STEP 2017 Solutions

Contents

Author's notes:

1. *At time of writing, I am not affiliated with Cambridge Assessment Admissions Testing. I did an undergraduate maths degree at Cambridge, so I sat the STEP II and III papers as an A-level student (in 2015), and I have also been one of a team of markers for the STEP exams (in 2019 and 2020). Any opinions given here are entirely my own, based on my own experiences of STEP.*

2. *These 'solutions' are not intended to be used as any sort of mark scheme. In terms of method, often there will be more than one correct way to answer a STEP question, and it is certainly not the case that the answers presented here are the only correct approaches to these questions. The worked solutions here were typed up after attempting the questions myself, and I have checked them against the official mark schemes published online. However, there is no guarantee that the solutions typed up here would achieve full marks. In particular, I have not provided diagrams for all questions due to the difficulties of typesetting them neatly. Many questions may ask the student to draw a diagram, and in these instances marks are usually awarded for this. Another point of consideration is explanation: sometimes marks are awarded for explicitly justifying an assumption used. I have tried to justify these as I think necessary, but there is no guarantee that these solutions justify all assumptions to the standards of the mark schemes.*

3. *If you are preparing to sit the STEP exams, I hope these can be of some help.*

STEP I

Section A: Pure Mathematics

Question 1

(i) Using $u = x\sin x + \cos x$, we have

$$\frac{du}{dx} = \sin x + x\cos x - \sin x = x\cos x \quad ,$$

giving

$$\begin{aligned}\int \frac{x}{x\tan x + 1}dx &= \int \frac{x\cos x}{x\sin x + \cos x}dx \\ &= \int \frac{1}{u}\frac{du}{dx}dx \\ &= \int \frac{1}{u}du \\ &= \log|u| + c \\ &= \log|x\sin x + \cos x| + c \quad ,\end{aligned}$$

where c is an arbitrary constant. Similarly, if we now let $v = x\cos x - \sin x$, we have

$$\frac{dv}{dx} = \cos x - x\sin x - \cos x = -x\sin x \quad ,$$

giving

$$\begin{aligned}\int \frac{x}{x\cot x - 1}dx &= \int \frac{x\sin x}{x\cos x - \sin x}dx \\ &= \int \frac{1}{v}\left(-\frac{dv}{dx}\right)dx \\ &= -\int \frac{1}{v}dv \\ &= -\log|v| + c \\ &= -\log|x\cos x - \sin x| + c \quad .\end{aligned}$$

(ii) Consider

$$\begin{aligned}\frac{d}{dx}\left(x\sec^2 x - \tan x\right) &= \sec^2 x + 2x\sec^2 x\tan x - \sec^2 x \\ &= 2x\sec^2 x\tan x \quad .\end{aligned}$$

Hence if we let $w = x\sec^2 x - \tan x$, we have

$$\begin{aligned}
\int \frac{x\sec^2 x \tan x}{x\sec^2 x - \tan x} dx &= \int \frac{1}{w} \cdot \frac{1}{2}\frac{dw}{dx} dx \\
&= \frac{1}{2}\int \frac{1}{w} dw \\
&= \frac{1}{2}\log|w| + c \\
&= \frac{1}{2}\log|x\sec^2 x - \tan x| + c \quad .
\end{aligned}$$

Sticking with the same substitution, we also have

$$\begin{aligned}
\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2} dx &= \int \frac{x\tan x}{(x\sec x - \sin x)^2} dx \\
&= \int \frac{x\sec^2 x\tan x}{(x\sec^2 x - \tan x)^2} dx \\
&= \int \frac{1}{w^2} \cdot \frac{1}{2}\frac{dw}{dx} dx \\
&= \frac{1}{2}\int \frac{1}{w^2} dw \\
&= -\frac{1}{2}\frac{1}{w} + c \\
&= -\frac{1}{2}\frac{1}{x\sec^2 x - \tan x} + c \quad .
\end{aligned}$$

Question 2

(i) We have $\frac{1}{t} \leqslant 1$ for $t \geqslant 1$, thus for $x \geqslant 1$ we have

$$\begin{aligned} \int_1^x \frac{1}{t} dt &\leqslant \int_1^x 1 dt \\ \Longrightarrow \quad \log x - \log 1 &\leqslant x - 1 \\ \Longrightarrow \quad \log x &\leqslant x - 1 \quad . \end{aligned}$$

Similarly, $\frac{1}{t} \geqslant 1$ for $0 < t \leqslant 1$, thus for $0 < x \leqslant 1$ we have

$$\begin{aligned} \int_x^1 \frac{1}{t} dt &\geqslant \int_x^1 1 dt \\ \Longrightarrow \quad \log 1 - \log x &\geqslant 1 - x \\ \Longrightarrow \quad \log x &\leqslant x - 1 \quad . \end{aligned} \tag{$*$}$$

(ii) We have $\frac{1}{t^2} \leqslant \frac{1}{t}$ for $t \geqslant 1$, thus for $x \geqslant 1$ we have

$$\begin{aligned} \int_1^x \frac{1}{t^2} dt &\leqslant \int_1^x \frac{1}{t} dt \\ \Longrightarrow \quad -\frac{1}{x} + \frac{1}{1} &\leqslant \log x - \log 1 \\ \Longrightarrow \quad 1 - \frac{1}{x} &\leqslant \log x \quad . \end{aligned}$$

Similarly, $\frac{1}{t^2} \geqslant \frac{1}{t}$ for $0 < t \leqslant 1$, thus for $0 < x \leqslant 1$ we have

$$\begin{aligned} \int_x^1 \frac{1}{t^2} dt &\geqslant \int_x^1 \frac{1}{t} dt \\ \Longrightarrow \quad -1 + \frac{1}{x} &\geqslant \log 1 - \log x \\ \Longrightarrow \quad 1 - \frac{1}{x} &\leqslant \log x \quad , \end{aligned} \tag{$**$}$$

and we deduce that $1 - \frac{1}{x} \leqslant \log x$ for $x > 0$.

(iii) Integrating $(*)$ we find that for $y > 1$

$$\begin{aligned} \int_1^y \log x \, dx &\leqslant \int_1^y (x-1) dx \\ y \log y - y - (1 \log 1 - 1) &\leqslant \frac{1}{2} y^2 - y - \frac{1}{2} + 1 \\ y \log y + 1 &\leqslant \frac{1}{2} y^2 + \frac{1}{2} \\ y \log y &\leqslant \frac{1}{2}(y^2 - 1) \\ \Longrightarrow \quad \frac{\log y}{y-1} &\leqslant \frac{y+1}{2y} \quad . \end{aligned}$$

Similarly, for $0 < y < 1$ we have

$$\begin{aligned}
\int_y^1 \log x \, dx &\leqslant \int_y^1 (x-1) dx \\
1 \log 1 - 1 - (y \log y - y) &\leqslant \frac{1}{2} - 1 - \frac{1}{2} y^2 + y \\
-1 - y \log y &\leqslant -\frac{1}{2} - \frac{1}{2} y^2 \\
-y \log y &\leqslant \frac{1}{2}(1 - y^2) \\
-\frac{\log y}{1-y} &\leqslant \frac{1+y}{2y} \\
\implies \qquad \frac{\log y}{y-1} &\leqslant \frac{y+1}{2y} \quad .
\end{aligned}$$

Likewise, integrating $(**)$ we find that for $y > 1$

$$\begin{aligned}
\int_1^y \log x \, dx &\geqslant \int_1^y \left(1 - \frac{1}{x}\right) dx \\
y \log y - y - (1 \log 1 - 1) &\geqslant y - \log y - 1 + 1 \log 1 \\
(y+1) \log y &\geqslant 2(y-1) \\
\implies \qquad \frac{\log y}{y-1} &\geqslant \frac{2}{y+1} \quad .
\end{aligned}$$

Similarly, for $0 < y < 1$ we have

$$\begin{aligned}
\int_y^1 \log x \, dx &\geqslant \int_y^1 \left(1 - \frac{1}{x}\right) dx \\
1 \log 1 - 1 - (y \log y - y) &\geqslant 1 - 1 \log 1 - y + \log y \\
-(y+1) \log y + y &\geqslant 2 - 2y + \log y \\
-(y+1) \log y &\geqslant 2(1-y) \\
-\frac{\log y}{1-y} &\geqslant \frac{2}{y+1} \\
\implies \qquad \frac{\log y}{y-1} &\geqslant \frac{2}{y+1} \quad .
\end{aligned}$$

Combining these results, we deduce that for $y > 0$, $y \neq 1$

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y} \quad .$$

Question 3

Differentiating the equation for the curve C, we have

$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{2a}{y} \quad .$$

Hence, we can find the equation of the tangent to C at $(x, y) = (ap^2, 2ap)$:

$$\begin{aligned} y - 2ap &= \frac{2a}{2ap}(x - ap^2) \\ 2apy - 4a^2p^2 &= 2ax - 2a^2p^2 \\ 2apy &= 2ax + 2a^2p^2 \\ y &= \frac{1}{p}x + ap \quad . \end{aligned}$$

Similarly, the equation of the tangent to C at $(x, y) = (aq^2, 2aq)$ is

$$y = \frac{1}{q}x + aq \quad .$$

The point R being the intersection of these two tangents lies at (x_R, y_R) satisfying

$$\begin{aligned} \frac{1}{p}x_R + ap &= \frac{1}{q}x_R + aq \\ \left(\frac{1}{p} - \frac{1}{q}\right) x_R &= a(q - p) \\ (q - p)x_R &= apq(q - p) \\ x_R &= apq \quad , \end{aligned}$$

giving

$$y_R = \frac{1}{p}(apq) + ap = a(p + q) \quad .$$

From the equations of the tangents, we can read off that S is the point $(0, ap)$ and T is the point $(0, aq)$.

At this point a diagram helps to visualise the two triangles being considered. Note $q < 0$.

The line PQ is given by

$$\frac{y - 2aq}{x - aq^2} = \frac{2ap - 2aq}{ap^2 - aq^2} \quad .$$

Let the point at which this line intersects the x-axis be $I = (x_I, 0)$, then we have

$$\begin{aligned}\frac{-2aq}{x_I - aq^2} &= \frac{2ap - 2aq}{ap^2 - aq^2} \\ \frac{-aq}{x_I - aq^2} &= \frac{1}{p+q} \\ -aq(p+q) &= x_I - aq^2 \\ x_I &= -apq \quad .\end{aligned}$$

Now we can find that the area of the triangle OPQ is

$$\begin{aligned}\text{area}(\Delta OPQ) &= \text{area}(\Delta OPI) + \text{area}(\Delta OIQ) \\ &= \frac{1}{2}(-apq)(2ap) + \frac{1}{2}(-apq)(-2aq) \\ &= a^2p(-q)(p-q) \quad .\end{aligned}$$

The area of the triangle RST is

$$\begin{aligned}\text{area}(\Delta RST) &= \frac{1}{2}(-apq)(ap - aq) \\ &= \frac{1}{2}a^2p(-q)(p-q) \\ &= \frac{1}{2}\text{area}(\Delta OPQ) \quad ,\end{aligned}$$

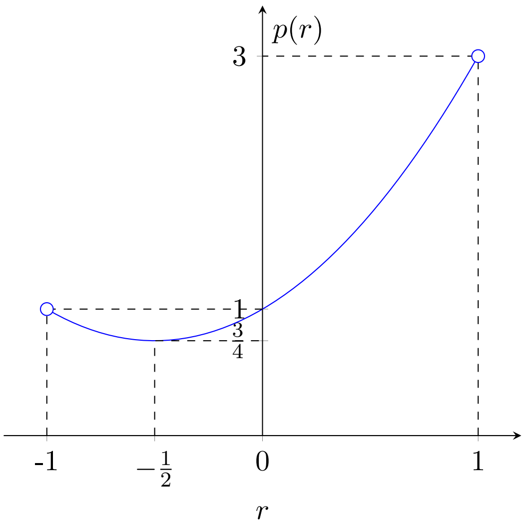
hence the area of triangle OPQ is twice the area of triangle RST.

Question 4

(i) Completing the square, we can write

$$p = 1 + r + r^2 = \left(r + \frac{1}{2}\right)^2 + \frac{3}{4} \quad ,$$

and, noting $-1 < r < 1$, our sketch is as follows:



We see that p takes its minimum value $p = \frac{3}{4}$ at $r = -\frac{1}{2}$, and $p \to 3$ as $r \to 1$. Thus $\frac{3}{4} \leqslant p < 3$, with strict upper bound, since $r = 1$ is not in the domain.

From the sketch then, if $1 < p < 3$, we have $0 < r < 1$ and the function $p(r)$ is one-to-one. Thus for p in this range, p determines r uniquely, and so the value of p determines the value of S uniquely.

If instead $\frac{3}{4} < p < 1$, then we have $-1 < r < 0$, $r \neq -\frac{1}{2}$, and from the sketch we see that for each value of p in this range, there are exactly two values of r which give this value of p. Since $r \neq 1$, we can rearrange

$$S = \frac{1}{1-r} \qquad \Longleftrightarrow \qquad r = 1 - \frac{1}{S} \quad ,$$

thus the function $S(r)$ is one-to-one over $|r| < 1$ and the two values of r giving a particular value of p in $\frac{3}{4} < p < 1$ will give distinct values of S.

Substituting in the expression for r in terms of S, we have

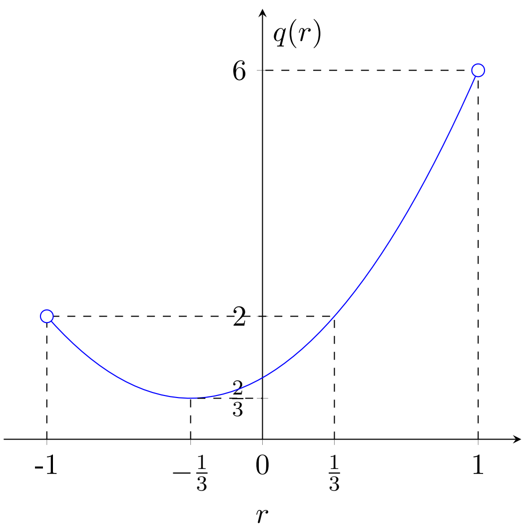
$$p = 1 + r + r^2 = 1 + \left(1 - \frac{1}{S}\right) + \left(1 - \frac{1}{S}\right)^2$$
$$= 1 + 1 - \frac{1}{S} + 1 - \frac{2}{S} + \frac{1}{S^2}$$
$$= 3 - \frac{3}{S} + \frac{1}{S^2}$$
$$\iff \qquad pS^2 = 3S^2 - 3S + 1$$
$$(p-3)S^2 + 3S - 1 = 0 \quad ,$$

as required.

(ii) Again completing the square, we have

$$q = 1 + 2r + 3r^2 = 3\left(r^2 + \frac{2}{3}r + \frac{1}{3}\right)$$
$$= 3\left(\left(r + \frac{1}{3}\right)^2 + \frac{2}{9}\right) \quad ,$$

and we can sketch q as a function of r:



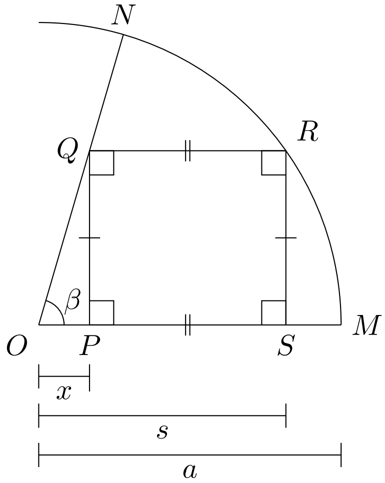
We can again rearrange to find r in terms of T, using the fact that $1 - r > 0$:

$$T = \frac{1}{(1-r)^2} \qquad \iff \qquad r = 1 - \frac{1}{\sqrt{T}} \quad .$$

In analogy with part (i), we see that for $2 < q < 6$, the value of q determines uniquely the value of T, and for $\frac{2}{3} < q < 2$ each value of q has two distinct corresponding values of r each of which determines a unique value of T. Again, by substituting in, we find that these values of T satisfy

$$
\begin{aligned}
q = 1 + 2r + 3r^2 &= 1 + 2\left(1 - \frac{1}{\sqrt{T}}\right) + 3\left(1 - \frac{1}{\sqrt{T}}\right)^2 \\
&= 1 + 2 - \frac{2}{\sqrt{T}} + 3 - \frac{6}{\sqrt{T}} + \frac{3}{T} \\
&= 6 - \frac{8}{\sqrt{T}} + \frac{3}{T} \\
\iff \qquad qT &= 6T - 8\sqrt{T} + 3 \\
(q-6)T - 3 &= -8\sqrt{T} \\
\implies \qquad (q-6)^2T^2 - 6(q-6)T + 9 &= 64T \\
(q-6)^2T^2 - (6q+28)T + 9 &= 0 \quad .
\end{aligned}
$$

Question 5



The area A of the rectangle $PQRS$ is

$$\begin{aligned} A &= |PS| \cdot |PQ| \\ &= (s-x) \cdot x \tan\beta \\ &= x(s-x)\tan\beta \quad . \end{aligned}$$

Since $PQRS$ is a rectangle, we have $|RS| = |PQ| = x\tan\beta$, and so using Pythagoras' theorem on triangle OSR we have

$$\begin{aligned} &|OS|^2 + |SR|^2 = |OR|^2 \\ &s^2 + x^2\tan^2\beta = a^2 \\ &\implies \qquad s = \sqrt{a^2 - x^2\tan^2\beta} \quad . \end{aligned}$$

Considering s as a function of x for fixed a, β, and differentiating $s^2 + x^2\tan^2\beta = a^2$ with respect to x, we find

$$\begin{aligned} &2s\frac{ds}{dx} + 2x\tan^2\beta = 0 \\ &\implies \qquad \frac{ds}{dx} = \frac{-x}{s}\tan^2\beta \quad . \end{aligned}$$

Now differentiating our expression for A with respect to x:

$$\begin{aligned} \frac{dA}{dx} &= \frac{d}{dx}(xs - x^2)\tan\beta \\ &= (s-2x)\tan\beta + x\tan\beta\frac{ds}{dx} \\ &= (s-2x)\tan\beta - \frac{x^2}{s}\tan^3\beta \quad . \end{aligned}$$

The greatest possible area must occur at a stationary value $\frac{dA}{dx} = 0$, thus

$$\begin{aligned}
(s-2x)\tan\beta &= \frac{x^2}{s}\tan^3\beta \\
\implies \qquad (s-2x) &= \frac{x^2}{s}\tan^2\beta \qquad\qquad (0<\beta<\tfrac{\pi}{2} \implies \tan\beta \neq 0) \\
s^2 - 2xs &= x^2\tan^2\beta \\
(s-x)^2 &= x^2(\tan^2\beta + 1) \\
(s-x)^2 &= x^2\sec^2\beta \\
\implies \qquad s - x &= x\sec\beta \quad ,
\end{aligned}$$

where the square roots are justified, since $0 < \beta < \frac{\pi}{2}$ ensures $\sec\beta > 0$, and we must have that $s \geqslant x$. Thus the maximum area occurs at

$$s - x = x\sec\beta \qquad \implies \qquad s = x(1+\sec\beta) \quad .$$

Substituting this result into $s^2 + x^2\tan^2\beta = a^2$, we find that the value of x giving the maximum area satisfies

$$\begin{aligned}
x^2(1+\sec\beta)^2 + x^2\tan^2\beta &= a^2 \\
x^2\left(1 + 2\sec\beta + \sec^2\beta + \tan^2\beta\right) &= a^2 \\
x^2\left(2\sec\beta + 2\sec^2\beta\right) &= a^2 \\
\implies \qquad x^2\sec\beta &= \frac{a^2}{2(1+\sec\beta)} \quad .
\end{aligned}$$

Hence the maximum value of the area is

$$\begin{aligned}
A &= x(x(1+\sec\beta) - x)\tan\beta = x^2\sec\beta\tan\beta = \frac{a^2\tan\beta}{2(1+\sec\beta)} \\
&= \frac{a^2\sin\beta}{2(\cos\beta + 1)} \\
&= \frac{a^2\sin\left(\frac{1}{2}\beta\right)\cos\left(\frac{1}{2}\beta\right)}{\cos^2\left(\frac{1}{2}\beta\right) - \sin^2\left(\frac{1}{2}\beta\right) + 1} \\
&= \frac{a^2\sin\left(\frac{1}{2}\beta\right)\cos\left(\frac{1}{2}\beta\right)}{2\cos^2\left(\frac{1}{2}\beta\right)} \\
&= \frac{1}{2}a^2\tan\left(\tfrac{1}{2}\beta\right) \quad .
\end{aligned}$$

When this occurs we have

$$\tan(\angle ROS) = \frac{x\tan\beta}{s} = \frac{x\tan\beta}{x(1+\sec\beta)} = \frac{\tan\beta}{1+\sec\beta} = \tan\left(\tfrac{1}{2}\beta\right) \quad .$$

Thus $\angle ROS = \frac{1}{2}\beta$ (where we may take arctans like this because both $\angle ROS$ and $\frac{1}{2}\beta$ are acute angles).

