)^n \left((-17u_0 + 3v_0)\left(\tfrac{49}{50}\right)^n + 18u_0 - 3v_0\right) \quad . \end{aligned}$$

Hence u_n has the same sign as

$$w_n := (-17u_0 + 3v_0)\left(\tfrac{49}{50}\right)^n + 18u_0 - 3v_0 \quad .$$

As $n \to \infty$, we have $w_n \to 18u_0 - 3v_0$, which is strictly negative. Thus for large enough n, u_n will become negative.

Question 10

A diagram is particularly helpful here.

Let C be the centre of mass of the combined disc and particle, then C lies on OP and

$$(M+m)(OC) = m(OP) = ma$$
$$OC = \frac{ma}{M+m} \quad .$$

In equilibrium, C lies vertically beneath A, giving

$$\sin\beta = \frac{OC}{OA} = \frac{m}{M+m} \quad .$$

By the cosine rule, on triangle AOP, we have

$$AP^2 = 2a^2 - 2a^2\cos\left(\tfrac{\pi}{2} - \beta\right)$$
$$\frac{AP^2}{a^2} = 2(1-\sin\beta)$$
$$= \frac{2M}{M+m}$$
$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}} \quad .$$

While rotating, the kinetic energy of the disc about the axis L is $\frac{1}{2}I\dot{\theta}^2$, and the kinetic energy of the particle about the axis L is

$$\frac{1}{2}m(AP)^2\dot{\theta}^2 = (1-\sin\beta)ma^2\dot{\theta}^2 \quad ;$$

relative to the equilibrium position (being zero), the gravitational potential energy of the combined disc and particle is

$$(M+m)g(AC)(1-\cos\theta) = (M+m)ga\cos\beta(1-\cos\theta) \quad .$$

Thus by conservation of energy, the given quantity (being the total kinetic and gravitational potential energy of the system) is constant during the motion.

Given $m = \frac{3}{2}M$, we have

$$\sin\beta = \frac{\frac{3}{2}M}{M+\frac{3}{2}M} = \frac{3}{5} \quad \Longrightarrow \quad \cos\beta = \frac{4}{5}$$

where we know $\cos\beta$ is positive, since β is acute. Substituting this, and $I = \frac{3}{2}Ma^2$ into the total energy, we get

$$E = \frac{3}{4}Ma^2\dot{\theta}^2 + \frac{2}{5}\cdot\frac{3}{2}Ma^2\dot{\theta}^2 + \frac{5}{2}Mga\cdot\frac{4}{5}\cos\theta$$
$$= \frac{27}{20}Ma^2\dot{\theta}^2 + 2Mga\cos\theta \quad .$$

Differentiating with respect to time, using the fact that E is constant throughout the motion, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{27}{20}Ma^2\dot{\theta}^2 + 2Mga\cos\theta\right) &= 0 \\ \frac{27}{10}Ma^2\dot{\theta}\ddot{\theta} - 2Mga\dot{\theta}\sin\theta &= 0 \\ \frac{27a}{20g}\ddot{\theta} - \sin\theta &= 0 \quad . \end{aligned}$$

For small oscillations $\sin\theta \approx \theta$, and we have

$$\ddot{\theta} \approx \frac{20g}{27a}\theta \quad ,$$

giving the period of small oscillations:

$$T = 2\pi\sqrt{\frac{27a}{20g}} = 3\pi\sqrt{\frac{3a}{5g}} \quad .$$

Question 11

Let the mass of the particle be m. As the particle moves in a circular arc, we have

$$F = \frac{mv^2}{b} \quad ,$$

where F is the total force on the particle directed toward O (the centre of the circular motion), and v is the (tangential) velocity of the particle. When the angle between the string and the upward vertical is θ, we have

$$F = \tau + mg\cos\theta \quad ,$$

where τ is the tension in the string. At $\theta = \alpha$, $v = V$, and the string goes slack, thus $\tau = 0$: this gives

$$\begin{aligned} \frac{mV^2}{b} &= mg\cos\alpha \\ V^2 &= bg\cos\alpha \\ V &= \sqrt{bg\cos\alpha} \quad . \end{aligned}$$