Question 6

(i) Suppose that either $f(x) \geqslant 0$ for all $x \in [0,1]$, or $f(x) \leqslant 0$ for all $x \in [0,1]$. Since f is continuous and we know that at some particular value of x we have $f(x) \neq 0$, then we have that

$$f(x) \geqslant 0 \text{ for all } x \in [0,1] \qquad \Longrightarrow \qquad \int_0^1 f(x)dx > 0 \ ,$$

and similarly

$$f(x) \leqslant 0 \text{ for all } x \in [0,1] \qquad \Longrightarrow \qquad \int_0^1 f(x)dx < 0 \ .$$

Thus, given that $\int_0^1 f(x)dx = 0$, we cannot have either $f(x) \geqslant 0$ for all x in $0 \leqslant x \leqslant 1$, or $f(x) \leqslant 0$ for all x in $0 \leqslant x \leqslant 1$. That is, we must have $f(x)$ taking both positive and negative values in the interval $0 \leqslant x \leqslant 1$.

(ii) We have

$$\begin{aligned}\int_0^1 (x-\alpha)^2 g(x)dx &= \int_0^1 \left(x^2 - 2\alpha x + \alpha^2\right) g(x)dx \\ &= \int_0^1 x^2 g(x)dx - 2\alpha \int_0^1 xg(x)dx + \alpha^2 \int_0^1 g(x)dx \\ &= \alpha^2 - 2\alpha \cdot \alpha + \alpha^2 \\ &= 0 \ .\end{aligned}$$

By the result of (i) then, we must have that $(x-\alpha)^2 g(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$. Since $(x-\alpha)^2$ is non-negative for all values of x, we must have that $g(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$. Hence $g(x) = 0$ for *some* x in the interval $0 \leqslant x \leqslant 1$.

Substituting in $g(x) = a + bx$, we find that

$$\begin{aligned} 1 &= \int_0^1 (a+bx)dx = \left[ax + \tfrac{1}{2}bx^2\right]_0^1 = a + \tfrac{1}{2}b \ , \\ \text{and} \quad \alpha &= \int_0^1 (ax+bx^2)dx = \left[\tfrac{1}{2}ax^2 + \tfrac{1}{3}bx^3\right]_0^1 = \tfrac{1}{2}a + \tfrac{1}{3}b \ , \\ \text{and} \quad \alpha^2 &= \int_0^1 (ax^2+bx^3)dx = \left[\tfrac{1}{3}ax^3 + \tfrac{1}{4}bx^4\right]_0^1 = \tfrac{1}{3}a + \tfrac{1}{4}b \ . \end{aligned}$$

Solving these simultaneous equations, we have

$$\begin{cases} 2a + b = 2 \\ 3a + 2b = 6\alpha \\ 4a + 3b = 12\alpha^2 \end{cases} \Longleftrightarrow \begin{cases} 2a + b = 2 \\ a + b = 6\alpha - 2 \\ a + b = 12\alpha^2 - 6\alpha \end{cases} \Longleftrightarrow \begin{cases} a = 4 - 6\alpha \\ b = 12\alpha - 6 \\ 12\alpha^2 - 12\alpha + 2 = 0 \end{cases} ,$$

thus we can find two particular functions $g(x)$ that satisfy the given conditions with each of two possible values of α:

$$12\alpha^2 - 12\alpha + 2 = 0 \qquad \Longrightarrow \qquad \alpha = \frac{12 \pm \sqrt{144-96}}{24} = \frac{1}{2} \pm \frac{\sqrt{3}}{6} \quad ,$$

giving two possibilities for $g(x)$. We have

$$a = 4 - 6\alpha = 4 - (3 \pm \sqrt{3}) = 1 \mp \sqrt{3} \quad ,$$

and

$$b = 12\alpha - 6 = 6 \pm 2\sqrt{3} - 6 = \pm 2\sqrt{3} \quad .$$

Thus our possible functions are

$$g_1(x) = 1 - \sqrt{3} + 2\sqrt{3}x \qquad \text{and} \qquad g_2(x) = 1 + \sqrt{3} - 2\sqrt{3}x \quad ,$$

and we can verify that $x = -\frac{a}{b}$ gives $g(-\frac{a}{b}) = a + b(-\frac{a}{b}) = 0$, that is

$$g_1\left(\frac{-1+\sqrt{3}}{2\sqrt{3}}\right) = 0 \qquad \text{and} \qquad g_2\left(\frac{-1-\sqrt{3}}{-2\sqrt{3}}\right) = 0 \quad ,$$

where

$$\frac{-1+\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} - \frac{\sqrt{3}}{6} \in [0,1] \quad ,$$

$$\text{and} \qquad \frac{-1-\sqrt{3}}{-2\sqrt{3}} = \frac{1}{2} + \frac{\sqrt{3}}{6} \in [0,1] \quad .$$

Thus each of these functions $g(x)$ satisfies the given conditions for a particular value of α, and each yields $g(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

(iii) Let $g(x) = h'(x)$. For $0 \leqslant t \leqslant 1$, we have that

$$\begin{aligned} h(t) - h(0) &= \int_0^t h'(x)dx \\ h(t) &= \int_0^t g(x)dx \quad , \end{aligned}$$

where we have used the fact that $h(0) = 0$. The property $h(1) = 1$ then yields

$$\int_0^1 g(x)dx = 1 \quad .$$

Integrating by parts, the other given properties of h yield that

$$\begin{aligned} \beta &= \int_0^1 h(x)dx = [xh(x)]_0^1 - \int_0^1 xh'(x)dx \\ &= 1 - \int_0^1 xg(x)dx \end{aligned}$$

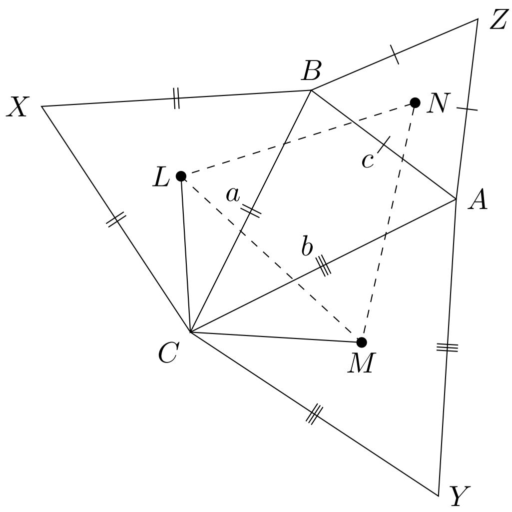
$$\Longleftrightarrow \qquad \int_0^1 xg(x)dx = 1 - \beta \quad ,$$

and

$$
\begin{aligned}
\frac{1}{2}\beta(2-\beta) &= \int_0^1 xh(x)dx \\
&= \left[\tfrac{1}{2}x^2h(x)\right]_0^1 - \frac{1}{2}\int_0^1 x^2h'(x)dx \\
&= \frac{1}{2} - \frac{1}{2}\int_0^1 x^2g(x)dx \\
\iff \qquad \int_0^1 x^2g(x)dx &= 1-\beta(2-\beta) \\
&= 1-2\beta+\beta^2 \\
\int_0^1 x^2g(x)dx &= (1-\beta)^2 \quad .
\end{aligned}
$$

Thus $g(x) = h'(x)$ satisfies the conditions in (ii) with $\alpha = 1-\beta$, and so by the result of part (ii) we deduce that $h'(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

Question 7



(i) Since triangle ACY is equilateral with M as its centre of rotational symmetry, we have $|CM| = |MA|$ and $\angle AMC = \frac{2\pi}{3}$. By the cosine rule on triangle ACM we get

$$\begin{aligned} |AC|^2 &= |CM|^2 + |MA|^2 - 2|CM||MA|\cos(\angle AMC) \\ b^2 &= |CM|^2 + |CM|^2 - 2|CM|^2 \cos\left(\tfrac{2\pi}{3}\right) \\ b^2 &= |CM|^2(1 + 1 - 2(-\tfrac{1}{2})) \\ b^2 &= 3|CM|^2 \\ \implies \quad |CM| &= \frac{b}{\sqrt{3}} \quad . \end{aligned}$$

Similarly, we have

$$|CL| = \frac{a}{\sqrt{3}} \quad .$$

(ii) Let $\angle ACB = \theta$, then the area of triangle ABC is

$$\Delta = \frac{1}{2}ab\sin\theta \quad ,$$

and by the cosine rule on triangle ABC we have

$$c^2 = a^2 + b^2 - 2ab\cos\theta \quad .$$

By symmetry of the equilateral triangles BXC and CYA, we have that

$$\angle MCA = \angle BCL = \frac{\pi}{6} \qquad \implies \qquad \angle MCL = \theta + \frac{\pi}{3} \quad .$$

Then by the cosine rule on triangle CML we have

$$
\begin{aligned}
|LM|^2 &= |CL|^2 + |CM|^2 - 2|CL||CM|\cos(\angle MCL) \\
|LM|^2 &= \frac{a^2}{3} + \frac{b^2}{3} - \frac{2ab}{3}\cos\left(\theta + \tfrac{\pi}{3}\right) \\
3|LM|^2 &= a^2 + b^2 - 2ab\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) \\
3|LM|^2 &= a^2 + b^2 - ab\cos\theta + \sqrt{3}ab\sin\theta \\
3|LM|^2 &= a^2 + b^2 - \frac{1}{2}(a^2 + b^2 - c^2) + 2\sqrt{3}\Delta \\
3|LM|^2 &= \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}c^2 + 2\sqrt{3}\Delta \\
6|LM|^2 &= a^2 + b^2 + c^2 + 4\sqrt{3}\Delta \quad .
\end{aligned}
$$

By symmetry, we also have

$$6|MN|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta$$

and

$$6|NL|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta \quad ,$$

that is: the triangle LMN is equilateral.

The area δ of triangle LMN is thus

$$
\begin{aligned}
\delta &= \frac{1}{2}|LM||MN|\sin\tfrac{\pi}{3} \\
&= \frac{\sqrt{3}}{4}|LM|^2 \\
&= \frac{\sqrt{3}}{24}(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta) \\
&= \frac{\sqrt{3}}{24}(a^2 + b^2 + c^2) + \frac{1}{2}\Delta \quad ,
\end{aligned}
$$

and the areas of the triangles are equal ($\delta = \Delta$) if and only if

$$
\begin{aligned}
\frac{\sqrt{3}}{24}(a^2 + b^2 + c^2) + \frac{1}{2}\Delta &= \Delta \\
\frac{\sqrt{3}}{24}(a^2 + b^2 + c^2) &= \frac{1}{2}\Delta \\
\sqrt{3}(a^2 + b^2 + c^2) &= 12\Delta \\
a^2 + b^2 + c^2 &= 4\sqrt{3}\Delta \quad .
\end{aligned}
$$

(iii) *Note: the angle C in the question refers to $\angle ACB$.* We have

$$
\begin{aligned}
& & (a-b)^2 &= -2ab(1-\cos(C-\tfrac{\pi}{3})) \\
&\iff & a^2+b^2 &= 2ab\cos(C-\tfrac{\pi}{3})) \\
&\iff & a^2+b^2 &= ab\cos C+\sqrt{3}ab\sin C \\
&\iff & \frac{1}{2}(a^2+b^2)+\frac{1}{2}(a^2+b^2-2ab\cos C) &= 2\sqrt{3}\cdot\frac{1}{2}ab\sin C \\
&\iff & \frac{1}{2}(a^2+b^2)+\frac{1}{2}c^2 &= 2\sqrt{3}\Delta \\
&\iff & a^2+b^2+c^2 &= 4\sqrt{3}\Delta \quad ,
\end{aligned}
$$

hence the two given conditions are equivalent. We deduce that triangles ABC and LMN have equal area if and only if

$$(a-b)^2 = -2ab(1-\cos(\theta-\tfrac{\pi}{3})) \quad .$$

The left-hand side of this equation is non-negative, whereas the right-hand side is non-positive. Thus this equation can hold if and only if

$$
\begin{aligned}
& a=b \quad \text{and} \quad \cos(\theta-\tfrac{\pi}{3})=1 \\
\iff \quad & a=b \quad \text{and} \quad \theta=\tfrac{\pi}{3} \quad ,
\end{aligned}
$$

since $0 \leqslant \theta \leqslant \pi$, being an interior angle of a triangle. The conditions $a=b$ and $\theta=\frac{\pi}{3}$ are equivalent to triangle ABC being equilateral. Hence triangles ABC and LMN have equal area if and only if triangle ABC is equilateral.

Question 8

First we check the basis case, $n = 1$: we have $a_1 = 1$, $b_1 = 2$, thus

$$a_1^2 + 2a_1b_1 - b_1^2 = 1 + 4 - 4 = 1 \quad .$$

Now suppose that $a_k^2 + 2a_kb_k - b_k^2 = 1$ for some $k \geqslant 1$. This gives that

$$\begin{aligned} a_{k+1}^2 + 2a_{k+1}b_{k+1} - b_{k+1}^2 &= (a_k + 2b_k)^2 + 2(a_k + 2b_k)(2a_k + 5b_k) - (2a_k + 5b_k)^2 \\ &= (1 + 4 - 4)a_k^2 + (4 + 2(4 + 5) - 20)a_kb_k + (4 + 20 - 25)b_k^2 \\ &= a_k^2 + 2a_kb_k - b_k^2 \\ &= 1 \qquad\qquad \text{(by the induction hypothesis)} \quad . \end{aligned}$$

By induction then, $a_n^2 + 2a_nb_n - b_n^2 = 1$ for all $n \geqslant 1$.

(i) First note that by the rules defining the sequences (and the initial values $a_1 = 1$, $b_1 = 2$) we know that a_n and b_n are positive integers for all $n \geqslant 1$.

By inspection we have $b_1 = 2 \geqslant 2 \cdot 5^0$. Now suppose that $b_k \geqslant 2 \cdot 5^{k-1}$ for some $k \geqslant 1$. We have

$$\begin{aligned} b_{k+1} &= 2a_k + 5b_k \\ &\geqslant 5b_k && (a_k \geqslant 0) \\ &\geqslant 5 \cdot 2 \cdot 5^{k-1} && \text{(by the induction hypothesis)} \\ \implies \quad b_{k+1} &\geqslant 2 \cdot 5^k \quad . \end{aligned}$$

By induction then, $b_n \geqslant 2 \cdot 5^{n-1}$ for all $n \geqslant 1$.

Further, by the result $(*)$

$$\begin{aligned} c_n^2 + 2c_n - 1 &= \frac{1}{b_n^2} \\ (c_n + 1)^2 &= 2 + \frac{1}{b_n^2} \\ c_n &= \sqrt{2 + \frac{1}{b_n^2}} - 1 \quad , \end{aligned}$$

where we take the positive square root because $a_n, b_n \geqslant 0 \implies c_n \geqslant 0$. As $n \to \infty$, the inequality $b_n \geqslant 2 \cdot 5^{n-1}$ shows that $b_n \to \infty$, and thus $\frac{1}{b_n^2} \to 0$. Hence

$$c_n \to \sqrt{2} - 1 \qquad \text{as } n \to 0 \quad .$$

(ii) Since $b_n \geqslant 0$, we have

$$\frac{1}{b_n^2} > 0 \quad \Longrightarrow \quad \sqrt{2 + \frac{1}{b_n^2}} > \sqrt{2} \quad \Longrightarrow \quad c_n > \sqrt{2} - 1 \quad .$$

Thus $\sqrt{2} < c_n + 1$, and

$$2 < \sqrt{2}(c_n + 1) \quad \Longrightarrow \quad \frac{2}{c_n + 1} < \sqrt{2} \quad .$$

That is:

$$\frac{2}{c_n + 1} < \sqrt{2} < c_n + 1 \quad .$$

We can compute

$$\begin{aligned} a_1 = 1\ ,\ b_1 = 2 \quad &\Longrightarrow \quad c_1 = \frac{1}{2} \quad , \\ a_2 = 1 + 4 = 5\ ,\ b_2 = 2 + 10 = 12 \quad &\Longrightarrow \quad c_2 = \frac{5}{12} \quad , \\ a_3 = 5 + 24 = 29\ ,\ b_3 = 10 + 60 = 70 \quad &\Longrightarrow \quad c_3 = \frac{29}{70} \quad , \end{aligned}$$

and substituting $c_3 = \frac{29}{70}$ into the derived inequality we have

$$\begin{aligned} \frac{2}{\frac{29}{70} + 1} < \sqrt{2} < \frac{29}{70} + 1 \\ \frac{140}{99} < \sqrt{2} < \frac{99}{70} \quad . \end{aligned}$$

Section B: Mechanics

Question 9

A diagram may or may not be useful in this question.

(i) The motion of the particle may be given parametrically by

$$x(t) = (u\cos\alpha)t \quad , \quad y(t) = (u\sin\alpha)t - \tfrac{1}{2}gt^2 \quad .$$

The time T at which the particle passes through point P is thus given by

$$\begin{aligned}(u\cos\alpha)T &= d\\ T &= \frac{d}{u\cos\alpha} \quad ,\end{aligned}$$

and substituting this into the vertical coordinate equation, we must have

$$\begin{aligned} d\tan\beta = y(T) &= u\sin\alpha\frac{d}{u\cos\alpha} - \frac{1}{2}g\frac{d^2}{u^2\cos^2\alpha}\\ &= d\tan\alpha - \frac{1}{2}\frac{gd^2}{u^2\cos^2\alpha}\\ \implies \qquad u^2(\tan\beta - \tan\alpha) &= -\frac{gd}{2\cos^2\alpha}\\ u^2 &= \frac{gd\sec^2\alpha}{2(\tan\alpha - \tan\beta)} \quad . \end{aligned} \qquad (*)$$

Since α is chosen such that u is as small as possible, we must have

$$\frac{du}{d\alpha} = 0 \qquad \implies \qquad \frac{d}{d\alpha}(u^2) = 0 \quad ,$$

thus

$$\begin{aligned} 0 &= \frac{d}{d\alpha}\left(\frac{\sec^2\alpha}{\tan\alpha - \tan\beta}\right)\\ &= \frac{(\tan\alpha - \tan\beta)\cdot 2\sec^2\alpha\tan\alpha - \sec^2\alpha\cdot\sec^2\alpha}{(\tan\alpha - \tan\beta)^2}\\ \implies \qquad 0 &= 2\tan\alpha(\tan\alpha - \tan\beta) - \sec^2\alpha \quad . \end{aligned}$$

Thus we have

$$\frac{\sec^2\alpha}{2(\tan\alpha - \tan\beta)} = \tan\alpha \quad ,$$

and substituting this into $(*)$ we find

$$u^2 = gd\tan\alpha \quad ,$$

as required.

Rearranging, we can find

$$\begin{aligned}
\tan\alpha - \tan\beta &= \frac{\sec^2\alpha}{2\tan\alpha} \\
\implies \tan\beta &= \tan\alpha - \frac{\sec^2\alpha}{2\tan\alpha} \\
&= \frac{2\tan^2\alpha - \sec^2\alpha}{2\tan\alpha} = \frac{2\sin^2\alpha - 1}{2\sin\alpha\cos\alpha} \\
&= \frac{\sin^2\alpha - \cos^2\alpha}{2\sin\alpha\cos\alpha} \\
&= -\cot(2\alpha) \\
&= -\tan(\tfrac{\pi}{2} - 2\alpha) \\
&= \tan(2\alpha - \tfrac{\pi}{2}) \quad .
\end{aligned}$$

Since β is acute and therefore α is also acute, this can only be satisfied by $\beta = 2\alpha - \frac{\pi}{2}$, that is: $2\alpha = \beta + \frac{\pi}{2}$.

(ii) As the particle passes through the point P, its horizontal velocity is

$$x'(T) = u\cos\alpha \quad ,$$

and its vertical velocity is

$$y'(T) = u\sin\alpha - gT = u\sin\alpha - \frac{gd}{u\cos\alpha} \quad .$$

Let θ be the angle relative to the horizontal the particle is travelling at when it passes through point P. Then we have

$$\begin{aligned}
\tan\theta &= \frac{u\sin\alpha - \frac{gd}{u\cos\alpha}}{u\cos\alpha} = \tan\alpha - \frac{gd}{u^2\cos^2\alpha} \\
&= \tan\alpha - \frac{gd}{gd\tan\alpha\cos^2\alpha} \\
&= \tan\alpha - \frac{1}{\sin\alpha\cos\alpha} \\
&= \frac{\sin^2\alpha - 1}{\sin\alpha\cos\alpha} \\
&= -\cot\alpha \\
&= \tan(\alpha - \tfrac{\pi}{2}) \quad ,
\end{aligned}$$

Since α is acute and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, this is satisfied by $\theta = \alpha - \frac{\pi}{2}$. Thus the particle travels through the point P at an angle $\frac{\pi}{2} - \alpha$ *below* the horizontal.