As the string goes slack the tangential velocity of the particle is directed at an angle α above the horizontal. The particle is now accelerating due to gravity only, thus we have parabolic motion. Taking O to be our origin, and supposing the circular motion was clockwise about O, the particle begins parabolic motion from initial coordinates

$$(x(0), y(0)) = (-b\sin\alpha, b\cos\alpha) \quad ,$$

and the initial velocity has horizontal and vertical components

$$(u_{\text{hor}}, u_{\text{vert}}) = (V\cos\alpha, V\sin\alpha) \quad ,$$

giving the trajectory:

$$(x(t), y(t)) = (-b\sin\alpha + Vt\cos\alpha, b\cos\alpha + Vt\sin\alpha - \tfrac{1}{2}gt^2) \quad .$$

The parabolic motion continues until time T when the string goes taut again at

$$\begin{aligned} b^2 &= x(T)^2 + y(T)^2 \\ &= (-b\sin\alpha + VT\cos\alpha)^2 + (b\cos\alpha + VT\sin\alpha - \tfrac{1}{2}gT^2)^2 \\ &= b^2\sin^2\alpha + V^2T^2\cos^2\alpha - 2bVT\sin\alpha\cos\alpha + b^2\cos^2\alpha + V^2T^2\sin^2\alpha + \frac{1}{4}g^2T^4 \\ &\qquad + 2bVT\sin\alpha\cos\alpha - bgT^2\cos\alpha - VgT^3\sin\alpha \quad ; \end{aligned}$$

and simplifying, we get

$$\begin{aligned}
V^2T^2 + \frac{1}{4}g^2T^4 - bgT^2\cos\alpha - VgT^3\sin\alpha &= 0 \\
\implies V^2 + \frac{1}{4}g^2T^2 - bg\cos\alpha - VgT\sin\alpha &= 0 \\
\frac{1}{4}g^2T^2 - VgT\sin\alpha &= 0 \\
gT &= 4V\sin\alpha \quad ,
\end{aligned}$$

as required.

Just before this point, the particle has horizontal velocity $v_{\mathrm{hor}} = V\cos\alpha$, and vertical velocity

$$v_{\mathrm{ver}} = V\sin\alpha - gT = -3V\sin\alpha \quad ,$$

hence the parabolic trajectory is at an angle β below the horizontal, with

$$\tan\beta = \frac{3V\sin\alpha}{V\cos\alpha} = 3\tan\alpha \quad .$$

At the point where the string goes taut again, the particle will instantaneously come to rest if and only if the parabolic trajectory is (tangentially) radial at that point: that is, if and only if

$$\begin{aligned}
\frac{y(T)}{x(T)} &= \frac{v_{\mathrm{vert}}}{v_{\mathrm{hor}}} \\
\frac{b\cos\alpha + VT\sin\alpha - \frac{1}{2}gT^2}{-b\sin\alpha + VT\cos\alpha} &= \frac{-3V\sin\alpha}{V\cos\alpha} \\
\left(b\cos\alpha + \frac{4}{g}V^2\sin^2\alpha - \frac{8}{g}V^2\sin^2\alpha\right)V\cos\alpha &= 3bV\sin^2\alpha - \frac{12}{g}V^3\sin^2\alpha\cos\alpha \\
bV\cos^2\alpha &= 3bV\sin^2\alpha - \frac{8}{g}V^3\sin^2\alpha\cos\alpha \\
bV\cos^2\alpha &= 3bV\sin^2\alpha - 8bV\sin^2\alpha\cos^2\alpha \\
1 - \sin^2\alpha &= 3\sin^2\alpha - 8\sin^2\alpha(1 - \sin^2\alpha) \\
\iff 8\sin^4\alpha - 4\sin^2\alpha - 1 &= 0 \\
\iff \sin^2\alpha &= \frac{4 + \sqrt{48}}{16} = \frac{1 + \sqrt{3}}{4}
\end{aligned}$$

(when solving the quadratic in $\sin^2$, we have taken the positive root, since $\sin^2\alpha \geqslant 0$).