Question 10

(i) Let v be the speed of particle P after the collision with P_1. By conservation of momentum we have

$$um = vm + u_1 \cdot \lambda m \quad ,$$

and by the law of restitution we have

$$u_1 - v = e(u - 0) \quad .$$

Solving these for v and u_1, we have $v = u_1 - eu$ hence

$$\begin{aligned} u &= v + \lambda u_1 \\ \implies \qquad \lambda u_1 + u_1 - eu &= u \\ u_1 &= \frac{1+e}{1+\lambda} u \quad , \end{aligned}$$

which gives

$$v = \left(\frac{1+e}{1+\lambda} - e \right) u = \frac{1 - e\lambda}{1+\lambda} u \quad .$$

Similarly, for the $(n+1)$-th collision particle P_n is travelling with velocity u_n towards particle P_{n+1} which is stationary. By conservation of momentum we have

$$\begin{aligned} u_n \cdot \lambda^n m &= v_n \cdot \lambda^n m + u_{n+1} \cdot \lambda^{n+1} m \\ u_n &= v_n + \lambda u_{n+1} \quad , \end{aligned}$$

and by the law of restitution

$$\begin{aligned} u_{n+1} - v_n &= e(u_n - 0) \\ v_n &= u_{n+1} - eu_n \quad . \end{aligned}$$

Combining these, we have

$$\begin{aligned} u_n &= (1+\lambda)u_{n+1} - eu_n \\ (1+\lambda)u_{n+1} &= (1+e)u_n \\ u_{n+1} &= \frac{1+e}{1+\lambda} u_n \quad , \end{aligned}$$

hence by induction

$$u_n = \left(\frac{1+e}{1+\lambda} \right)^n u \quad ,$$

and this gives

$$\begin{aligned} v_n &= \left(\frac{1+e}{1+\lambda} \right)^{n+1} u - e \left(\frac{1+e}{1+\lambda} \right)^n u \\ &= \left(\frac{1+e}{1+\lambda} \right)^n \left(\frac{1+e}{1+\lambda} - e \right) u \\ &= \left(\frac{1+e}{1+\lambda} \right)^n \left(\frac{1-e\lambda}{1+\lambda} \right) u \quad . \end{aligned}$$

(ii) If $e > \lambda$, we have that

$$\frac{1+e}{1+\lambda} > 1 \quad .$$

In particular, this gives that $v < v_1 < v_2 < v_3 < \dots$, and so (for $n \geqslant 1$) after two collisions each particle P_n is travelling faster away from the origin than the particle before it. Thus each particle (after P) is involved in exactly two collisions.

(iii) If $e = \lambda$ then $u_n = u$ for all n and

$$\begin{aligned} v_n &= \frac{1-e\lambda}{1+\lambda}u = \frac{1-e^2}{1+e}u \\ &= (1-e)u \quad . \end{aligned}$$

The initial kinetic energy of the system is

$$E_0 = \frac{1}{2}mu^2 \quad .$$

After n collisions ($n \geqslant 1$), particle P_n is travelling with velocity $u_n = u$ and particles $P, P_1, P_2, \dots, P_{n-1}$ are all travelling with velocity $v = (1-e)u$ giving a total kinetic energy

$$\begin{aligned} E_n &= \frac{1}{2}\lambda^n mu^2 + \sum_{k=0}^{n-1} \frac{1}{2}\lambda^k m(1-e)^2u^2 = \frac{1}{2}mu^2 \left(e^n + (1-e)^2 \sum_{k=0}^{n-1} e^k \right) \\ &= \frac{1}{2}mu^2 \left(e^n + (1-e)^2 \frac{1-e^n}{1-e} \right) \\ &= \frac{1}{2}mu^2 \left(e^n + (1-e)(1-e^n) \right) \\ &= \frac{1}{2}mu^2 \left(1 - e + e^{n+1} \right) \quad . \end{aligned}$$

Thus the fraction of the initial kinetic energy lost is

$$\begin{aligned} \frac{E_0 - E_n}{E_0} &= \frac{\frac{1}{2}mu^2 - \frac{1}{2}mu^2 \left(1 - e + e^{n+1}\right)}{\frac{1}{2}mu^2} \\ &= 1 - 1 + e - e^{n+1} \\ &= e - e^{n+1} \quad , \end{aligned}$$

and since $0 < e < 1$, as $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{E_0 - E_n}{E_0} = e \quad .$$

(iv) If $\lambda e = 1$, then we have $v = 0$ and $v_n = 0$ for all n. Particle P stops after the first collision and all the subsequent particles stop after their second collision. After n collisions ($n \geqslant 1$), particle P_n is travelling with velocity u_n and all other particles are stationary giving a total kinetic energy of

$$\begin{aligned} E_n &= \frac{1}{2}\lambda^n m \left(\frac{1+e}{1+\lambda}\right)^{2n} u^2 = \frac{1}{2}e^{-n} m \left(\frac{1+e}{1+e^{-1}}\right)^{2n} u^2 \\ &= \frac{1}{2}e^{-n} m \left(\frac{e+e^2}{e+1}\right)^{2n} u^2 \\ &= \frac{1}{2}e^{-n} m \cdot e^{2n} u^2 \\ &= \frac{1}{2}e^{n} m u^2 \quad , \end{aligned}$$

and the fraction of the initial kinetic energy lost is

$$\begin{aligned} \frac{E_0 - E_n}{E_0} &= \frac{\frac{1}{2}mu^2 - \frac{1}{2}e^n mu^2}{\frac{1}{2}mu^2} \\ &= 1 - e^n \quad . \end{aligned}$$

Since $0 < e < 1$, as $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{E_0 - E_n}{E_0} = 1 \quad ,$$

that is as $n \to \infty$ all of the kinetic energy is lost.

Question 11

Let R_A and R_B be the normal reaction forces on the rod from the sides of the box at A and B respectively. Similarly, let F_A and F_B be the frictional forces on the ends of the rod from the sides of the box at A and B respectively. By the way the rod is nearly slipping, F_A is directed down the plane towards O, while F_B is directed up the side of the box away from O. Since the friction is limiting at A and B, we have

$$F_A = \tan\gamma\ R_A \qquad \text{and} \qquad F_B = \tan\gamma\ R_B \quad .$$

Resolving forces on the rod parallel to OB, we have

$$\begin{aligned} F_B + R_A &= W\cos\alpha \\ \implies \tan\gamma\ R_B + R_A &= W\cos\alpha \quad . \end{aligned} \tag{1}$$

Resolving forces on the rod parallel to OA, we have

$$\begin{aligned} R_B - F_A &= W\sin\alpha \\ \implies R_B - \tan\gamma\ R_A &= W\sin\alpha \quad . \end{aligned} \tag{2}$$

Dividing equation (2) by equation (1) we may eliminate W to get

$$\begin{aligned} \frac{R_B - \tan\gamma\ R_A}{\tan\gamma\ R_B + R_A} &= \frac{W\sin\alpha}{W\cos\alpha} \\ \frac{R_B - \tan\gamma\ R_A}{\tan\gamma\ R_B + R_A} &= \tan\alpha \\ R_B - \tan\gamma\ R_A &= \tan\alpha\tan\gamma\ R_B + \tan\alpha\ R_A \\ (1-\tan\alpha\tan\gamma)R_B &= (\tan\alpha + \tan\gamma)R_A \\ R_B &= \frac{\tan\alpha+\tan\gamma}{1-\tan\alpha\tan\gamma}R_A \\ R_B &= \tan(\alpha+\gamma)R_A \quad . \end{aligned}$$

Taking moments about the midpoint of the rod, we have

$$\begin{aligned} LR_A\sin\beta - LF_B\sin\beta &= LF_A\cos\beta + LR_B\cos\beta \\ R_A\sin\beta - R_B\tan\gamma\sin\beta &= R_A\tan\gamma\cos\beta + R_B\cos\beta \\ R_A\tan\beta - R_B\tan\gamma\tan\beta &= R_A\tan\gamma + R_B \\ \tan\beta &= \frac{R_A\tan\gamma + R_B}{R_A - R_B\tan\gamma} \\ &= \frac{R_A\tan\gamma + \tan(\alpha+\gamma)R_A}{R_A - \tan(\alpha+\gamma)R_A\tan\gamma} \\ &= \frac{\tan\gamma + \tan(\alpha+\gamma)}{1-\tan(\alpha+\gamma)\tan\gamma} \\ \tan\beta &= \tan(\alpha+2\gamma) \quad . \end{aligned}$$

From this we deduce that $\beta = \alpha + 2\gamma + n\pi$ for some integer n. Since $\frac{\pi}{2} > \beta > \alpha > 0$ we must have that $\frac{\pi}{2} > \beta - \alpha > 0$ and so $\frac{\pi}{2} > 2\gamma + n\pi > 0$. Since we are given $0 < 2\gamma < \pi$, this can only be satisfied if $n = 0$, and so $\beta = \alpha + 2\gamma$.

Section C: Probability and Statistics

Question 12

To clarify the question: if any player picks the winning number, the organiser pays out a total of £J, which is divided between all players who picked the winning number.

(i) The probability that any given player picks the winning number is $\frac{1}{N}$, hence the probability that all of the N players do not pick the winning number is

$$p = \left(1 - \frac{1}{N}\right)^N \quad .$$

If no player picks the winning number, the organiser's profit is cN. If any player picks the winning number, the organiser's profit is $cN - J$. Thus the organiser's expected profit is

$$\begin{aligned} E &= pcN + (1-p)(cN - J) \\ &= cN - J + pJ \\ &= cN - J + \left(1 - \frac{1}{N}\right)^N J \quad . \end{aligned}$$

If $2Nc = J$, using the provided approximation, the organiser's expected profit is

$$\begin{aligned} cN - J + \left(1 - \frac{1}{N}\right)^N J &\approx cN - J + e^{-1}J \\ &= \left(\frac{1}{2} - 1 + \frac{1}{e}\right) J \\ &= \left(\frac{1}{e} - \frac{1}{2}\right) J \quad . \end{aligned}$$

Since $e > 2$ we have $\frac{1}{2} > \frac{1}{e}$, and so this quantity is negative. Thus in this case, the organiser will expect to make a loss.

(ii) The probability that a given player picks a popular number is $\gamma N \cdot \frac{a}{N}$, while the probability that they pick an unpopular number is $(1-\gamma)N \cdot \frac{b}{N}$. Since these events are mutually exclusive and exhaustive, we have

$$\begin{aligned} \gamma N \cdot \frac{a}{N} + (1-\gamma)N \cdot \frac{b}{N} &= 1 \\ \gamma a + (1-\gamma)b &= 1 \quad . \end{aligned}$$

The probability that organiser draws a popular number is γ, and in this case, the probability that no player has picked the winning number is

$$p_1 = \left(1 - \frac{a}{N}\right)^N \quad ,$$

and accordingly the organiser's expected profit is

$$E_1 = cN - J + \left(1 - \frac{a}{N}\right)^N J \quad .$$

The probability that organiser draws an unpopular number is $1 - \gamma$, and in this case, the probability that no player has picked the winning number is

$$p_2 = \left(1 - \frac{b}{N}\right)^N \quad ,$$

and accordingly the organiser's expected profit is

$$E_2 = cN - J + \left(1 - \frac{b}{N}\right)^N J \quad .$$

In total then, the organiser's expected profit is

$$\begin{aligned}\gamma E_1 + (1-\gamma)E_2 &= \gamma(cN - J) + \gamma J \left(1 - \frac{a}{N}\right)^N \\ &\qquad + (1-\gamma)(cN - J) + (1-\gamma)J\left(1 - \frac{a}{N}\right)^N \\ &= cN - J + \gamma J \left(1 - \frac{a}{N}\right)^N + (1-\gamma)J\left(1 - \frac{a}{N}\right)^N \quad ,\end{aligned}$$

and by the given result $(*)$, this is approximately

$$cN - J + \gamma J e^{-a} + (1-\gamma)Je^{-b} \quad ,$$

that is: $A = \gamma J$, $B = (1-\gamma)J$, $C = cN - J$.

If $\gamma = \frac{1}{8}$ and $a = 9b$, we have

$$\begin{aligned}\gamma a + (1-\gamma)b &= 1 \\ \frac{9}{8}b + \frac{7}{8}b &= 1 \\ 2b &= 1 \quad ,\end{aligned}$$

thus $b = \frac{1}{2}$ and $a = \frac{9}{2}$. If $2Nc = J$, then the organisers expected profit is approximately

$$\begin{aligned}cN - J + \gamma J e^{-a} + (1-\gamma)Je^{-b} &= \left(-\frac{1}{2} + \frac{1}{8}e^{-\frac{9}{2}} + \frac{7}{8}e^{-\frac{1}{2}}\right) J \\ &= \frac{1}{8}e^{-\frac{1}{2}}\left(-4\sqrt{e} + e^{-4} + 7\right) J \\ &> \frac{1}{8}e^{-\frac{1}{2}}\left(-4\sqrt{e} + 7\right) J \quad .\end{aligned}$$

We have $(4\sqrt{e})^2 = 16e < 16 \cdot 3 = 48$, thus

$$7^2 > (4\sqrt{e})^2 \qquad \Longrightarrow \qquad 7 > 4\sqrt{e} \quad ,$$

and the approximate expected profit is positive.

Question 13

Since the first slice of bread cannot be the second of a pair that is used to make a sandwich, we have

$$s_1 = 0 \quad .$$

For $r \geqslant 2$, in order for the r-th slice of bread to be used to make toast, we must have had that either (i) the $(r-1)$-th slice was used to make toast, or (ii) the $(r-1)$-th slice was used as the second slice to make a sandwich. Thus the probability, t_r, that r-th slice of bread will be used to make toast is the probability that one of the exclusive events (i) or (ii) occurs, and that we then choose to make toast for the next snack. That is, for $2 \leqslant r \leqslant n-1$:

$$t_r = (t_{r-1} + s_{r-1})p \quad .$$

For $r = n$, we have to choose toast if we have only one slice left, thus

$$t_n = t_{n-1} + s_{n-1} \quad .$$

For $r \geqslant 2$, the r-th slice of bread will be used as the second slice of a sandwich if and only if the $(r-1)$-th slice is used as the first slice of the sandwich. The possible uses for the $(r-1)$-th slice are (a) as toast (with probability t_{r-1}), (b) as the second slice of a sandwich (with probability s_{r-1}), or (c) as the first slice of a sandwich. As explained, the probability of (c) is the same as the probability that the r-th slice is used as the second slice of a sandwich, which is s_r. Since (a), (b), (c) are mutually exclusive and exhaustive, for $2 \leqslant r \leqslant n$ we have that

$$\begin{aligned} t_{r-1} + s_{r-1} + s_r &= 1 \\ s_r &= 1 - (s_{r-1} + t_{r-1}) \quad . \end{aligned} \tag{$*$}$$

For $2 \leqslant r \leqslant n-1$, rearranging the second of these two results we have

$$t_{r-1} = 1 - s_r - s_{r-1} \quad ,$$

and substituting this into the first result

$$\begin{aligned} t_r &= ((1 - s_r - s_{r-1}) + s_{r-1})p \\ t_r &= (1 - s_r)p \quad . \end{aligned}$$

This result even holds for $r = 1$ (with $s_1 = 0$), since the probability that we make toast with the first slice is exactly the probability that the first snack we make is toast, which occurs with probability p. Thus for $2 \leqslant r \leqslant n-1$, we have

$$t_{r-1} = (1 - s_{r-1})p \quad ,$$

and substituting this into $(*)$, we get

$$\begin{aligned} s_r &= 1 - (s_{r-1} + (1 - s_{r-1})p) \\ &= 1 - (1-p)s_{r-1} - p \\ &= (1-p)(1 - s_{r-1}) \\ s_r &= q(1 - s_{r-1}) \quad . \end{aligned} \tag{$**$}$$

Consider $r = 1$:

$$\frac{q + (-q)^1}{1+q} = \frac{q - q}{1+q} = 0 = s_1 \quad ,$$

hence the result holds for the basis case $r = 1$. Now suppose that

$$s_{k-1} = \frac{q + (-q)^{k-1}}{1+q}$$

for some $1 \leqslant k \leqslant n-2$, then by $(**)$

$$\begin{aligned} s_k &= q(1 - s_{k-1}) \\ &= q\left(1 - \frac{q + (-q)^{k-1}}{1+q}\right) \\ &= q\frac{1 - (-q)^{k-1}}{1+q} \\ &= \frac{q - q(-q)^{k-1}}{1+q} \\ s_k &= \frac{q + (-q)^k}{1+q} \quad , \end{aligned}$$

hence the result holds for all $1 \leqslant r \leqslant n-1$. When $r = n$, the result $(**)$ does not hold, and so this expression does not necessarily extend to s_n.

Using our earlier result that $t_r = (1 - s_r)p$, which holds for $1 \leqslant r \leqslant n-1$, we have

$$\begin{aligned} t_r &= \left(1 - \frac{q + (-q)^r}{1+q}\right) p \\ &= p - p\frac{q + (-q)^r}{1+q} \quad . \end{aligned}$$

For s_n, we have

$$\begin{aligned} s_n &= 1 - (s_{n-1} + t_{n-1}) \\ &= 1 - \left(\frac{q + (-q)^{n-1}}{1+q} + p - p\frac{q + (-q)^r}{1+q}\right) \\ &= 1 - p - (1-p)\frac{q + (-q)^{n-1}}{1+q} \\ &= q\left(1 - \frac{q + (-q)^{n-1}}{1+q}\right) \\ &= q\frac{1 - (-q)^{n-1}}{1+q} \\ &= \frac{q + (-q)^n}{1+q} \quad . \end{aligned}$$

As mentioned earlier, we have $t_n = t_{n-1} + s_{n-1}$, thus

$$\begin{aligned} t_n &= t_{n-1} + s_{n-1} \\ &= 1 - s_n \\ &= 1 - \frac{q + (-q)^n}{1+q} \\ &= \frac{1 - (-q)^n}{1+q} \quad . \end{aligned}$$

STEP II

Section A: Pure Mathematics

Question 1

(i) Integrating by parts, we have

$$\begin{aligned} I_n &= \int_0^1 x^n \arctan x dx \\ &= \left[\frac{x^{n+1}}{n+1} \arctan x\right]_0^1 - \frac{1}{n+1}\int_0^1 x^{n+1}\frac{1}{1+x^2}dx \\ &= \frac{1}{n+1}\arctan 1 - 0 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx \\ (n+1)I_n &= \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx \quad . \end{aligned}$$

Substituting $n = 0$, this gives

$$\begin{aligned} I_0 &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx \\ &= \frac{\pi}{4} - \left[\frac{1}{2}\log(1+x^2)\right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\log 2 \quad . \end{aligned}$$

(ii) By shifting the index $n \mapsto n+2$ in the result from (i), we have

$$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx \quad ,$$

thus

$$\begin{aligned} (n+3)I_{n+2} + (n+1)I_n &= \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx + \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx \\ &= \frac{\pi}{2} - \int_0^1 \frac{x^{n+3}+x^{n+1}}{1+x^2}dx \\ &= \frac{\pi}{2} - \int_0^1 x^{n+1}dx \\ &= \frac{\pi}{2} - \frac{1}{n+2} \quad . \end{aligned}$$

Substituting $n = 0$ and $n = 2$ into this result, we find

$$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2} \quad ,$$

and

$$5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4} \quad .$$

Subtracting the first of these results from the second, and using our expression for I_0, we obtain

$$\begin{aligned} 5I_4 - I_0 &= -\frac{1}{4} + \frac{1}{2} \\ 5I_4 &= \frac{1}{4} + I_0 \\ &= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{2}\log 2 \\ I_4 &= \frac{1}{20}(1 + \pi - 2\log 2) \quad . \end{aligned}$$

(iii) Substituting $n = 1$ and our expression for I_4 into the given equation, we get

$$\begin{aligned} A - \frac{1}{2}\left((-1)^1 \cdot \frac{1}{1} + (-1)^2 \cdot \frac{1}{2}\right) &= 5I_4 \\ A - \frac{1}{2}\left(-1 + \frac{1}{2}\right) &= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{2}\log 2 \\ A + \frac{1}{4} &= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{2}\log 2 \\ A &= \frac{\pi}{4} - \frac{1}{2}\log 2 \quad . \end{aligned}$$

That is, the result holds for the basis cases $n = 1$ with this constant. Now suppose that the result holds with this value of A with $n = k$ for some $k \geqslant 1$. Substituting $n = 4k$ into the result in (ii), we have

$$(4k+3)I_{4k+2} + (4k+1)I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2} \quad ,$$

and by shifting the index $4k \mapsto 4k+2$

$$(4k+5)I_{4k+4} + (4k+3)I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4} \quad .$$

Thus

$$\begin{aligned} (4k+5)I_{4k+4} - (4k+1)I_{4k} &= \frac{1}{4k+2} - \frac{1}{4k+4} \\ (4(k+1)+1)I_{4(k+1)} &= (4k+1)I_{4k} + \frac{1}{4k+2} - \frac{1}{4k+4} \quad . \end{aligned}$$

Now using the induction result for $n = k$, we get

$$
\begin{aligned}
(4(k+1)+1)I_{4(k+1)} &= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r\frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4} \\
&= A - \frac{1}{2}\left(\sum_{r=1}^{2k}(-1)^r \cdot \frac{1}{r} - \frac{1}{2k+1} + \frac{1}{2k+2}\right) \\
&= A - \frac{1}{2}\sum_{r=1}^{2k+2}(-1)^r \cdot \frac{1}{r} \quad ,
\end{aligned}
$$

which proves the induction result for $n = k + 1$. Hence by induction, the result holds for all $n \geqslant 1$.

Question 2

We have

$$\begin{aligned} x_{n+2} &= \frac{a x_{n+1} - 1}{x_{n+1} + b} \\ &= \frac{a \cdot \frac{a x_n - 1}{x_n + b} - 1}{\frac{a x_n - 1}{x_n + b} + b} \\ &= \frac{a(a x_n - 1) - (x_n + b)}{a x_n - 1 + b(x_n + b)} \\ &= \frac{(a^2 - 1) x_n - (a + b)}{(a + b) x_n + (b^2 - 1)} \quad . \end{aligned}$$

(i) For the sequence to have period 2 we must have $x_{n+2} = x_n$ and $x_{n+1} \neq x_n$ for all $n = 0, 1, 2, \ldots$. We have $x_{n+2} = x_n$ if and only if

$$\begin{aligned} x_n &= \frac{(a^2 - 1) x_n - (a + b)}{(a + b) x_n + (b^2 - 1)} \\ (a + b) x_n^2 + (b^2 - 1) x_n &= (a^2 - 1) x_n - (a + b) \\ (a + b) x_n^2 + (b^2 - a^2) x_n + (a + b) &= 0 \\ (a + b)(x_n^2 + (b - a) x_n + 1) &= 0 \quad . \end{aligned}$$

We have $x_{n+1} = x_n$ if and only if

$$\begin{aligned} x_n &= \frac{a x_n - 1}{x_n + b} \\ x_n^2 + b x_n &= a x_n - 1 \\ x_n^2 + (b - a) x_n + 1 &= 0 \quad . \end{aligned}$$

Hence in order for $x_{n+2} = x_n$ and $x_{n+1} \neq x_n$ for all $n = 0, 1, 2, \ldots$, it is necessary that $a + b = 0$.

(ii) For the sequence to have period 4, it must not have period 2, hence we have that $a + b \neq 0$ is a necessary condition. We have

$$\begin{aligned} x_{n+4} &= \frac{(a^2 - 1) x_{n+2} - (a + b)}{(a + b) x_{n+2} + (b^2 - 1)} \\ &= \frac{(a^2 - 1) \frac{(a^2 - 1) x_n - (a + b)}{(a + b) x_n + (b^2 - 1)} - (a + b)}{(a + b) \frac{(a^2 - 1) x_n - (a + b)}{(a + b) x_n + (b^2 - 1)} + (b^2 - 1)} \\ &= \frac{(a^2 - 1)^2 x_n - (a^2 - 1)(a + b) - (a + b)^2 x_n - (b^2 - 1)(a + b)}{(a + b)(a^2 - 1) x_n - (a + b)^2 + (b^2 - 1)(a + b) x_n + (b^2 - 1)^2} \quad . \end{aligned}$$

Let

$$A = \frac{a^2 - 1}{a + b} \qquad \text{and} \qquad B = \frac{b^2 - 1}{a + b} \quad ,$$

then dividing the numerator and denominator of the fraction by $(a+b)^2$, we find

$$\begin{aligned} x_{n+4} &= \frac{A^2 x_n - A - x_n - B}{A x_n - 1 + B x_n + B^2} \\ &= \frac{(A^2-1)x_n - (A+B)}{(A+B)x_n + (B^2-1)} \quad . \end{aligned}$$

By direct comparison with part (i), a necessary condition for the sequence to have period 4 is $A+B=0$. For sufficiency, we must also ensure that the sequence does not have period 2, thus $a+b \neq 0$, and we must ensure that the sequence is not constant, thus $x_1 \neq x_0$.

Assuming $a+b \neq 0$, we have

$$\begin{aligned} A+B=0 \qquad &\Longleftrightarrow \qquad \frac{a^2-1+b^2-1}{a+b} = 0 \\ &\Longleftrightarrow \qquad a^2+b^2 = 2 \quad , \end{aligned}$$

and assuming $x_0 + b \neq 0$, we also have

$$\begin{aligned} x_1 \neq x_0 \qquad &\Longleftrightarrow \qquad x_0 \neq \frac{a x_0 - 1}{x_0 + b} \\ &\Longleftrightarrow \qquad x_0^2 + b x_0 \neq a x_0 - 1 \\ &\Longleftrightarrow \qquad x_0^2 + (b-a)x_0 + 1 \neq 0 \quad . \end{aligned}$$

Hence, our necessary and sufficient conditions for the sequence to have period 4 are:

$$a^2+b^2=2 \qquad \text{and} \qquad a+b \neq 0 \qquad \text{and} \qquad x_0^2 + (b-a)x_0 + 1 \neq 0 \quad .$$

Question 3

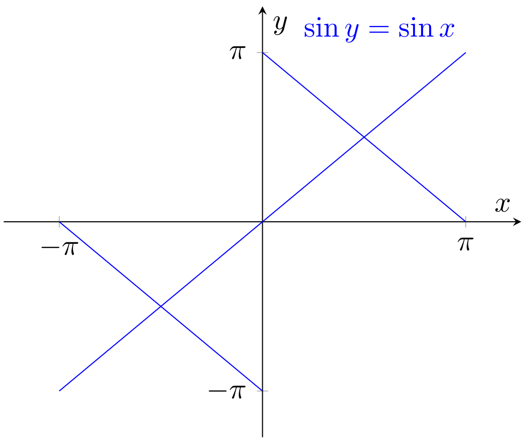
(i) By the periodicity and symmetry of sin, we have

$$\sin y = \sin x \quad \iff \quad \begin{cases} y = 2n\pi + x & \text{for some } n \in \mathbb{Z}, \text{ or} \\ y = (2n+1)\pi - x & \text{for some } n \in \mathbb{Z} \end{cases} .$$

Thus, in $-\pi \leqslant x \leqslant \pi$, $-\pi \leqslant y \leqslant \pi$, we have three branches of solutions:

$$y = x \quad , \quad y = \pi - x \quad , \quad y = -\pi - x \ ,$$

and our sketch is as follows.



(ii) Differentiating

$$\begin{aligned} (\cos y)y' &= \frac{1}{2}\cos x \\ y' &= \frac{\frac{1}{2}\cos x}{\sqrt{1-\sin^2 y}} \qquad (0 \leqslant y \leqslant \tfrac{\pi}{2}, \text{ so we take the positive square root}) \\ &= \frac{\frac{1}{2}\cos x}{\sqrt{1-\frac{1}{4}\sin^2 x}} \\ &= \frac{\cos x}{\sqrt{4-\sin^2 x}} \ . \end{aligned}$$

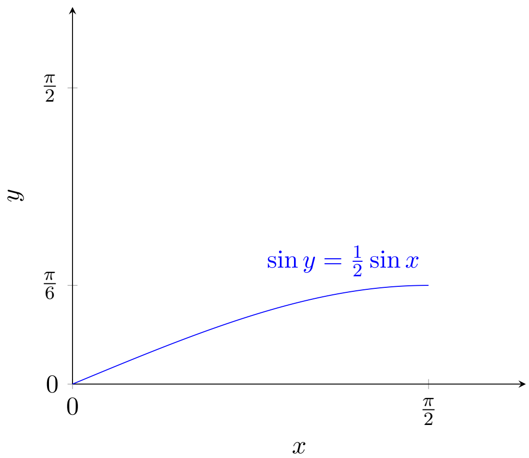
Rearranging, we have

$$\begin{aligned} (y')^2 &= \frac{\cos^2 x}{4-\sin^2 x} \\ &= \frac{1-\sin^2 x}{4-\sin^2 x} \\ &= 1 - \frac{3}{4-\sin^2 x} \ , \end{aligned}$$

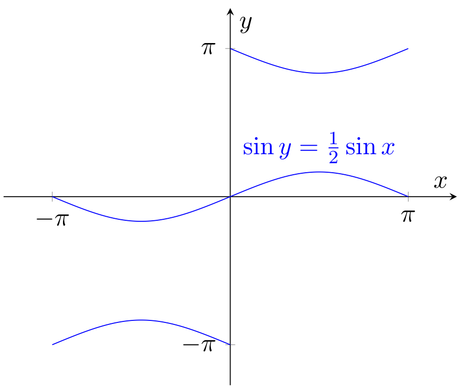
and now differentiating again

$$\begin{aligned}
2(y')y'' &= \frac{3(-2\sin x\cos x)}{(4-\sin^2 x)^2} \\
y'' &= \frac{-3\sin x\cos x}{(4-\sin^2 x)^2}\cdot\frac{1}{y'} = \frac{-3\sin x\cos x}{(4-\sin^2 x)^2}\cdot\frac{\sqrt{4-\sin^2 x}}{\cos x} \\
&= \frac{-3\sin x}{(4-\sin^2 x)^{\frac{3}{2}}} \quad .
\end{aligned}$$

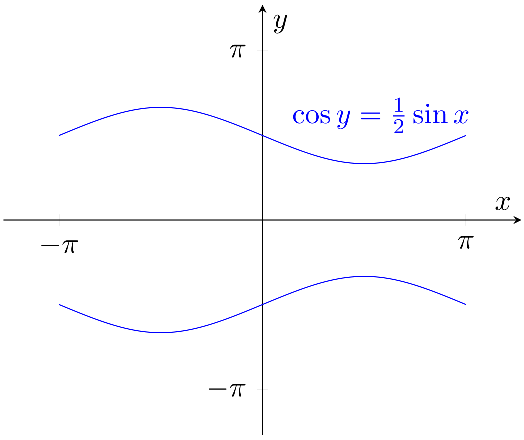
For $0 \leqslant x \leqslant \frac{\pi}{2}$, $0 \leqslant y \leqslant \frac{\pi}{2}$ there is only one curve of solutions from $x = 0, y = 0$ to $x = \frac{\pi}{2}$, $y = \arcsin\frac{1}{2} = \frac{\pi}{6}$. We note that $y'' \leqslant 0$ for all $0 \leqslant x \leqslant \frac{\pi}{2}$, hence the gradient decreases across this domain from $y'(0) = \frac{1}{\sqrt{4-0^2}} = \frac{1}{2}$ to $y'(\frac{\pi}{2}) = 0$. In particular, we note that $x = \frac{\pi}{2}$ gives the maximum over this domain. Our sketch is as follows.



Using the periodicity and symmetry of sin, we get the following sketch for all the points $\sin y = \frac{1}{2}\sin x$ in $-\pi \leqslant x \leqslant \pi$, $-\pi \leqslant y \leqslant \pi$.



(iii) By using the identity $\cos(\frac{\pi}{4} + t) = \sin(\frac{\pi}{4} - t)$, we can immediately get the graph of $\cos y = \frac{1}{2}\sin x$ by reflecting the graph for $\sin y = \frac{1}{2}\sin x$ across the line $y = \frac{\pi}{4}$ (and recalling that the graph is 2π-periodic in both x- and y- directions).



Question 4

(i) Setting $f(x) = 1$ in $(*)$ we have

$$\left(\int_a^b g(x)\ dx\right)^2 \leqslant \left(\int_a^b 1\ dx\right)\left(\int_a^b g(x)^2\ dx\right)$$
$$\left(\int_a^b g(x)\ dx\right)^2 \leqslant (b-a)\int_a^b g(x)^2\ dx \quad .$$

If we set $a = 0$, $b = t$, $g(x) = e^x$, we get

$$\left(\int_0^t e^x\ dx\right)^2 \leqslant t\int_0^t e^{2x}\ dx$$
$$\left([e^x]_0^t\right)^2 \leqslant t\left[\frac{1}{2}e^{2x}\right]_0^t$$
$$(e^t-1)^2 \leqslant \frac{1}{2}(e^{2t}-1)$$
$$(e^t-1)^2 \leqslant \frac{1}{2}(e^t+1)(e^t-1)$$
$$\frac{e^t-1}{e^t+1} \leqslant \frac{1}{2}t \quad .$$

(ii) Setting $f(x) = x$, $a = 0$, $b = 1$ in $(*)$ we have

$$\left(\int_0^1 xg(x)\ dx\right)^2 \leqslant \left(\int_0^1 x^2\ dx\right)\left(\int_0^1 g(x)^2\ dx\right)$$
$$\left(\int_0^1 xg(x)\ dx\right)^2 \leqslant \frac{1}{3}\int_0^1 g(x)^2\ dx \quad .$$

If we now set $g(x) = e^{-\frac{1}{4}x^2}$ (so that $g(x)^2 = e^{-\frac{1}{2}x^2}$), we have

$$\left(\int_0^1 xe^{-\frac{1}{4}x^2}\ dx\right)^2 \leqslant \frac{1}{3}\int_0^1 e^{-\frac{1}{2}x^2}\ dx$$
$$\left(\left[-2e^{-\frac{1}{4}x^2}\right]_0^1\right)^2 \leqslant \frac{1}{3}\int_0^1 e^{-\frac{1}{2}x^2}\ dx$$
$$4\left(e^{-\frac{1}{4}}-1\right)^2 \leqslant \frac{1}{3}\int_0^1 e^{-\frac{1}{2}x^2}\ dx$$
$$12\left(e^{-\frac{1}{4}}-1\right)^2 \leqslant \int_0^1 e^{-\frac{1}{2}x^2}\ dx \quad .$$

(iii) Setting $f(x) = 1$, $a = 0$, $b = \frac{\pi}{2}$, we have

$$\left(\int_0^{\frac{\pi}{2}} g(x) \ dx \right)^2 \leqslant \frac{\pi}{2} \int_0^{\frac{\pi}{2}} g(x)^2 \ dx \quad .$$

If we set $g(x) = \sqrt{\sin x}$ we find

$$\begin{aligned}
\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right)^2 &\leqslant \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin x \ dx \\
\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right)^2 &\leqslant \frac{\pi}{2} \left[-\cos x \right]_0^{\frac{\pi}{2}} \\
\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right)^2 &\leqslant \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx &\leqslant \sqrt{\frac{\pi}{2}} \quad .
\end{aligned}$$

If instead we set $a = 0$, $b = \frac{\pi}{2}$, and $g(x)^2 = \sqrt{\sin x}$ in $(*)$, we get

$$\left(\int_0^{\frac{\pi}{2}} f(x) \sin(x)^{\frac{1}{4}} \ dx \right)^2 \leqslant \left(\int_0^{\frac{\pi}{2}} f(x)^2 \ dx \right) \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \quad .$$

In order to evaluate the left-hand integral, we try $f(x) = \cos x$ (since $\cos x = \frac{d}{dx}(\sin x)$), and this gives

$$\begin{aligned}
\left(\int_0^{\frac{\pi}{2}} \cos(x) \sin(x)^{\frac{1}{4}} \ dx \right)^2 &\leqslant \left(\int_0^{\frac{\pi}{2}} \cos^2(x) \ dx \right) \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \\
\left(\left[\frac{4}{5} \sin(x)^{\frac{5}{4}} \right]_0^{\frac{\pi}{2}} \right)^2 &\leqslant \left(\int_0^{\frac{\pi}{2}} \frac{1}{2} \left(\cos(2x) + 1 \right) \ dx \right) \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \\
\left(\frac{4}{5} \right)^2 &\leqslant \frac{1}{2} \left[\frac{1}{2} \sin(2x) + x \right]_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \\
\frac{64}{25} &\leqslant \left[\sin(2x) + 2x \right]_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \\
\frac{64}{25} &\leqslant \pi \left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx \right) \\
\implies \qquad \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx &\geqslant \frac{64}{25\pi} \quad .
\end{aligned}$$

Question 5

(i) Differentiating, the gradient of C at the point $(at^2, 2at)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} \quad .$$

Hence the gradient of the normal to C at the point P is $-p$, and the equation of this line is

$$y - 2ap = -p(x - ap^2) \quad .$$

Given that this line intersects C again at $(an^2, 2an)$ where $n \neq p$ (and also $p \neq 0$), we have

$$\begin{aligned} 2an - 2ap &= -p(an^2 - ap^2) \\ 2(n-p) &= -p(n-p)(n+p) \\ -\frac{2}{p} &= n + p \\ \implies \qquad n &= -\left(p + \frac{2}{p}\right) \quad . \end{aligned}$$

(ii) By Pythagoras' theorem, the distance $|PN|$ is given by

$$\begin{aligned} |PN|^2 &= (an^2 - ap^2)^2 + (2an - 2ap)^2 \\ &= a^2(n-p)^2(n+p)^2 + 4a^2(n-p)^2 \\ &= a^2(n-p)^2\left((n+p)^2 + 4\right) \\ &= a^2\left(-2p - \frac{2}{p}\right)^2 \left(\left(-\frac{2}{p}\right)^2 + 4\right) \\ &= 16a^2\left(\frac{p^2+1}{p}\right)^2\left(\frac{p^2+1}{p^2}\right) \\ &= 16a^2\frac{(p^2+1)^3}{p^4} \quad . \end{aligned}$$

Differentiating, we find

$$\begin{aligned} \frac{d}{dp}|PN|^2 &= 16a^2\frac{d}{dp}\left(\frac{p^6 + 3p^4 + 3p^2 + 1}{p^4}\right) = 16a^2\frac{d}{dp}\left(p^2 + 3 + 3p^{-2} + p^{-4}\right) \\ &= 16a^2\left(2p - 6p^{-3} - 4p^{-5}\right) \\ &= \frac{32a^2}{p^5}(p^6 - 3p^2 - 2) \\ &= \frac{32a^2}{p^5}(p^2 - 2)(p^4 + 2p^2 + 1) \\ &= \frac{32a^2}{p^5}(p^2+1)^2(p^2-2) \quad . \end{aligned}$$

Thus

$$\frac{d}{dp}|PN|^2 = 0 \qquad \Longleftrightarrow \qquad p^2 = 2 \ .$$

Further, since $p^2 < 2 \implies \frac{d}{dp}|PN|^2 < 0$ and $p^2 > 2 \implies \frac{d}{dp}|PN|^2 > 0$, these points are minima.

(iii) We know that $q \neq p$ and $q \neq n$, and since Q is a point on the circumference of a circle with diameter PN, the angle at Q subtended by the diameter is a right angle, that is: QP and QN are perpendicular. Thus

$$\begin{aligned}
(\text{gradient of } QP) \cdot (\text{gradient of } QN) &= -1 \\
\frac{2aq - 2ap}{aq^2 - ap^2} \cdot \frac{2aq - 2an}{aq^2 - an^2} &= -1 \\
\frac{2}{q+p} \cdot \frac{2}{q+n} &= -1 \\
4 &= -np - (n+p)q - q^2 \\
4 &= \left(p + \frac{2}{p}\right) p + \frac{2}{p} q - q^2 \\
4 &= p^2 + 2 - q^2 - \frac{2q}{p} \\
2 &= p^2 - q^2 + \frac{2q}{p} \quad .
\end{aligned}$$

When $p^2 = 2$, we have that

$$\begin{aligned}
2 &= 2 - q^2 + \frac{2q}{p} \\
q^2 &= \frac{2q}{p} \\
&= \frac{p^2 q}{p} \\
pq^2 - p^2 q &= 0 \\
pq(q-p) &= 0 \quad .
\end{aligned}$$

Since $p \neq 0$ and $q \neq p$, we deduce that $p^2 = 2$ if and only if $q = 0$; that is: $|PN|$ is minimised when $q = 0$, which is when Q is at the origin.

Question 6

(i) For $n = 1$ we have

$$S_n = 1 \quad , \quad 2\sqrt{1} - 1 = 1 \ ,$$

hence the result holds in the basis case $n = 1$. Now suppose that $S_n \leqslant 2\sqrt{n} - 1$ for some $n = k$, $k \geqslant 1$;

$$\begin{aligned} S_{k+1} &= \sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} \\ &= S_k + \frac{1}{\sqrt{k+1}} \\ &\leqslant 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \ . \end{aligned}$$

We now want to show that this is less than or equal to $2\sqrt{k+1} - 1$. We have

$$\begin{aligned} \left(2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}\right) - \left(2\sqrt{k+1} - 1\right) &= 2\sqrt{k} + \frac{1}{\sqrt{k+1}} - 2\sqrt{k+1} \\ &= \frac{2\sqrt{k^2+k} + 1 - 2(k+1)}{\sqrt{k+1}} \\ &= \frac{2\sqrt{k^2+k} - (2k+1)}{\sqrt{k+1}} \\ &= \frac{\sqrt{4k^2+4k} - \sqrt{4k^2+4k+1}}{\sqrt{k+1}} \\ &\leqslant 0 \ , \end{aligned}$$

since $k \geqslant 1 \implies 4k^2 + 4k \geqslant 0$. Thus

$$\begin{aligned} 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} &\leqslant 2\sqrt{k+1} - 1 \\ \implies \quad S_{k+1} &\leqslant 2\sqrt{k+1} - 1 \ , \end{aligned}$$

and so the result holds for all $n \geqslant 1$.

(ii) For $k \geqslant 0$, we have

$$\begin{aligned} (4k+1)\sqrt{k+1} - (4k+3)\sqrt{k} &= \sqrt{(4k+1)^2(k+1)} - \sqrt{(4k+3)^2k} \\ &= \sqrt{(16k^2+8k+1)(k+1)} - \sqrt{(16k^2+24k+9)k} \\ &= \sqrt{16k^3+24k^2+9k+1} - \sqrt{16k^3+24k^2+9k} \\ &> 0 \ , \end{aligned}$$

since $k \geqslant 0 \implies 16k^3 + 24k^2 + 9k \geqslant 0$. Hence we deduce that

$$(4k+1)\sqrt{k+1} > (4k+3)\sqrt{k} \quad \text{for } k \geqslant 0 \ .$$

If we substitute $n = 1$, we have

$$S_1 = 1 \quad , \quad 2\sqrt{1} + \frac{1}{2\sqrt{1}} - C = \frac{5}{2} - C \ ,$$

thus we must have

$$\begin{aligned} 1 &\geqslant \frac{5}{2} - C \\ C &\geqslant \frac{3}{2} \ . \end{aligned}$$

That is, for the given inequality to hold for all n, C cannot be less than $\frac{3}{2}$. We now need to verify if the inequality does in fact hold for all n if we set $C = \frac{3}{2}$.