Section C: Probability and Statistics

Question 12

(i) We will have that $Y_k \leqslant y$ if and only if *at least* k of the n randomly generated numbers are less than or equal to y, thus

$$\begin{aligned}\mathbb{P}\{Y_k \leqslant y\} &= \sum_{m=k}^{n} \mathbb{P}\{\text{Exactly } m \text{ numbers } \leqslant y \text{ and } n-m \text{ numbers } > y\} \\ &= \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m} \quad .\end{aligned}$$

(ii) Expanding the binomial coefficients as fractions in factorials, we have

$$m\binom{n}{m} = m\frac{n!}{m!(n-m)!} = n\frac{(n-1)!}{(m-1)!(n-m)!} = n\binom{n-1}{m-1} \quad ,$$

and

$$(n-m)\binom{n}{m} = (n-m)\frac{n!}{m!(n-m)!} = n\frac{(n-1)!}{m!(n-m-1)!} = n\binom{n-1}{m} \quad .$$

We have that the distribution function of Y_k is

$$F(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m} \quad ,$$

thus the probability density function of Y_k is

$$\begin{aligned}f(y) &= \frac{d}{dy}\left(\sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}\right) \\ &= \sum_{m=k}^{n} m\binom{n}{m} y^{m-1}(1-y)^{n-m} - \sum_{m=k}^{n-1} (n-m)\binom{n}{m} y^m (1-y)^{n-m-1} \\ &= \sum_{m=k}^{n} n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k}^{n-1} n\binom{n-1}{m} y^m (1-y)^{n-m-1} \\ &= \sum_{m=k}^{n} n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k+1}^{n} n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} \\ &= n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \quad .\end{aligned}$$

We deduce that

$$\int_0^1 y^{k-1}(1-y)^{n-k}dy = \left(n\binom{n-1}{k-1}\right)^{-1} \int_0^1 f(y)dy$$
$$= \left(n\binom{n-1}{k-1}\right)^{-1} \quad ,$$

where we have used the fact that f is a probability density function on $[0,1]$, thus $\int_0^1 f(y)dy = 1$.

(iii) We have

$$\begin{aligned}
\mathbb{E}(Y_k) &= \int_0^1 yf(y)dy \\
&= n\binom{n-1}{k-1}\int_0^1 y^k(1-y)^{n-k}dy \\
&= n\binom{n-1}{k-1}\int_0^1 y^{(k+1)-1}(1-y)^{(n+1)-(k+1)}dy \\
&= n\binom{n-1}{k-1}\left((n+1)\binom{(n+1)-1}{(k+1)-1}\right)^{-1} = n\binom{n-1}{k-1}\left((n+1)\binom{n}{k}\right)^{-1} \\
&= n\frac{(n-1)!}{(k-1)!(n-k)!}\frac{1}{n+1}\frac{k!(n-k)!}{n!} \\
&= \frac{k}{n+1} \quad ,
\end{aligned}$$

where we used the result of part (ii) (with $k+1$ and $n+1$ instead of k and n) to evaluate the integral.

Question 13

We have that

$$G(t) = \sum_{n=0}^{\infty} \mathbb{P}\{X = n\}t^n = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \dots \quad ,$$

where we are denoting $\mathbb{P}\{X = n\}$ by p_n; so

$$\begin{aligned} G(1) &= p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + \dots \\ \text{and} \;\; G(-1) &= p_0 - p_1 + p_2 - p_3 + p_4 - p_5 + p_6 - \dots \quad . \end{aligned}$$

Combining these then:

$$\frac{1}{2}(G(1) + G(-1)) = p_0 + p_2 + p_4 + p_6 + \cdots = \mathbb{P}\{X = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6 \dots\} \quad .$$

Now given that X has a Poisson distribution, so

$$p_n = e^{-\lambda}\frac{\lambda^n}{n!} \quad ,$$

we have

$$G(t) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}t^n = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!} = e^{-\lambda}e^{\lambda t} = e^{-\lambda(1-t)} \quad .$$

(i) Let $G_Y(t)$ be the probability generating function of Y, then

$$\begin{aligned} G_Y(t) &= kp_0 + kp_2t^2 + kp_4t^4 + kp_6t^6 + \dots \\ &= k(p_0 + p_2t^2 + p_4t^4 + p_6t^6 + \dots) \\ &= \frac{k}{2}\big((p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4 + p_5t^5 + p_6t^6 + \dots) \\ &\qquad\qquad + (p_0 - p_1t + p_2t^2 - p_3t^3 + p_4t^4 - p_5t^5 + p_6t^6 - \dots)\big) \\ &= \frac{k}{2}(G(t) + G(-t)) \\ &= \frac{k}{2}e^{-\lambda}(e^{\lambda t} + e^{-\lambda t}) \\ &= ke^{-\lambda}\cosh(\lambda t) \quad . \end{aligned}$$

To eliminate k, we note that $G_Y(1) = 1$ (summing the probabilities of all possible values of Y), hence

$$ke^{-\lambda}\cosh\lambda = 1 \quad \Longrightarrow \quad ke^{-\lambda} = \frac{1}{\cosh\lambda} \quad ,$$

giving

$$G_Y(t) = \frac{\cosh(\lambda t)}{\cosh\lambda} \quad .$$

To find the expectation, we use the probability generating function:

$$\mathbb{E}(Y) = G'_Y(1) = \left.\frac{\lambda \sinh(\lambda t)}{\cosh \lambda}\right|_{t=1} = \frac{\lambda \sinh \lambda}{\cosh \lambda} = \lambda \tanh \lambda \quad .$$

Since $\tanh \lambda < 1$ for all $\lambda > 0$, we deduce that $\mathbb{E}(Y) < \lambda$ for $\lambda > 0$.

(ii) We repeat this approach for Z: let $G_Z(t)$ be the probability generating function of Z, we have

$$\begin{aligned} G_Z(t) &= c(p_0 + p_4 t^4 + p_8 t^8 + p_{12} t^1 2 + \dots) \\ &= \frac{c}{2}\big((p_0 + p_2 t^2 + p_4 t^4 + p_6 t^6 + p_8 t^8 + p_{10} t^{10} + p_{12} t^{12} \dots) \\ &\qquad + (p_0 - p_2 t^2 + p_4 t^4 - p_6 t^6 + p_8 t^8 - p_{10} t^{10} + p_{12} t^{12} \dots)\big) \\ &= \frac{c}{2}(G_Y(t) + G_Y(it)) \\ &= \frac{c}{2}\frac{\cosh(\lambda t) + \cosh(\lambda i t)}{\cosh \lambda} = \frac{c}{2}\frac{\cosh(\lambda t) + \cos(\lambda t)}{\cosh \lambda} \quad . \end{aligned}$$

To eliminate c, we again use the fact that $G_Z(1) = 1$, thus

$$\frac{c}{2}\frac{\cosh \lambda + \cos \lambda}{\cosh \lambda} = 1 \quad \Longrightarrow \quad \frac{c}{2}\frac{1}{\cosh \lambda} = \frac{1}{\cosh \lambda + \cos \lambda} \quad ,$$

giving

$$G_Z(t) = \frac{\cosh(\lambda t) + \cos(\lambda t)}{\cosh \lambda + \cos \lambda} \quad .$$

From this we find $\mathbb{E}(Z)$:

$$\mathbb{E}(Z) = G'_Z(1) = \left.\frac{\lambda \sinh(\lambda t) - \lambda \sin(\lambda t)}{\cosh \lambda + \cos \lambda}\right|_{t=1} = \frac{\lambda \sinh \lambda - \lambda \sin \lambda}{\cosh \lambda + \cos \lambda} \quad .$$

We have

$$\frac{\mathbb{E}(Z)}{\lambda} = \frac{\sinh \lambda - \sin \lambda}{\cosh \lambda + \cos \lambda} \quad .$$

For all $\lambda > 0$ we have $\cosh \lambda + \cos \lambda \geqslant \cosh \lambda - 1 > 0$, hence we have

$$\begin{aligned} \mathbb{E}(Z) < \lambda \quad &\Longleftrightarrow \quad \sinh \lambda - \sin \lambda < \cosh \lambda + \cos \lambda \\ &\Longleftrightarrow \quad -\sin \lambda - \cos \lambda < \cosh \lambda - \sinh \lambda \\ &\Longleftrightarrow \quad -\sin \lambda - \cos \lambda < e^{-\lambda} \quad . \end{aligned}$$

For example take $\lambda = \pi$: this gives $-\sin \pi - \cos \pi = 1$; but $\pi > 0$, thus $e^{-\pi} < 1$. Hence the above inequality does *not* hold for all $\lambda > 0$.