As shown, the result holds for $n = 1$, now suppose that

$$S_k \geqslant 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2} \ ,$$

for some $k \geqslant 1$. Then we have

$$\begin{aligned} S_{k+1} &= S_k + \frac{1}{\sqrt{k+1}} \\ &\geqslant 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2} + \frac{1}{\sqrt{k+1}} \ . \end{aligned}$$

We now want to show that this is greater than or equal to $2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - \frac{3}{2}$. We have

$$\begin{aligned} &\left(2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2} + \frac{1}{\sqrt{k+1}}\right) - \left(2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - \frac{3}{2}\right) \\ &= 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - 2\sqrt{k+1} - \frac{1}{2\sqrt{k+1}} \\ &= 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{2\sqrt{k+1}} - 2\sqrt{k+1} \\ &= \frac{4k\sqrt{k+1} + \sqrt{k+1} + \sqrt{k} - 4(k+1)\sqrt{k}}{\sqrt{k}\sqrt{k+1}} \\ &= \frac{(4k+1)\sqrt{k+1} - (4k+3)\sqrt{k}}{\sqrt{k}\sqrt{k+1}} \\ &\geqslant 0 \ , \end{aligned}$$

by the inequality shown above. We thus have

$$\begin{aligned} 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2} + \frac{1}{\sqrt{k+1}} &\geqslant 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - \frac{3}{2} \\ \implies \qquad S_{k+1} &\geqslant 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - \frac{3}{2} \ , \end{aligned}$$

and so the result holds with $C = \frac{3}{2}$ for all $n \geqslant 1$. We conclude that $C = \frac{3}{2}$ is the smallest number for which the result holds for all $n \geqslant 1$.

Question 7

(i) For $0 < x < 1$ we have $\log x < 0$, so multiplying the inequality $x < 1$ by $-\log x$ we have

$$\begin{aligned} -x\log x &< -\log x \\ \Longrightarrow \quad x\log x &> \log x \\ \log f(x) &> \log x \\ \Longrightarrow \quad f(x) &> x \quad . \end{aligned}$$

Further, for $0 < x < 1$ we have

$$\begin{aligned} 0 &< -x\log x \\ \Longrightarrow \quad 0 &> x\log x \\ \log 1 &> \log f(x) \\ \Longrightarrow \quad 1 &> f(x) \quad . \end{aligned}$$

Multiplying through the inequality $x < f(x) < 1$ by $-\log x$, we have

$$\begin{aligned} -x\log x &< -f(x)\log x < -\log x \\ \Longrightarrow \quad x\log x &> f(x)\log x > \log x \\ \log f(x) &> \log g(x) > \log x \\ \Longrightarrow \quad f(x) &> g(x) > x \quad . \end{aligned}$$

For $x > 1$, we have $f(x) > x$ and $g(x) > f(x)$.

Additionally, we note that $f(1) = g(1) = 1$.

(ii) Taking logs and differentiating

$$\begin{aligned} \log f(x) &= x\log x \\ \Longrightarrow \quad \frac{f'(x)}{f(x)} &= \log x + 1 \\ f'(x) &= x^x(\log x + 1) \quad . \end{aligned}$$

Since $x > 0$, we have $f'(x) = 0 \iff \log x + 1 = 0$. Thus the value of x for which $f'(x) = 0$ is

$$\begin{aligned} \log x &= -1 \\ x &= e^{-1} \quad . \end{aligned}$$

(iii) Given that $\lim_{x\to 0} x\log x = 0$, we have

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} x^x \\ &= \lim_{x\to 0} \exp(x\log x) \\ &= \exp\left(\lim_{x\to 0} x\log x\right) \\ &= \exp(0) \\ &= 1 \quad .\end{aligned}$$

We have $\log g(x) = f(x)\log x$, and we have $f(x) \to 1$ and $\log x \to -\infty$ as $x \to 0$. Thus as $x \to 0$ we have

$$\begin{aligned} & \log g(x) = f(x)\log x \to -\infty \\ \implies \quad & \log g(x) \to -\infty \\ \implies \quad & g(x) \to 0 \quad .\end{aligned}$$

(iv) For $x > 0$, let $h(x) = \frac{1}{x} + \log x$. By differentiating

$$\begin{aligned}h'(x) &= -\frac{1}{x^2} + \frac{1}{x} \\ &= \frac{x-1}{x^2} \quad ,\end{aligned}$$

and

$$\begin{aligned}h''(x) &= \frac{2}{x^3} - \frac{1}{x^2} \\ &= \frac{-(x+2)}{x^3} \quad .\end{aligned}$$

We see that $h''(x) > 0$ for all $x > 0$, and so the stationary point at $x = 1$, $h(1) = 1$ is a minimum. $h(x)$ is continuous and well-defined for all $x > 0$, and so we have $h(x) \geqslant 1$ for all $x > 0$.

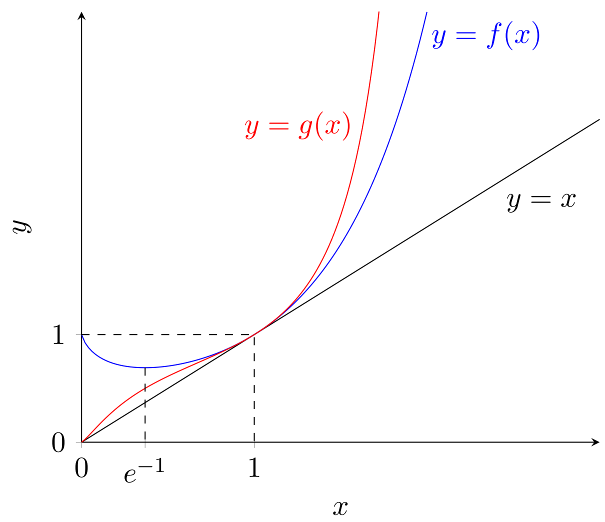
Taking logs and differentiating we have

$$\begin{aligned}\log g(x) &= f(x)\log x \\ \frac{g'(x)}{g(x)} &= \frac{1}{x} f(x) + f'(x)\log x \\ &= \frac{1}{x}\cdot x^x + x^x(1+\log x)\log x \\ &= f(x)\left(\frac{1}{x} + \log x + (\log x)^2\right) \\ g'(x) &= f(x)g(x)\left(h(x) + (\log x)^2\right) \quad .\end{aligned}$$

Since $(\log x)^2 \geqslant 0$, $h(x) \geqslant 1$, and $f(x) \geqslant x$ and $g(x) \geqslant x$, we deduce that for $x > 0$

$$\begin{aligned} & g'(x) \geqslant x \cdot x(1+0) \\ & g'(x) \geqslant x^2 \\ \implies \quad & g'(x) > 0 \quad . \end{aligned}$$

Our sketch is as follows.



Question 8

The line through A that is perpendicular to BC is given by

$$\boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{u} \quad , \quad \lambda \in \mathbb{R} \quad . \tag{1}$$

The line through B perpendicular to AC is similarly given by

$$\boldsymbol{x} = \boldsymbol{b} + \mu \boldsymbol{v} \quad , \quad \mu \in \mathbb{R} \quad . \tag{2}$$

Let $\boldsymbol{x} = \boldsymbol{p}$ be the position vector of the intersection P; we want to solve (1) and (2) simultaneously for $\boldsymbol{p}$, eliminating λ and μ. Since $\boldsymbol{v}$ is perpendicular to CA, we have $\boldsymbol{v} \cdot (\boldsymbol{a} - \boldsymbol{c}) = 0$. Taking a dot product with $\boldsymbol{a} - \boldsymbol{c}$ in (2), we thus have

$$\boldsymbol{p} \cdot (\boldsymbol{a} - \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) \quad . \tag{3}$$

Taking the same dot product in (1), we have

$$\boldsymbol{p} \cdot (\boldsymbol{a} - \boldsymbol{c}) = \boldsymbol{a} \cdot (\boldsymbol{a} - \boldsymbol{c}) + \lambda \boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c}) \quad ,$$

thus by (3) we find

$$\begin{aligned} \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) &= \boldsymbol{a} \cdot (\boldsymbol{a} - \boldsymbol{c}) + \lambda \boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c}) \\ \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) - \boldsymbol{a} \cdot (\boldsymbol{a} - \boldsymbol{c}) &= \lambda \boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c}) \\ \implies \lambda &= \frac{(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{a} - \boldsymbol{c})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \quad . \end{aligned}$$

Note the condition that triangle ABC is non-right-angled ensures $(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{a} - \boldsymbol{c}) \neq 0$ thus $\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c}) \neq 0$. Substituting this expression for λ into (1), the position vector of the intersection is

$$\boldsymbol{p} = \boldsymbol{a} + \frac{(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{a} - \boldsymbol{c})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \boldsymbol{u} \quad .$$

Subtracting the vector $\boldsymbol{c}$ and taking a dot product with $(\boldsymbol{b} - \boldsymbol{a})$, we find that

$$\begin{aligned} (\boldsymbol{p} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) &= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) + \frac{(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{a} - \boldsymbol{c})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \boldsymbol{u} \cdot (\boldsymbol{b} - \boldsymbol{a}) \\ &= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \left(1 + \frac{\boldsymbol{u} \cdot (\boldsymbol{b} - \boldsymbol{a})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \right) \\ &= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \frac{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c} + \boldsymbol{b} - \boldsymbol{a})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \\ &= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \frac{\boldsymbol{u} \cdot (\boldsymbol{b} - \boldsymbol{c})}{\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{c})} \quad . \end{aligned}$$

Since $\boldsymbol{u}$ is perpendicular to BC, we have $\boldsymbol{u} \cdot (\boldsymbol{b} - \boldsymbol{c}) = 0$, and thus we have $(\boldsymbol{p} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}) = 0$. That is: the line PC is perpendicular to the line AB.

Section B: Mechanics

Question 9

(i) Let F_p be the frictional force between each cylinder and the plank. Taking moments about the axis of one of the cylinders:

$$\begin{aligned} F_p \cdot r &= F \cdot r \\ F_p &= F \quad . \end{aligned}$$

Resolving forces on one of the cylinders horizontally, we then have

$$\begin{aligned} F\cos\theta + F &= R\sin\theta \\ R\sin\theta &= F(1+\cos\theta) \quad . \end{aligned}$$

Since the system is in equilibrium, considering the friction between the cylinder and the plank, we have $F \leqslant \mu R$, where $\mu = \frac{1}{2}$ is the coefficient of friction:

$$\begin{aligned} F &\leqslant \frac{1}{2}R \\ 2F(1+\cos\theta) &\leqslant R(1+\cos\theta) \qquad\qquad (1+\cos\theta \geqslant 0) \\ 2R\sin\theta &\leqslant R(1+\cos\theta) \\ 2\sin\theta &\leqslant 1+\cos\theta \quad . \end{aligned}$$

(ii) Resolving forces on one of the cylinders vertically we have

$$F\sin\theta + R\cos\theta = N - W \quad ,$$

and resolving forces on the plank vertically we have

$$2F\sin\theta + 2R\cos\theta = kW \quad .$$

Combining these and eliminating W, we get

$$\begin{aligned} kF\sin\theta + kR\cos\theta &= kN - kW \\ \implies \qquad kF\sin\theta + kR\cos\theta &= kN - (2F\sin\theta + 2R\cos\theta) \\ kN &= (k+2)(F\sin\theta + R\cos\theta) \\ kN &= (k+2)\left(F\sin\theta + \frac{F(1+\cos\theta)}{\sin\theta}\cos\theta\right) \\ N &= \left(1+\frac{2}{k}\right)\frac{\sin^2\theta + \cos\theta + \cos^2\theta}{\sin\theta}F \\ N &= \left(1+\frac{2}{k}\right)\frac{1+\cos\theta}{\sin\theta}F \quad . \end{aligned}$$

The condition that the cylinder does not slip on the floor is $F \leqslant \frac{1}{2}N$:

$$\begin{aligned} & 2F \leqslant N \\ \iff \quad & 2F \leqslant \left(1 + \frac{2}{k}\right) \frac{1+\cos\theta}{\sin\theta} F \\ \iff \quad & 2\sin\theta \leqslant \left(1 + \frac{2}{k}\right)(1+\cos\theta) \qquad (\theta \text{ is acute, so } \sin\theta \geqslant 0) \\ \iff \quad & 2\sin\theta - (1+\cos\theta) \leqslant \frac{2}{k}(1+\cos\theta) \quad . \end{aligned}$$

We know already that $2\sin\theta \leqslant 1+\cos\theta$, hence the left-hand side here is non-positive, while the right-hand side is non-negative ($1+\cos\theta \geqslant 0$ for any θ). Hence the condition that the cylinder does not slip on the floor is satisfied without any further restrictions on θ.

(iii) Since θ is acute, we know $\sin\theta \geqslant 0$ and so we can square both sides of the inequality $2\sin\theta \leqslant 1+\cos\theta$:

$$\begin{aligned} & (2\sin\theta)^2 \leqslant (1+\cos\theta)^2 \\ & 4\sin^2\theta \leqslant 1 + 2\cos\theta + \cos^2\theta \\ \iff \quad & 5\cos^2\theta + 2\cos\theta - 3 \geqslant 0 \\ & (5\cos\theta - 3)(\cos\theta + 1) \geqslant 0 \\ \iff \quad & 5\cos\theta - 3 \geqslant 0 \quad . \end{aligned}$$

Hence $\cos\theta \geqslant \frac{3}{5}$. Since θ is acute we can thus deduce

$$\sin\theta \leqslant \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad .$$

The horizontal distance of the centre of the plank from the centre of each cylinder is $r\sin\theta + a$. Thus the horizontal distance between the centres of the two cylinders is $2(r\sin\theta + a)$. This must be at least $2r$ (in which case the cylinders are touching), hence we have

$$\begin{aligned} & 2r \leqslant 2(r\sin\theta + a) \\ & r \leqslant r\sin\theta + a \\ \implies \quad & r \leqslant \frac{4}{5}r + a \\ & \frac{1}{5}r \leqslant a \\ & r \leqslant 5a \quad . \end{aligned}$$

Question 10

Let F denote the driving force from the engine. By Newton's second law

$$\begin{aligned} ma &= F - (Av^2 + R) \\ \implies \quad F &= ma + R + Av^2 \quad . \end{aligned}$$

The work done by the engine for the first half of the journey, W_1, is then

$$W_1 = \int_{x=0}^{x=d} F dx = \int_0^d (ma + R + Av^2) dx \quad .$$

We have

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \quad ,$$

hence

$$\frac{dv}{dx} = \frac{a}{v} \quad .$$

The car starts from rest, thus when $x = 0$ we have $v = 0$. Let $v = w$ be the car's velocity at $x = d$, then we have

$$\begin{aligned} w^2 &= 0^2 + 2ad \\ w &= \sqrt{2ad} \quad . \end{aligned}$$

Changing variable in the integral expression for W_1 then:

$$\begin{aligned} W_1 &= \int_0^d (ma + R + Av^2) dx \\ &= \int_{v=0}^{v=w} (ma + R + Av^2) \frac{dx}{dv} dv \\ &= \int_0^w (ma + R + Av^2) \frac{v}{a} dv \quad . \end{aligned}$$

(i) We can then evaluate the integral for W_1 explicitly:

$$\begin{aligned} W_1 &= \frac{1}{a} \int_0^w ((ma + R)v + Av^3) dv \\ &= \frac{1}{a} \left[\frac{1}{2}(ma + R)v^2 + \frac{1}{4}Av^4 \right]_0^w \\ &= \frac{1}{a} \left(\frac{1}{2}(ma + R)w^2 + \frac{1}{4}Aw^4 \right) \\ &= \frac{1}{a} \left((ma + R)ad + Aa^2d^2 \right) \\ &= (ma + R)d + Aad^2 \quad . \end{aligned}$$

Let W_2 denote the work done by the engine for the second half of the journey. The force from the engine is now

$$F = -ma + R + Av^2 \quad ,$$

and the condition that $R > ma$ ensures that this is positive throughout the second half of the journey. By changing the sign of a and changing the integral limits (since we are now decelerating), we get an integral expression for W_2:

$$W_2 = \int_{v=w}^{v=0} (-ma + R + Av^2)\frac{v}{-a}dv = \int_0^w (-ma + R + Av^2)\frac{v}{a}dv \quad .$$

By comparison with the integral for W_1 we get

$$W_2 = (-ma + R)d + Aad^2 \quad ,$$

and thus the total work done by the engine for the whole journey is

$$W_1 + W_2 = 2(Rd + Aad^2) = 2Rd + 2Aad^2 \quad .$$

(ii) Now instead $R < ma$, hence there is a speed V at which $F = 0$. Specifically $F = 0$ at $AV^2 = ma - R$, thus

$$V = \sqrt{\frac{ma - R}{A}} \quad ,$$

This occurs during the second half of the journey if this speed V is less than w:

$$\sqrt{\frac{ma - R}{A}} < \sqrt{2ad} \qquad \iff \qquad ma - R < 2Aad$$
$$\iff \qquad ma - 2Aad < R \quad ,$$

which is the other condition on R that we are given. Thus the integral expression for W_2 only holds for $V < v < w$ and as v drops below V no more work is done. In this case the work done for the second half of the journey is

$$\begin{aligned}
W_2 &= \int_{v=V}^{v=w} (-ma + R + Av^2)\frac{v}{a}dv = \frac{1}{a}\int_V^w ((-ma + R)v + Av^3)dv \\
&= \frac{1}{a}\left[\frac{1}{2}(-ma + R)v^2 + \frac{1}{4}Av^4\right]_V^w \\
&= \frac{1}{a}\left(\frac{1}{2}(-ma + R)(w^2 - V^2) + \frac{1}{4}A(w^4 - V^4)\right) \\
&= \frac{1}{a}\left(\frac{1}{2}(-ma + R)\left(2ad - \frac{ma - R}{A}\right) + \frac{1}{4}A\left(4a^2d^2 - \frac{(ma - R)^2}{A^2}\right)\right) \\
&= (-ma + R)d + \frac{(ma - R)^2}{2Aa} + Aad^2 - \frac{(ma - R)^2}{4Aa} \\
&= (-ma + R)d + Aad^2 + \frac{(ma - R)^2}{4Aa} \quad .
\end{aligned}$$

The work done for the first half of the journey is unchanged, and thus the total work done for the whole journey is now

$$
\begin{aligned}
W_1 + W_2 &= (ma + R)d + Aad^2 + (-ma + R)d + Aad^2 + \frac{(ma - R)^2}{4Aa} \\
&= 2Rd + 2Aad^2 + \frac{(ma - R)^2}{4Aa} \quad .
\end{aligned}
$$

Question 11

(i) Let P be the point at the top of the first wall and Q be the point at the top of the second wall. Let m be the mass of the particle. The initial energy is

$$E = \frac{1}{2}m(\sqrt{5ag})^2 = \frac{5}{2}mag \quad .$$

Given that the particle passes through P (it just clears the first wall), let v be the speed of the particle at P. At the top of the wall, the gravitational potential energy is $mg \cdot 2a$, thus by conservation of energy, the kinetic energy of the particle at P is

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{5}{2}mag - 2mag \\ &= \frac{1}{2}mag \quad , \end{aligned}$$

and this gives

$$\begin{aligned} v^2 &= ag \\ v &= \sqrt{ag} \quad . \end{aligned}$$

Let θ be the angle of the particle's trajectory above the horizontal as it passes through P – note this angle is necessarily acute. Fixing coordinates such that P is at the origin, the particle's motion from P to Q is described parametrically by

$$x(t) = \sqrt{ag}\cos\theta\, t \quad , \quad y(t) = \sqrt{ag}\sin\theta\, t - \frac{1}{2}gt^2 \quad .$$

The particle passes through point Q (it just clears the second wall) when $x = a$, $y = 0$. By considering the vertical equation, this occurs at time T where

$$\begin{aligned} \sqrt{ag}\sin\theta\, T - \frac{1}{2}gT^2 &= 0 \\ \sqrt{ag}\sin\theta - \frac{1}{2}gT &= 0 \qquad \text{(when } T = 0 \text{ the particle is at point } P\text{)} \\ \frac{1}{2}\sqrt{g}T &= \sqrt{a}\sin\theta \\ T &= 2\sqrt{\frac{a}{g}}\sin\theta \quad . \end{aligned}$$

Substituting this into the horizontal equation, we get

$$\begin{aligned} a &= \sqrt{ag}\cos\theta\, T = \sqrt{ag}\cos\theta \cdot 2\sqrt{\frac{a}{g}}\sin\theta \\ a &= 2a\cos\theta\sin\theta \\ \implies \sin(2\theta) &= 1 \quad . \end{aligned}$$

Since θ is acute, we deduce that $\theta = \frac{\pi}{4}$.

Now fixing coordinates such that A is at the origin, let ϕ be the (acute) angle above the horizontal at which the particle is projected, and let d be the horizontal distance from A to the first wall. The motion from A to P is given parametrically by

$$x(t) = \sqrt{5ag}\cos\phi\, t \qquad , \qquad y(t) = \sqrt{5ag}\sin\phi z, t - \frac{1}{2}gt^2 \quad .$$

The horizontal velocity is constant: $\dot{x}(t) = \sqrt{5ag}\cos\phi$, and we know from above that the particle's horizontal velocity at P is $\sqrt{ag}\cos(\frac{\pi}{4}) = \frac{1}{2}\sqrt{2ag}$, hence

$$\begin{aligned} \sqrt{5ag}\cos\phi &= \frac{1}{2}\sqrt{2ag} \\ \cos\phi &= \frac{\sqrt{2}}{2\sqrt{5}} = \frac{\sqrt{10}}{10} \quad . \end{aligned}$$

From this we deduce that

$$\sin\phi = \sqrt{1 - \frac{10}{100}} = \frac{\sqrt{90}}{10} = \frac{3\sqrt{10}}{10} \quad ,$$

and so our vertical equation of motion is

$$\begin{aligned} y(t) &= \sqrt{5ag}\frac{3\sqrt{10}}{10}t - \frac{1}{2}gt^2 \\ &= \frac{3\sqrt{2}}{2}\sqrt{ag}t - \frac{1}{2}gt^2 \quad . \end{aligned}$$

Substituting in $y = 2a$, when the particle reaches the top of the walls, we get

$$\begin{aligned} 2a &= \frac{3\sqrt{2}}{2}\sqrt{ag}t - \frac{1}{2}gt^2 \\ gt^2 - 3\sqrt{2ag}t + 4a &= 0 \\ t^2 - 3\sqrt{\frac{2a}{g}}t + \frac{2a}{g} &= 0 \\ \left(t - \sqrt{\frac{2a}{g}}\right)\left(t - 2\sqrt{\frac{2a}{g}}\right) &= 0 \quad . \end{aligned}$$

The earlier time here $t = \sqrt{\frac{2a}{g}}$ corresponds to the particle passing through P, while the second $t = 2\sqrt{\frac{2a}{g}}$ corresponds to the particle passing through Q. Thus we get that the horizontal distance travelled by the time the particle passes through P is

$$d = \sqrt{5ag}\cos\phi \cdot \sqrt{\frac{2a}{g}} = \sqrt{5ag} \cdot \frac{\sqrt{10}}{10}\sqrt{\frac{2a}{g}} = a \quad .$$

(ii) Let m denote the mass of this second particle and let u denote its speed when it passes over the first wall, by conservation of energy

$$\begin{aligned}\frac{1}{2}m(\sqrt{5ag})^2 &= \frac{1}{2}mu^2 + mg(2a+h)\\ ag &= u^2 + 2g(2a+h)\\ u^2 &= g(a-2h) \quad .\end{aligned}$$

Let β be the angle of this second particle's trajectory relative to the horizontal as it passes vertically above P. Fixing coordinates such that P is at the origin, the motion of this second particle (starting from when it is vertically above P) is given by

$$x(t) = u\cos\beta\, t \qquad , \qquad y(t) = h + u\sin\beta\, t - \frac{1}{2}gt^2 \quad .$$

The time T at which the particle has moved a distance a horizontally is given by

$$a = u\cos\beta\, T \qquad \Longrightarrow \qquad T = \frac{a}{u\cos\beta} \quad ,$$

and the height of the particle above the tops of the walls at this time is

$$\begin{aligned}y(T) &= h + a\tan\beta - \frac{1}{2}g\frac{a^2}{u^2\cos^2\beta}\\ &= h + a\tan\beta - \frac{1}{2}g\frac{a^2}{g(a-2h)}\sec^2\beta\\ &= h + a\tan\beta - \frac{1}{2}\frac{a^2}{a-2h}(\tan^2\beta+1) \quad .\end{aligned}$$

If we suppose that it is possible for the particle to clear the second wall, then there must be some angle β such that it can pass through Q and $y(T) = 0$. This would give

$$\begin{aligned}h + a\tan\beta - \frac{1}{2}\frac{a^2}{a-2h}(\tan^2\beta+1) &= 0\\ \frac{1}{2}a^2(\tan^2\beta+1) - a(a-2h)\tan\beta - (a-2h)h &= 0\\ a^2\tan^2\beta - (2a^2-4ah)\tan\beta + a^2 - 2ah + 4h^2 &= 0 \quad .\end{aligned}$$

This is a quadratic equation in $\tan\beta$ with discriminant

$$\begin{aligned}&(2a^2-4ah)^2 - 4a^2(a^2-2ah+4h^2)\\ &\qquad = 4a^4 - 16a^3h + 16a^2h^2 - 4a^4 + 8a^3h - 16a^2h^2\\ &\qquad = -8a^3h \quad ,\end{aligned}$$

which is negative. Thus there is no solution for β such that the particle passes through Q. Hence the particle cannot clear the second wall.

Section C: Probability and Statistics

Question 12

(i) Using the independence of X and Y, we have

$$\begin{aligned}\mathbb{P}(X+Y=r) &= \sum_{s=0}^{r} \mathbb{P}(X=s)\cdot\mathbb{P}(Y=r-s) \\ &= \sum_{s=0}^{r} \frac{e^{-\lambda}\lambda^s}{s!}\frac{e^{-\mu}\mu^{r-s}}{(r-s)!} \\ &= \frac{e^{-(\lambda+\mu)}}{r!}\sum_{s=0}^{r}\frac{r!}{s!(r-s)!}\lambda^s\mu^{r-s} \\ &= \frac{e^{-(\lambda+\mu)}}{r!}(\lambda+\mu)^r \quad ,\end{aligned}$$

where we recognise the last sum as the binomial expansion for $(\lambda+\mu)^r$. We recognise this result as the probability mass function for a Poisson random variable, that is: $X+Y$ has a Poisson distribution with parameter $\lambda+\mu$.

(ii) Now given that $X+Y=k$, we have

$$\begin{aligned}\mathbb{P}(X=r \mid X+Y=k) &= \frac{\mathbb{P}(X=r \text{ and } X+Y=k)}{\mathbb{P}(X+Y=k)} \\ &= \frac{\mathbb{P}(X=r \text{ and } Y=k-r)}{\mathbb{P}(X+Y=k)} \\ &= \frac{\mathbb{P}(X=r)\cdot\mathbb{P}(Y=k-r)}{\mathbb{P}(X+Y=k)} \\ &= \frac{e^{-\lambda}\lambda^r}{r!}\cdot\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}\Bigg/\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^k}{k!} \\ &= \frac{k!}{r!(k-r)!}\frac{\lambda^r\mu^{k-r}}{(\lambda+\mu)^k} \\ &= \frac{k!}{r!(k-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{k-r} \\ &= \binom{k}{r}p^r(1-p)^{k-r} \quad ,\end{aligned}$$

where $p=\frac{\lambda}{\lambda+\mu}$. This is the probability mass function for a Binomial(k,p) random variable.

(iii) This corresponds to the probability that when $X+Y=1$ we have $X=1$. Setting $r=1$, $k=1$ in the result from (ii) then, the probability that Adam catches the first fish is

$$\mathbb{P}(X=1 \mid X+Y=1) = \frac{\lambda}{\lambda+\mu} \quad .$$

(iv) Since their combined total follows a Poisson distribution with parameter $\lambda + \mu$, the expected time until either Adam or Eve catches a fish is $\frac{1}{\lambda+\mu}$. If Adam catches the first fish (which occurs with probability $p = \frac{\lambda}{\lambda+\mu}$), then the remaining waiting time is the expected time until Eve catches a fish, which is $\frac{1}{\mu}$. Likewise, if Eve catches the first fish (which occurs with probability $1 - p = \frac{\mu}{\lambda+\mu}$), then the remaining waiting time is the expected time until Adam catches a fish, which is $\frac{1}{\lambda}$. The expected time t until both Adam and Eve have caught one fish is thus

$$\begin{aligned} t &= \frac{1}{\lambda+\mu} + \left(\frac{\lambda}{\lambda+\mu} \cdot \frac{1}{\mu} + \frac{\mu}{\lambda+\mu} \cdot \frac{1}{\lambda} \right) \\ &= \frac{1}{\lambda+\mu} + \frac{\lambda+\mu-\mu}{(\lambda+\mu)\mu} + \frac{\lambda+\mu-\lambda}{(\lambda+\mu)\lambda} \\ &= \frac{1}{\lambda+\mu} + \frac{1}{\mu} - \frac{1}{\lambda+\mu} + \frac{1}{\lambda} - \frac{1}{\lambda+\mu} \\ &= \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu} \quad . \end{aligned}$$

Question 13

(i) The probability that the m-th attempt to open the door fails is $\frac{n-1}{n}$, while the probability that it succeeds is $\frac{1}{n}$. Thus, the probability that exactly k attempts are required to open the door is the probability that the first $k-1$ attempts fail and the k-th attempt is successful. Let p_k denote this probability, then we have

$$p_k = \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n} \quad , \qquad k = 1, 2, 3, \ldots \quad .$$

Hence we find the expected number of attempts required is

$$\begin{aligned} \mathbb{E} = \sum_{k=1}^{\infty} k p_k &= \sum_{k=1}^{\infty} k \frac{(n-1)^{k-1}}{n^k} \\ &= \frac{1}{n} \sum_{k=1}^{\infty} k \left(1 - \frac{1}{n}\right)^{k-1} \\ &= p \sum_{k=1}^{\infty} k q^{k-1} \quad , \end{aligned}$$

where $p = \frac{1}{n}$ and $q = 1 - p$. Expanding this, we get $p(1 + 2q + 3q^2 + 4q^3 + \ldots)$, and we recognise the expression in the brackets as the binomial expansion of $(1-q)^{-2}$. Thus

$$\mathbb{E} = p(1-q)^{-2} = p \cdot p^{-2} = \frac{1}{p} = n \quad .$$

(ii) Now at the m-th attempt there are $n - (m-1)$ keys remaining, thus the probability that the m-th attempt to open the door fails is $\frac{n-m}{n-(m-1)}$ and the probability it succeeds is $\frac{1}{n-(m-1)}$. The probability that exactly k attempts are required to open the door is thus

$$p_k = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \ldots \cdot \frac{n-(k-1)}{n-k} \cdot \frac{1}{n-(k-1)} = \frac{1}{n} \quad , \qquad k = 1, 2, \ldots, n \quad .$$

We note that at most n attempts will be required, since each key is tried at most once so the correct key will be found on the n-th attempt at the latest. This gives the expected number of attempts required to open the door

$$\begin{aligned} \mathbb{E} = \sum_{k=1}^{n} k p_k = \sum_{k=1}^{n} \frac{k}{n} &= \frac{1}{n} \sum_{k=1}^{n} k \\ &= \frac{1}{n} \cdot \frac{1}{2} n(n+1) \\ &= \frac{1}{2}(n+1) \quad . \end{aligned}$$

(iii) Now at the m-th attempt there are $n+m-1$ keys in the bag, thus the probability that the m-th attempt to open the door fails is $\frac{n+m-2}{n+m-1}$ and the probability it succeeds is $\frac{1}{n+m-1}$. The probability that exactly k attempts are required to open the door is thus

$$\begin{aligned} p_k &= \frac{n-1}{n} \cdot \frac{n}{n+1} \cdot \frac{n+1}{n+2} \cdot \dots \cdot \frac{n+k-3}{n+k-2} \cdot \frac{1}{n+k-1} \\ &= \frac{n-1}{(n+k-2)(n+k-1)} \\ &= (n-1)\left(\frac{1}{n+k-2} - \frac{1}{n+k-1}\right) \quad , \qquad k = 1, 2, 3, \dots \quad . \end{aligned}$$

The expected number of attempts required to open the door is thus

$$\begin{aligned} \mathbb{E} = \sum_{k=1}^{\infty} k p_k &= (n-1) \sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} - \frac{k}{n+k-1}\right) \\ &= (n-1)\left(\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right) \right. \\ &\qquad\qquad \left. + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) + \dots\right) \\ &= (n-1)\left(\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} + \dots\right) \\ &= (n-1) \sum_{m=n-1}^{\infty} \frac{1}{m} \\ &= (n-1)\left(\sum_{m=1}^{\infty} \frac{1}{m} - \sum_{m=1}^{n-2} \frac{1}{m}\right) \quad . \end{aligned}$$

In this final expression, the second sum is finite, while the first sum is infinite (by the result provided in the question). Thus the expected number of attempts required to open the door is infinite.

STEP III

Section A: Pure Mathematics

Question 1

(i) For positive integers n, r, we have

$$\begin{aligned}
\frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r}-\frac{1}{{}^{n+r}C_r}\right) &= \frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!}-\frac{n!r!}{(n+r)!}\right)\\
&= \frac{r+1}{r}\cdot\frac{(n+r)r!(n-1)!-n!r!}{(n+r)!}\\
&= \frac{(r+1)!}{(n+r)!}\cdot\frac{(n+r)(n-1)!-n!}{r}\\
&= \frac{(r+1)!}{(n+r)!}\cdot\frac{n!+r(n-1)!-n!}{r}\\
&= \frac{(r+1)!(n-1)!}{(n+r)!}\\
&= \frac{1}{{}^{n+r}C_{r+1}} \quad .
\end{aligned}$$

Using this result we find

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{{}^{n+r}C_{r+1}} &= \sum_{n=1}^{\infty}\frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r}-\frac{1}{{}^{n+r}C_r}\right)\\
&= \frac{r+1}{r}\left(\left(\frac{1}{{}^{r}C_r}-\frac{1}{{}^{r+1}C_r}\right)+\left(\frac{1}{{}^{r+1}C_r}-\frac{1}{{}^{r+2}C_r}\right)\right.\\
&\qquad\qquad\left.+\left(\frac{1}{{}^{r+2}C_r}-\frac{1}{{}^{r+3}C_r}\right)+\dots\right)\\
&= \frac{r+1}{r}\frac{1}{{}^{r}C_r}\\
&= \frac{r+1}{r} \quad ,
\end{aligned}$$

where we can 'telescope' the sum because $\frac{1}{{}^{r+n}C_r} \to 0$ as $n \to \infty$.

Setting $r = 2$, we get

$$\sum_{n=1}^{\infty}\frac{1}{{}^{n+2}C_3} = \frac{3}{2} \qquad \Longrightarrow \qquad \begin{aligned}\sum_{n=2}^{\infty}\frac{1}{{}^{n+2}C_3} &= \frac{3}{2}-\frac{1}{{}^{3}C_3}\\ &= \frac{1}{2} \quad .\end{aligned}$$

(ii) For $n \geqslant 3$, we have

$$\begin{aligned}
\frac{1}{{}^{n+1}C_3} &= \frac{3!(n-2)!}{(n+1)!} \\
&= \frac{3!}{(n+1)n(n-1)} \\
&= \frac{3!}{n^3} \cdot \frac{n^2}{n^2-1} \\
\Longrightarrow \qquad \frac{1}{{}^{n+1}C_3} &> \frac{3!}{n^3} \quad ,
\end{aligned}$$

since $n^2 > n^2 - 1 > 0$ and so $\frac{n^2}{n^2-1} > 1$. We also have

$$\begin{aligned}
\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} &= \frac{20 \cdot 3!(n-2)!}{(n+1)!} - \frac{5!(n-3)!}{(n+2)!} \\
&= \frac{5!}{(n+1)n(n-1)} - \frac{5!}{(n+2)(n+1)n(n-1)(n-2)} \\
&= \frac{5!}{(n+1)n(n-1)}\left(1 - \frac{1}{n^2-4}\right) \\
&= \frac{5!}{n^3}\frac{n^2}{n^2-1}\frac{n^2-5}{n^2-4} \\
&= \frac{5!}{n^3}\frac{n^4-5n^2}{n^4-5n^2+4} \\
\Longrightarrow \qquad \frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} &< \frac{5!}{n^3} \quad ,
\end{aligned}$$

since $0 < n^4 - 5n^2 < n^4 - 5n^2 + 4$ and so $0 < \frac{n^4-5n^2}{n^4-5n^2+4} < 1$.

Summing the first inequality over $n \geqslant 3$, we get

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{3!}{n^3} &< \sum_{n=3}^{\infty} \frac{1}{{}^{n+1}C_3} \\
\Longrightarrow \qquad \sum_{n=3}^{\infty} \frac{1}{n^3} &< \frac{1}{6}\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_3} = \frac{1}{12} \quad .
\end{aligned}$$

Adding $\frac{1}{1^3} + \frac{1}{2^3}$ to both sides, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^3} &< \frac{1}{12} + 1 + \frac{1}{8} = 1 + \frac{20}{96} \\
\sum_{n=1}^{\infty} \frac{1}{n^3} &< \frac{116}{96} \quad .
\end{aligned}$$

Similarly summing the second inequality over $n \geqslant 3$, we get

$$\begin{aligned} & \sum_{n=3}^{\infty} \frac{5!}{n^3} > \sum_{n=3}^{\infty} \frac{20}{{}^{n+1}C_3} - \sum_{n=3}^{\infty} \frac{1}{{}^{n+2}C_5} \\ \Longrightarrow \quad & \sum_{n=3}^{\infty} \frac{5!}{n^3} > \sum_{n=2}^{\infty} \frac{20}{{}^{n+2}C_3} - \sum_{n=1}^{\infty} \frac{1}{{}^{n+4}C_5} \quad . \end{aligned}$$

From the result in (i) we have

$$\sum_{n=2}^{\infty} \frac{20}{{}^{n+2}C_3} = \frac{20}{2} \quad ,$$

and setting $r = 4$ we have

$$\sum_{n=1}^{\infty} \frac{1}{{}^{n+4}C_5} = \frac{5}{4} \quad ,$$

thus

$$\begin{aligned} & \sum_{n=3}^{\infty} \frac{5!}{n^3} > 10 - \frac{5}{4} = \frac{35}{4} \\ \Longrightarrow & \sum_{n=3}^{\infty} \frac{1}{n^3} > \frac{35}{4 \cdot 5!} = \frac{7}{96} \quad . \end{aligned}$$

Adding $\frac{1}{1^3} + \frac{1}{2^3}$ to both sides, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^3} > \frac{7}{96} + 1 + \frac{1}{8} = 1 + \frac{19}{96} \\ & \sum_{n=1}^{\infty} \frac{1}{n^3} > \frac{115}{96} \quad , \end{aligned}$$

that is:

$$\frac{115}{96} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{116}{96} \quad .$$

Question 2

(i) Let z' be the complex number representing the image of P under the transformation R. We have

$$z' - a = e^{i\theta}(z-a) \qquad \Longrightarrow \qquad z' = e^{i\theta}z + a(1-e^{i\theta}) \ \ .$$

(ii) Let z'' be the complex number representing the image of P under the transformation R followed by S. Similar to the result of (i), we have

$$z'' = e^{i\phi}z' + b(1-e^{i\phi}) \ \ ,$$

thus by substituting in the expression for z'

$$\begin{aligned} z'' &= e^{i\phi}\left(e^{i\theta}z + a(1-e^{i\theta})\right) + b(1-e^{i\phi}) \\ &= e^{i(\theta+\phi)}z + ae^{i\phi}(1-e^{i\theta}) + b(1-e^{i\phi}) \ \ . \end{aligned}$$

By comparing to the result in (i), if $\theta+\phi \neq 2\pi n$ for any $n \in \mathbb{Z}$, this is a rotation by an angle $\theta+\phi$ about the point represented by complex number c where

$$\begin{aligned} c(1-e^{i(\theta+\phi)}) &= ae^{i\phi}(1-e^{i\theta}) + b(1-e^{i\phi}) \\ c(e^{-i(\theta+\phi)/2} - e^{i(\theta+\phi)/2}) &= ae^{i\phi/2}(e^{-i\theta/2} - e^{i\theta/2}) + be^{-i\theta/2}(e^{-i\phi/2} - e^{i\phi/2}) \\ -2ic\sin\tfrac{1}{2}(\theta+\phi) &= -2iae^{i\phi/2}\sin\tfrac{1}{2}\theta - 2ibe^{-i\theta/2}\sin\tfrac{1}{2}\phi \\ c\sin\tfrac{1}{2}(\theta+\phi) &= ae^{i\phi/2}\sin\tfrac{1}{2}\theta + be^{-i\theta/2}\sin\tfrac{1}{2}\phi \ \ . \end{aligned}$$

If $\theta+\phi \neq 2\pi n$ for any $n \in \mathbb{Z}$ then we can solve this to find c, otherwise $\sin\frac{1}{2}(\theta+\phi) = 0$. In the case $\theta+\phi = 2\pi$ we have

$$\begin{aligned} z'' &= e^{2\pi i}z + ae^{i\phi}(1-e^{-i\phi}) + b(1-e^{i\phi}) \\ &= z + ae^{i\phi} - a + b - be^{i\phi} \\ &= z + (b-a)(1-e^{i\phi}) \ \ , \end{aligned}$$

so this is a translation by $(b-a)(1-e^{i\phi})$.

(iii) Let w be the complex number representing the image of P under transformation S followed by R, then by comparing with the expression for z'' we see that

$$w = e^{i(\theta+\phi)}z + be^{i\theta}(1-e^{i\phi}) + a(1-e^{i\theta}) \ \ .$$

We have that $RS = SR$ if and only if $w = z''$ for any z, which holds if and only if

$$\begin{aligned} be^{i\theta}(1-e^{i\phi}) + a(1-e^{i\theta}) &= ae^{i\phi}(1-e^{i\theta}) + b(1-e^{i\phi}) \\ a(1-e^{i\theta} - e^{i\phi}(1-e^{i\theta})) &= b(1-e^{i\phi} - e^{i\theta}(1-e^{i\phi})) \\ a(1-e^{i\theta})(1-e^{i\phi}) &= b(1-e^{i\phi})(1-e^{i\theta}) \\ (a-b)(1-e^{i\theta})(1-e^{i\phi}) &= 0 \ \ . \end{aligned}$$

Thus $RS = SR$ if and only if $a = b$, or $\theta = 2\pi n$ for some $n \in \mathbb{Z}$, or $\phi = 2\pi n$ for some $n \in \mathbb{Z}$.

Question 3

Applying Vieta's formulae to the quartic equation for x, we have

$$\begin{cases} \alpha+\beta+\gamma+\delta = -p \\ \alpha\beta+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma = q \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r \\ \alpha\beta\gamma\delta = s \end{cases} ,$$

while applying the formulae to the cubic for y the coefficient of y^2 yields

$$\alpha\beta+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma = -A \quad ,$$

thus we deduce $A = -q$. Now we are given the quartic equation

$$x^4 + 3x^2 - 6x + 10 = 0 \quad .$$

(i) With the given values of p, q, r, s, we get the cubic equation:

$$y^3 - 3y^2 - 40y + 84 = 0 \quad ,$$

which factorises neatly:

$$\begin{aligned} y^3 - 3y^2 - 40y + 84 &= 0 \\ (y-2)(y^2 - y - 42) &= 0 \\ (y-2)(y-7)(y+6) &= 0 \quad , \end{aligned}$$

thus the largest root of the cubic equation is $\alpha\beta + \gamma\delta = 7$.

(ii) Using Vieta's equation for q with the above result

$$\begin{aligned} & \alpha\beta+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma = q \\ \implies \quad & 7+\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma = 3 \\ & \alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma = -4 \\ & (\alpha+\beta)(\gamma+\delta) = -4 \quad . \end{aligned}$$

Using the result from (i) we have $\gamma\delta = 7 - \alpha\beta$, hence Vieta's equation for s yields

$$\begin{aligned} & \alpha\beta\gamma\delta = s \\ \implies \quad & \alpha\beta(7-\alpha\beta) = 10 \\ & (\alpha\beta)^2 - 7\alpha\beta + 10 = 0 \\ & (\alpha\beta - 5)(\alpha\beta - 2) = 0 \quad . \end{aligned}$$

This gives the possibilities $\alpha\beta = 5$, $\gamma\delta = 2$ or $\alpha\beta = 2$, $\gamma\delta = 5$. Given that $\alpha\beta > \gamma\delta$, we deduce that $\alpha\beta = 5$, $\gamma\delta = 2$.

(iii) From Vieta's equation for p

$$\begin{aligned} & \alpha+\beta+\gamma+\delta = -p \\ \implies \quad & \alpha+\beta+\gamma+\delta = 0 \\ & \alpha+\beta = -(\gamma+\delta) \quad . \end{aligned}$$

Combined with the result that $(\alpha+\beta)(\gamma+\delta) = -4$, we deduce that

$$\alpha+\beta = 2 \quad , \quad \gamma+\delta = -2 \qquad \text{or} \qquad \alpha+\beta = -2 \quad , \quad \gamma+\delta = 2 \quad .$$

Now using Vieta's equation for r and the results that $\alpha\beta = 5$, $\gamma\delta = 2$:

$$\begin{aligned} & \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r \\ \implies \quad & 5\gamma+5\delta+2\alpha+2\beta = 6 \\ & 5(\gamma+\delta)+2(\alpha+\beta) = 6 \quad . \end{aligned}$$

Hence we deduce $\alpha+\beta = -2$, $\gamma+\delta = 2$. Accordingly, α and β are the roots of

$$\begin{aligned} (t-\alpha)(t-\beta) &= 0 \\ t^2-(\alpha+\beta)t+\alpha\beta &= 0 \\ t^2+2t+5 &= 0 \quad . \end{aligned}$$

Solving this:

$$\begin{aligned} t^2+2t+5 &= 0 \\ (t+1)^2+4 &= 0 \\ t &= -1 \pm 2i \quad , \end{aligned}$$

hence $\alpha, \beta = -1 \pm 2i$. Similarly γ and δ are the roots of

$$\begin{aligned} (t-\gamma)(t-\delta) &= 0 \\ t^2-(\gamma+\delta)t+\gamma\delta &= 0 \\ t^2-2t+2 &= 0 \quad . \end{aligned}$$

Solving this:

$$\begin{aligned} t^2-2t+5 &= 0 \\ (t-1)^2+1 &= 0 \\ t &= 1 \pm i \quad , \end{aligned}$$

hence $\gamma, \delta = 1 \pm i$, and we have solved the quartic:

$$\begin{aligned} & x^4+3x^2-6x+10 = 0 \\ \iff \quad & x = -1 \pm 2i \quad \text{or} \quad x = 1 \pm i \quad . \end{aligned}$$

Question 4

(i) Using the identity $a^z = e^{z \ln a}$ and setting $z = \log_a f(x)$ we have

$$\begin{aligned} a^{\log_a f(x)} &= e^{\ln a \log_a f(x)} \\ f(x) &= e^{\ln a \log_a f(x)} \\ \ln f(x) &= \ln a \log_a f(x) \quad . \end{aligned}$$

Thus we have

$$\begin{aligned} F(y) &= e^{\frac{1}{y}\int_0^y \ln f(x) dx} \\ &= e^{\frac{1}{y}\int_0^y \ln a \log_a f(x) dx} \\ &= e^{\ln a \left(\frac{1}{y}\int_0^y \log_a f(x) dx\right)} \\ &= a^{\frac{1}{y}\int_0^y \log_a f(x) dx} \quad . \end{aligned}$$

(ii) Since $f(x) > 0$ and $g(x) > 0$, we have $h(x) > 0$, thus we can write

$$\begin{aligned} H(y) &= e^{\frac{1}{y}\int_0^y \ln h(x) dx} \\ &= e^{\frac{1}{y}\int_0^y \ln(f(x)g(x)) dx} \\ &= e^{\frac{1}{y}\int_0^y (\ln f(x) + \ln g(x)) dx} \\ &= e^{\frac{1}{y}\int_0^y \ln f(x) dx + \frac{1}{y}\int_0^y \ln g(x) dx} \\ &= e^{\frac{1}{y}\int_0^y \ln f(x) dx} e^{\frac{1}{y}\int_0^y \ln g(x) dx} \\ &= F(y)G(y) \quad . \end{aligned}$$

(iii) Setting $f(x) = b^x$ and using the alternate form for $F(y)$ in (i) with $a = b$, we find

$$\begin{aligned} F(y) &= b^{\frac{1}{y}\int_0^y \log_b(b^x) dx} \\ &= b^{\frac{1}{y}\int_0^y x dx} \\ &= b^{\frac{1}{y}\cdot\frac{1}{2}y^2} \\ &= b^{\frac{1}{2}y} \\ &= \sqrt{b^y} \quad . \end{aligned}$$

(iv) We have

$$\begin{aligned} F(y) &= \sqrt{f(y)} \\ \implies \qquad \ln F(y) &= \frac{1}{2}\ln f(y) \\ \frac{1}{y}\int_0^y \ln f(x) dx &= \frac{1}{2}\ln f(y) \\ 2\int_0^y \ln f(x) dx &= y \ln f(y) \quad . \end{aligned}$$

Differentiating both sides with respect to y, we get

$$\begin{aligned} 2\ln f(y) &= \ln f(y) + \frac{yf'(y)}{f(y)} \\ \ln f(y) &= \frac{yf'(y)}{f(y)} \\ \frac{1}{y} &= \frac{1}{\ln f(y)}\frac{f'(y)}{f(y)} \\ \frac{1}{y} &= (\ln f(y))^{-1}\frac{d}{dy}(\ln f(y)) \quad . \end{aligned}$$

Now we can integrate both sides with respect to y:

$$\begin{aligned} \frac{1}{y} &= (\ln f(y))^{-1}\frac{d}{dy}(\ln f(y)) \\ \implies \qquad \ln y &= \ln(\ln f(y)) + c \\ \implies \qquad y &= \alpha \ln f(y) \quad , \end{aligned}$$

where $c \in \mathbb{R}$ is a constant of integration and $\alpha = e^c > 0$. Exponentiating both sides, we get

$$\begin{aligned} y &= \alpha \ln f(y) \\ \implies \qquad e^y &= f(y)^\alpha \\ f(y) &= e^{\frac{1}{\alpha}y} \\ f(y) &= b^y \qquad \text{where} \qquad b = e^{\frac{1}{\alpha}} = e^{e^{-c}} > 0 \quad . \end{aligned}$$

That is, we deduce $f(x) = b^x$ where b is some positive constant.

Question 5

Writing x and y as functions of θ:

$$\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \quad , \end{aligned}$$

we can differentiate to find

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\ &= \frac{f(\theta) + f'(\theta)\tan\theta}{f'(\theta) - f(\theta)\tan\theta} \quad . \end{aligned}$$

This equation gives the gradient of a tangent to the curve $r = f(\theta)$ at a given value of θ. Using the corresponding expression for the gradient of a tangent to $r = g(\theta)$, the two curves meet at right angles when

$$\begin{aligned} \frac{f + f'\tan\theta}{f' - f\tan\theta} \cdot \frac{g + g'\tan\theta}{g' - g\tan\theta} &= -1 \\ (f + f'\tan\theta)(g + g'\tan\theta) &= -(f' - f\tan\theta)(g' - g\tan\theta) \\ fg + (f'g + fg')\tan\theta + f'g'\tan^2\theta &= -f'g' + (fg' + f'g)\tan\theta - fg\tan^2\theta \\ (fg + f'g')(1 + \tan^2\theta) &= 0 \\ fg + f'g' &= 0 \quad . \end{aligned}$$

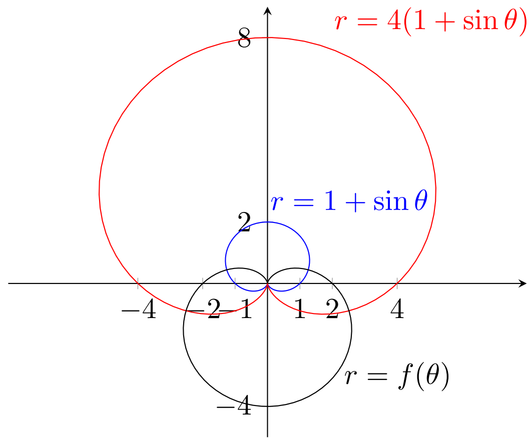
That is, the curves meet at right angles when

$$f(\theta)g(\theta) + f'(\theta)g'(\theta) = 0 \quad .$$

Setting $g(\theta) = a(1 + \sin\theta)$, the above result gives

$$\begin{aligned} a(1 + \sin\theta)f(\theta) + a\cos\theta f'(\theta) &= 0 \\ f'(\theta) &= -\frac{1 + \sin\theta}{\cos\theta} f(\theta) \\ \frac{f'(\theta)}{f(\theta)} &= -\frac{1 + \sin\theta}{\cos\theta} \cdot \frac{1 - \sin\theta}{1 - \sin\theta} \\ \frac{f'(\theta)}{f(\theta)} &= \frac{-\cos\theta}{1 - \sin\theta} \\ \log|f(\theta)| - \log|f(-\tfrac{1}{2}\pi)| &= \log|1 - \sin\theta| - \log|1 - \sin(-\tfrac{1}{2}\pi)| \\ \frac{f(\theta)}{f(-\frac{1}{2}\pi)} &= \frac{1 - \sin\theta}{1 - \sin(-\frac{1}{2}\pi)} \\ \frac{f(\theta)}{4} &= \frac{1 - \sin\theta}{2} \\ f(\theta) &= 2(1 - \sin\theta) \quad . \end{aligned}$$

Our sketch of the three curves is as follows:



Question 6

(i) Substituting $v = \frac{1}{u}$ in the integral, we have $\frac{du}{dv} = -\frac{1}{v^2}$ and thus

$$\begin{aligned} T(x) &= \int_{\infty}^{x^{-1}} \frac{1}{1+v^{-2}} \cdot \frac{-1}{v^2} dv = \int_{x^{-1}}^{\infty} \frac{1}{1+v^2} dv \\ &= \int_0^{\infty} \frac{1}{1+v^2} dv - \int_0^{x^{-1}} \frac{1}{1+v^2} dv \\ &= T_\infty - T(x^{-1}) \quad . \end{aligned}$$

(ii) Rearranging, assuming $1 - au \neq 0$ (that is: $u \neq a^{-1}$), we have

$$\begin{aligned} v = \frac{u+a}{1-au} \qquad &\Longrightarrow \qquad v - auv = u + a \\ &\qquad\qquad v - u = a(1+uv) \\ &\Longrightarrow \qquad a = \frac{v-u}{1+uv} \quad . \end{aligned}$$

Now differentiating with respect to u, noting that a is a constant:

$$\begin{aligned} 0 &= \frac{(1+uv)\left(\frac{dv}{du} - 1\right) - (v-u)\left(v + u\frac{dv}{du}\right)}{(1+uv)^2} \\ \iff \qquad (v-u)\left(v + u\frac{dv}{du}\right) &= (1+uv)\left(\frac{dv}{du} - 1\right) \\ (1+uv)\frac{dv}{du} - 1 - uv &= (v-u)v + (v-u)u\frac{dv}{du} \\ \frac{dv}{du} - 1 &= v^2 - u^2\frac{dv}{du} \\ (1+u^2)\frac{dv}{du} &= 1 + v^2 \\ \frac{dv}{du} &= \frac{1+v^2}{1+u^2} \quad . \end{aligned}$$

Using this substitution in the integral for $T(x)$, when $u = 0$ we get $v = a$, thus for $a > 0$ and provided $1 - ax > 0$ (that is: $x < a^{-1}$) we have

$$\begin{aligned} T(x) &= \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+u^2} \frac{du}{dv} dv = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+u^2} \frac{1+u^2}{1+v^2} dv \\ &= \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv \\ &= \int_0^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv - \int_0^{a} \frac{1}{1+v^2} dv \\ &= T\left(\frac{x+a}{1-ax}\right) - T(a) \quad . \end{aligned}$$

Substituting $T(x) = T_\infty - T(x^{-1})$ and $T(a) = T_\infty - T(a^{-1})$, we get

$$\begin{aligned} T_\infty - T(x^{-1}) &= T\left(\frac{x+a}{1-ax}\right) - T_\infty + T(a^{-1}) \\ T(x^{-1}) &= 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1}) \quad . \end{aligned}$$

Now letting $y = x^{-1}$ and $b = a^{-1}$ we deduce

$$\begin{aligned} T(y) &= 2T_\infty - T\left(\frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}}\right) - T(b) \\ T(y) &= 2T_\infty - T\left(\frac{b+y}{by-1}\right) - T(b) \quad . \end{aligned}$$

(iii) Let $y = b = \sqrt{3}$ (which satisfy $b > 0$, $y > b^{-1}$), then we get

$$\begin{aligned} T(\sqrt{3}) &= 2T_\infty - T\left(\frac{2\sqrt{3}}{3-1}\right) - T(\sqrt{3}) \\ 3T(\sqrt{3}) &= 2T_\infty \\ T(\sqrt{3}) &= \tfrac{2}{3}T_\infty \quad . \end{aligned}$$

Let $x = a = \sqrt{2} - 1$ (so $a^{-1} = \sqrt{2} + 1 > x$), then we get

$$\begin{aligned} T(x) &= T\left(\frac{x+a}{1-ax}\right) - T(a) \\ T(\sqrt{2}-1) &= T\left(\frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2}\right) - T(\sqrt{2}-1) \\ 2T(\sqrt{2}-1) &= T\left(\frac{2(\sqrt{2}-1)}{1-(3-2\sqrt{2})}\right) \\ &= T\left(\frac{2(\sqrt{2}-1)}{2\sqrt{2}-2}\right) \\ &= T(1) \\ T(\sqrt{2}-1) &= \tfrac{1}{2}T(1) \quad . \end{aligned}$$

Now setting $x = 1$ in the very first result:

$$\begin{aligned} T(x) &= T_\infty - T(x^{-1}) \\ T(1) &= T_\infty - T(1) \\ T(1) &= \tfrac{1}{2}T_\infty \quad , \end{aligned}$$

thus we deduce that

$$T(\sqrt{2}-1) = \tfrac{1}{2}T(1) = \tfrac{1}{4}T_\infty \quad .$$

Question 7

By direct substitution, we find

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{a^2(1-t^2)^2}{a^2(1+t^2)^2} + \frac{4b^2t^2}{b^2(1+t^2)^2} \\
&= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} \\
&= \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} \\
&= \frac{1+2t^2+t^4}{(1+t^2)^2} \\
&= 1 \quad ,
\end{aligned}$$

thus the point T lies on the given ellipse.

(i) Differentiating the equation for the ellipse

$$\begin{aligned}
\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} &= 0 \\
\frac{y}{b^2}\frac{dy}{dx} &= -\frac{x}{a^2} \\
\frac{dy}{dx} &= -\frac{b^2x}{a^2y} \quad .
\end{aligned}$$

Now substituting in the coordinates of T, the gradient of the ellipse at T is

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{b^2}{a^2}\frac{a(1-t^2)}{2bt} \\
&= -\frac{b}{a}\frac{1-t^2}{2t} \quad .
\end{aligned}$$

The tangent to the ellipse at T is thus given by

$$\begin{aligned}
y - \frac{2bt}{1+t^2} &= -\frac{b}{a}\frac{1-t^2}{2t}\left(x - \frac{a(1-t^2)}{1+t^2}\right) \\
2at(1+t^2)y - 4abt^2 &= -b(1-t^2)((1+t^2)x - a(1-t^2)) \\
2at(1+t^2)y + b(1-t^2)(1+t^2)x &= 4abt^2 + ab(1-t^2)^2 \\
(2aty + b(1-t^2)x)(1+t^2) &= ab(1+t^2)^2 \\
2aty + b(1-t^2)x &= ab(1+t^2) \quad .
\end{aligned}$$

Thus the coordinates (X, Y) of the point on this tangent satisfy

$$\begin{aligned}
2atY + b(1-t^2)X &= ab(1+t^2) \\
abt^2 + bt^2X - 2atY - bX + ab &= 0 \\
(a+X)bt^2 - 2aYt + b(a-X) &= 0 \quad .
\end{aligned}$$

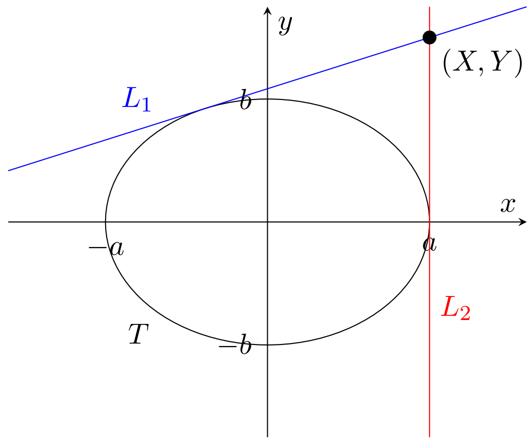
This is a quadratic equation in t with discriminant

$$\Delta = (-2aY)^2 - 4(a+X)(a-X)b^2 = 4(a^2Y^2 - (a^2-X^2)b^2) \ ,$$

thus if $a^2Y^2 > (a^2-X^2)b^2$ then there are two distinct values of t for which this holds and so there are two distinct lines L through (X,Y) that are tangent to the ellipse. We have

$$\begin{aligned} a^2Y^2 > (a^2-X^2)b^2 \qquad &\iff \qquad \frac{Y^2}{b^2} > 1 - \frac{X^2}{a^2} \\ &\iff \qquad \frac{X^2}{a^2} + \frac{Y^2}{b^2} > 1 \ , \end{aligned}$$

which is exactly the condition that (X,Y) lies outside the ellipse. We note that if $X^2 = a^2$ then one of these tangent lines is vertical:



(ii) Given that (X,Y) lies on both tangents we have

$$\begin{aligned} &(a+X)bp^2 - 2aYp + b(a-X) = 0 \ , &(1)\\ \text{and} \qquad &(a+X)bq^2 - 2aYq + b(a-X) = 0 \ . &(2) \end{aligned}$$

Multiplying equation (1) by q and equation (2) by p and subtracting one from the other we can eliminate Y to find

$$\begin{aligned} (a+X)b(p^2q - pq^2) + b(a-X)(q-p) &= 0 \\ (a+X)pq - (a-X) &= 0 \\ (a+X)pq &= a - X \ . \end{aligned}$$

Instead if we subtract (2) from (1) we find

$$\begin{aligned} (a+X)b(q^2-p^2) - 2aY(q-p) &= 0 \\ (a+X)b(q+p) - 2aY &= 0 \\ q + p &= \frac{2aY}{b(a+X)} \ . \end{aligned}$$

Substituting the given coordinates into the tangent equations we get

$$abp^2 - 2ay_1p + ab = 0 \quad , \tag{3}$$

$$\text{and} \qquad abq^2 - 2ay_2q + ab = 0 \quad . \tag{4}$$

Multiplying equation (3) by q and equation (4) by p and summing the two we find

$$\begin{aligned} ab(p^2q + pq^2) - 2apq(y_1 + y_2) + ab(p+q) &= 0 \\ abpq(p+q) - 2apq \cdot 2b + ab(p+q) &= 0 \\ pq(p+q) - 4pq + (p+q) &= 0 \quad . \end{aligned}$$

Now substituting in $pq = \frac{a-X}{a+X}$, $p + q = \frac{2aY}{b(a+X)}$ we get

$$\begin{aligned} \frac{a-X}{a+X} \cdot \frac{2aY}{b(a+X)} - 4\frac{a-X}{a+X} + \frac{2aY}{b(a+X)} &= 0 \\ (a-X) \cdot 2aY - 4b(a-X)(a+X) + 2aY(a+X) &= 0 \\ 4a^2Y + 4b(X^2 - a^2) &= 0 \\ bX^2 + a^2Y &= a^2b \\ \frac{X^2}{a^2} + \frac{Y}{b} &= 1 \quad . \end{aligned}$$

Question 8

Rearranging the sum directly

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} a_m b_{m+1} - \sum_{m=1}^{n} a_m b_m \\ &= \sum_{m=1}^{n} a_m b_{m+1} + a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=2}^{n+1} a_m b_m \\ &= \sum_{m=1}^{n} a_m b_{m+1} + a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} a_{m+1} b_{m+1} \\ &= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n}(a_{m+1}-a_m)b_{m+1} \quad .\end{aligned}$$

(i) Setting $b_m = \sin mx$ and $a_m = 1$, the identity above gives

$$\sum_{m=1}^{n}(\sin(m+1)x - \sin mx) = \sin(n+1)x - \sin x \quad .$$

If we now use the noted trigonometric identity on the left-hand side, we get

$$\sum_{m=1}^{n} 2\cos\left(m+\tfrac{1}{2}\right)x \sin\tfrac{1}{2}x = \sin(n+1)x - \sin x$$

$$\sum_{m=1}^{n} \cos\left(m+\tfrac{1}{2}\right)x = \tfrac{1}{2}\left(\sin(n+1)x - \sin x\right)\operatorname{cosec}\tfrac{1}{2}x \quad .$$

(ii) Setting $a_m = m$, the identity gives

$$\sum_{m=1}^{n} m(b_{m+1}-b_m) = (n+1)b_{n+1} - b_1 - \sum_{m=1}^{n} b_{m+1} \quad .$$

We set $b_{m+1} = \sin mx - \sin(m+1)x$, such that the sum on the right-hand side will telescope:

$$\begin{aligned}\sum_{m=1}^{n} m(b_{m+1}-b_m) &= (n+1)(\sin nx - \sin(n+1)x) - (0 - \sin x) \\ &\qquad - \sum_{m=1}^{n}(\sin mx - \sin(m+1)x) \\ &= -(n+1)\sin(n+1)x + (n+1)\sin nx + \sin x \\ &\qquad - (\sin x - \sin(n+1)x) \\ &= -n\sin(n+1)x + (n+1)\sin nx \quad .\end{aligned}$$

Using the first noted trigonometric identity on the terms of the left-hand sum:

$$\begin{aligned} b_{m+1} - b_m &= \sin mx - \sin(m+1)x - (\sin(m-1)x - \sin mx) \\ &= -2\cos\left(m + \tfrac{1}{2}\right)x \sin\tfrac{1}{2}x + 2\cos\left(m - \tfrac{1}{2}\right)x \sin\tfrac{1}{2}x \\ &= 2\sin\tfrac{1}{2}x\left(\cos\left(m - \tfrac{1}{2}\right)x - \cos\left(m + \tfrac{1}{2}\right)x\right) \quad . \end{aligned}$$

Now using the other noted trigonometric identity where a difference of two cos terms is expressed as a product of two sin terms, we have $A = mx$, $B = \frac{1}{2}x$:

$$\begin{aligned} b_{m+1} - b_m &= 2\sin\tfrac{1}{2}x\left(\cos\left(m - \tfrac{1}{2}\right)x - \cos\left(m + \tfrac{1}{2}\right)x\right) \\ &= 2\sin\tfrac{1}{2}x\left(2\sin mx \sin\tfrac{1}{2}x\right) \\ &= 4\sin^2\tfrac{1}{2}x \sin mx \quad . \end{aligned}$$

Substituting this back into the full equation then

$$\begin{aligned} \sum_{m=1}^{n} m(b_{m+1} - b_m) &= -n\sin(n+1)x + (n+1)\sin nx \\ 4\sin^2\tfrac{1}{2}x \sum_{m=1}^{n} m\sin mx &= -n\sin(n+1)x + (n+1)\sin nx \\ \sum_{m=1}^{n} m\sin mx &= \left(-\tfrac{1}{4}n\sin(n+1)x + \tfrac{1}{4}(n+1)\sin nx\right)\operatorname{cosec}^2\tfrac{1}{2}x \quad , \end{aligned}$$

thus $p = -\frac{1}{4}n$, $q = \frac{1}{4}(n+1)$.

Section B: Mechanics

Question 9

Considering the forces on particle A, we get the equation of motion

$$m\ddot{y} = mg - T \quad .$$

Similarly, considering the horizontal forces on particle B, we get the equation of motion

$$2m\ddot{x} = T \quad .$$

Adding these two equations together, and using the fact that both particles are released from rest ($\dot{x}(0) = \dot{y}(0) = 0$) at $x(0) = y(0) = 0$, $t = 0$, we can solve to get

$$\begin{aligned}
\ddot{y} + 2\ddot{x} &= g \qquad \text{(after cancelling factors of } m\text{)} \\
\implies \quad \dot{y} + 2\dot{x} - (\dot{y}(0) + 2\dot{x}(0)) &= gt \\
\dot{y} + 2\dot{x} &= gt \qquad (*) \\
\implies y + 2x - (y(0) + 2x(0)) &= \frac{1}{2}gt^2 \\
y + 2x &= \tfrac{1}{2}gt^2 \quad .
\end{aligned}$$

When B reaches the edge of the table $t = \sqrt{\frac{6a}{g}}$, $x = a$, and we have

$$\begin{aligned}
y &= \tfrac{1}{2}gt^2 - 2x \\
\implies \quad y &= \tfrac{1}{2}g\frac{6a}{g} - 2a = a \quad ,
\end{aligned}$$

so at this point the spring is at its natural length and there is no elastic potential energy. The total energy of the system at time T is

$$E = \frac{1}{2} \cdot 2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy = m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mga \quad .$$

The initial energy of the system is zero, hence by conservation of energy

$$2\dot{x}^2 + \dot{y}^2 - 2ga = 0 \quad ,$$

after cancelling factors of $\frac{1}{2}m$. Now using $(*)$ evaluated at $t = T = \sqrt{\frac{6a}{g}}$, we find

$$\begin{aligned}
2\dot{x}^2 + (gT - 2\dot{x})^2 = 2ga \qquad \implies \qquad 2\dot{x}^2 + \left(\sqrt{6ga} - 2\dot{x}\right)^2 &= 2ga \\
6\dot{x}^2 - 4\sqrt{6ga}\dot{x} + 4ga &= 0 \\
\dot{x}^2 - 2 \cdot \frac{\sqrt{6ga}}{3}\dot{x} + \frac{2}{3}ga &= 0 \\
\left(\dot{x} - \frac{\sqrt{6ga}}{3}\right)^2 &= 0 \quad .
\end{aligned}$$

Hence the velocity of particle B at time T is $\dot{x} = \frac{\sqrt{6ga}}{3} = \sqrt{\frac{2ga}{3}}$.

Question 10

The moment of inertia of the the rod about P is $\frac{1}{3}m \cdot (3a)^2 = 3ma^2$, by conservation of energy then

$$\begin{aligned}
\frac{1}{2} \cdot (3ma^2)\dot{\theta}^2 + \frac{1}{2}m(l\dot{\theta})^2 - \left(\frac{3}{2}a\sin\theta\right)mg - (l\sin\theta)mg &= 0 \\
3ma^2\dot{\theta}^2 + ml^2\dot{\theta}^2 - (3a+2l)\sin\theta\, mg &= 0 \\
\implies \qquad (3a^2+l^2)\dot{\theta}^2 &= g(3a+2l)\sin\theta \quad .
\end{aligned}$$

Differentiating with respect to time

$$\begin{aligned}
2(3a^2+l^2)\dot{\theta}\ddot{\theta} &= g(3a+2l)\cos\theta\,\dot{\theta} \\
\implies \qquad 2(3a^2+l^2)\ddot{\theta} &= g(3a+2l)\cos\theta \\
\ddot{\theta} &= \frac{g(3a+2l)\cos\theta}{2(3a^2+l^2)} \quad .
\end{aligned}$$

Resolving forces on the particle perpendicular to the rod, we have

$$mg\cos\theta - R = ml\ddot{\theta} \quad ,$$

where R is the normal reaction force of the rod on the particle. This gives

$$\begin{aligned}
R &= mg\cos\theta - ml\ddot{\theta} \\
&= mg\cos\theta - ml\frac{g(3a+2l)\cos\theta}{2(3a^2+l^2)} \\
&= mg\cos\theta\left(1 - \frac{l(3a+2l)}{2(3a^2+l^2)}\right) \\
&= mg\cos\theta\frac{6a^2+2l^2-3al-2l^2}{2(3a^2+l^2)} \\
&= mg\cos\theta\frac{6a^2-3al}{2(3a^2+l^2)} \\
&= \frac{3}{2}mga\cos\theta\frac{2a-l}{3a^2+l^2} \quad .
\end{aligned}$$

Since $l < 2a$ we see that $R > 0$ whenever $\cos\theta > 0$, that is $R > 0$ for $\theta < \frac{\pi}{2}$.

Now resolving forces on the particle parallel to the rod, we have

$$F - mg\sin\theta = ml\dot{\theta}^2 \quad ,$$

where F is the frictional force of the rod on the particle.

Substituting in for $\dot{\theta}$:

$$
\begin{aligned}
F &= mg\sin\theta + ml\left(\frac{g(3a+2l)}{3a^2+l^2}\sin\theta\right) \\
&= mg\sin\theta\left(1+\frac{l(3a+2l)}{3a^2+l^2}\right) \\
&= mg\sin\theta\frac{3a^2+l^2+l(3a+2l)}{3a^2+l^2} \\
&= 3mg\sin\theta\frac{a^2+al+l^2}{3a^2+l^2} \quad .
\end{aligned}
$$

When the particle is on the point of slipping we have $F = \mu R$, thus

$$
\begin{aligned}
3mg\sin\theta\frac{a^2+al+l^2}{3a^2+l^2} &= \mu\cdot\frac{3}{2}mga\cos\theta\frac{2a-l}{3a^2+l^2} \\
\tan\theta(a^2+al+l^2) &= \frac{1}{2}\mu a(2a-l) \\
\tan\theta &= \frac{\mu a(2a-l)}{2(a^2+al+l^2)} \quad .
\end{aligned}
$$

At the moment the rod is released, its equation of motion neglecting the particle is

$$
\begin{aligned}
\frac{3}{2}amg &= 3ma^2\ddot{\theta} \\
\Longrightarrow \qquad \ddot{\theta} &= \frac{g}{2a} \quad ,
\end{aligned}
$$

thus the instantaneous acceleration of the point on the rod just beneath the particle is

$$
l\ddot{\theta} = \frac{lg}{2a} \quad .
$$

If $l > 2a$ then the instantaneous acceleration of the rod away from the particle is greater than g, the free-fall acceleration of the particle under its own weight. Thus if $l > 2a$ the particle loses contact with the rod as soon as the rod is released.

Question 11

(i) By conservation of linear momentum

$$\begin{aligned} 0 &= Mu + nm(-v) \\ \implies \qquad u &= \frac{nmv}{M} \quad . \end{aligned}$$

Hence the kinetic energy of the system after the guns are fired is

$$\begin{aligned} K &= \frac{1}{2}Mu^2 + n \cdot \frac{1}{2}mv^2 \\ &= \frac{1}{2}M\frac{n^2m^2v^2}{M^2} + \frac{1}{2}nmv^2 \\ &= \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right) \quad . \end{aligned}$$

(ii) By conservation of linear momentum after the r-th gun is fired, for $1 \leqslant r \leqslant n$

$$\begin{aligned} (M + (n - (r-1))m)u_{r-1} &= (M + (n-r)m)u_r - m(v - u_{r-1}) \\ (M + (n-r)m)u_{r-1} &= (M + (n-r)m)u_r - mv \\ (M + (n-r)m)(u_r - u_{r-1}) &= mv \\ u_r - u_{r-1} &= \frac{mv}{M + (n-r)m} \quad . \end{aligned}$$

Summing this for $1 \leqslant r \leqslant n$, we get

$$\begin{aligned} \sum_{r=1}^{n}(u_r - u_{r-1}) &= mv\sum_{r=1}^{n}\frac{1}{M + (n-r)m} \\ u_n - u_0 &= mv\sum_{r=n-1}^{r=0}\frac{1}{M + rm} \\ u_n &= mv\sum_{r=0}^{n-1}\frac{1}{M + rm} \\ \implies \qquad u_n &< mv\sum_{r=0}^{n-1}\frac{1}{M} = \frac{nmv}{M} \\ \implies \qquad u_n &< u \quad . \end{aligned}$$

(iii) The increase in kinetic energy after the r-th gun is fired is

$$
\begin{aligned}
K_r - K_{r-1} &= \frac{1}{2}(M+(n-r)m)u_r^2 + \frac{1}{2}m(v-u_{r-1})^2 \\
&\qquad\qquad\qquad\qquad - \frac{1}{2}(M+(n-r+1)m)u_{r-1}^2 \\
&= \frac{1}{2}(M+(n-r)m)(u_r^2 - u_{r-1}^2) + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}mu_{r-1}^2 \\
&= \frac{1}{2}(M+(n-r)m)(u_r - u_{r-1})(u_r + u_{r-1}) + \frac{1}{2}m(v^2 - 2u_{r-1}v) \\
&= \frac{1}{2}mv(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mu_{r-1}v \\
&= \frac{1}{2}mv(u_r - u_{r-1}) + \frac{1}{2}mv^2 \quad .
\end{aligned}
$$

Now summing this for $1 \leqslant r \leqslant n$, we get

$$
\begin{aligned}
\sum_{r=1}^{n}(K_r - K_{r-1}) &= \frac{1}{2}mv\sum_{r=1}^{n}(u_r - u_{r-1}) + \frac{1}{2}nmv^2 \\
K_n - K_0 &= \frac{1}{2}mv(u_n - u_0) + \frac{1}{2}nmv^2 \\
K_n &= \frac{1}{2}mvu_n + \frac{1}{2}nmv^2 \\
&= \frac{1}{2}mv(u_n + v) \quad .
\end{aligned}
$$

Using the fact that $u_n < u$ we thus get

$$
\begin{aligned}
K_n &< \frac{1}{2}mv(u+v) \\
K_n &< \frac{1}{2}mv\left(v + \frac{nmv}{M}\right) \\
K_n &< \frac{1}{2}mv^2\left(1 + \frac{nm}{M}\right) \\
\implies \qquad K_n &< \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right) \quad ,
\end{aligned}
$$

since $n > 1$. This final expression on the right-hand side is our expression for K found above. Hence we deduce $K_n < K$.

Section C: Probability and Statistics

Question 12

(i) By summing over the possible values of y, we can find

$$\begin{aligned}\mathbb{P}(X=x)=\sum_{y=1}^{n}\mathbb{P}(X=x,Y=y)=\sum_{y=1}^{n}k(x+y)&\\ &=nkx+k\sum_{y=1}^{n}y\\ &=nkx+k\cdot\frac{1}{2}n(n+1)\quad.\end{aligned}$$

Using the fact that $\sum_{x=1}^{n}\mathbb{P}(X=x)=1$ we can find k in terms of n:

$$\begin{aligned}\sum_{x=1}^{n}\left(nkx+k\cdot\frac{1}{2}n(n+1)\right)&=1\\ nk\sum_{x=1}^{n}x+\frac{k}{2}n^2(n+1)&=1\\ \frac{k}{2}n^2(n+1)+\frac{k}{2}n^2(n+1)&=1\\ k&=\frac{1}{n^2(n+1)}\quad.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{P}(X=x)=nkx+k\cdot\frac{1}{2}n(n+1)&=\frac{1}{2}nk(2x+n+1)\\ &=\frac{2x+n+1}{2n(n+1)}\quad.\end{aligned}$$

By symmetry, we also have

$$\mathbb{P}(Y=y)=\frac{2y+n+1}{2n(n+1)}\quad.$$

For example

$$\mathbb{P}(X=1)=\frac{n+3}{2n(n+1)}\quad,\quad\mathbb{P}(Y=1)=\frac{n+3}{2n(n+1)}\quad,$$

and

$$\mathbb{P}(X=1,Y=1)=2k=\frac{2}{n^2(n+1)}\quad.$$

We show that X and Y are not independent using this example: we have

$$\begin{aligned}
&\mathbb{P}(X=1)\cdot\mathbb{P}(Y=1)=\mathbb{P}(X=1,Y=1) \\
\iff\quad &\frac{(n+3)^2}{4n^2(n+1)^2}=\frac{2}{n^2(n+1)} \\
\iff\quad &(n+3)^2=8(n+1) \\
\iff\quad &n^2-2n+1=0 \\
&(n-1)^2=0 \quad .
\end{aligned}$$

Since $n \geqslant 2$, this does not hold, thus $\mathbb{P}(X=1)\cdot\mathbb{P}(Y=1)\neq\mathbb{P}(X=1,Y=1)$ and so X and Y are not independent.

(ii) The expectation of X is

$$\begin{aligned}
\mathbb{E}(X)=\sum_{x=1}^{n}x\mathbb{P}(X=x)&=\sum_{x=1}^{n}x\cdot\frac{2x+n+1}{2n(n+1)} \\
&=\frac{1}{2n(n+1)}\left(2\sum_{x=1}^{n}x^2+(n+1)\sum_{x=1}^{n}x\right) \\
&=\frac{1}{2n(n+1)}\left(\frac{1}{3}n(n+1)(2n+1)+\frac{1}{2}n(n+1)^2\right) \\
&=\frac{1}{2}\left(\frac{1}{3}(2n+1)+\frac{1}{2}(n+1)\right) \\
&=\frac{7n+5}{12} \quad ,
\end{aligned}$$

and likewise (by symmetry)

$$\mathbb{E}(Y)=\frac{7n+5}{12} \quad .$$

The expectation of the product XY is

$$\begin{aligned}
\mathbb{E}(XY)&=\sum_{x=1}^{n}\sum_{y=1}^{n}xy\mathbb{P}(X=x,Y=y)=\sum_{x=1}^{n}\sum_{y=1}^{n}xy\cdot k(x+y) \\
&=\frac{1}{n^2(n+1)}\sum_{x=1}^{n}\left(x^2\sum_{y=1}^{n}y+x\sum_{y=1}^{n}y^2\right) \\
&=\frac{1}{n^2(n+1)}\sum_{x=1}^{n}\left(x^2\cdot\frac{1}{2}n(n+1)+x\cdot\frac{1}{6}n(n+1)(2n+1)\right) \\
&=\frac{1}{n^2(n+1)}\left(\frac{1}{6}n(n+1)(2n+1)\cdot\frac{1}{2}n(n+1)+\frac{1}{2}n(n+1)\cdot\frac{1}{6}n(n+1)(2n+1)\right) \\
&=2\cdot\left(\frac{1}{6}(2n+1)\cdot\frac{1}{2}(n+1)\right) \\
&=\frac{1}{6}(n+1)(2n+1) \quad .
\end{aligned}$$

Hence the covariance of X and Y is

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y) \\
&= \frac{1}{6}(n+1)(2n+1) - \left(\frac{7n+5}{12}\right)^2 \\
&= \frac{1}{144}\big(24(n+1)(2n+1) - (7n+5)^2\big) \\
&= \frac{1}{144}\big(48n^2+72n+24-(49n^2+70n+25)\big) \\
&= \frac{1}{144}\big(-n^2+2n-1\big) \\
&= -\frac{(n-1)^2}{144} \quad ,
\end{aligned}$$

which is strictly negative, since $n \geqslant 2$.

Question 13

Expanding and simplifying

$$\begin{aligned} V(x) = \mathbb{E}((X-x)^2) &= \mathbb{E}(X^2 - 2xX + x^2) = \mathbb{E}(X^2) - 2x\mathbb{E}(X) + x^2 \\ &= (\mathrm{Var}(X) + \mathbb{E}(X)^2) - 2x\mathbb{E}(X) + x^2 \\ &= \sigma^2 + \mu^2 - 2x\mu + x^2 \\ &= \sigma^2 + (x-\mu)^2 \quad . \end{aligned}$$

We then have

$$\begin{aligned} \mathbb{E}(Y) = \mathbb{E}(V(X)) &= \mathbb{E}(\sigma^2 + (X-\mu)^2) = \sigma^2 + \mathbb{E}((X-\mu)^2) \\ &= \sigma^2 + \mathrm{Var}(X) \\ &= 2\sigma^2 \quad . \end{aligned}$$

If X is uniformly distributed on $0 \leqslant x \leqslant 1$, then we have

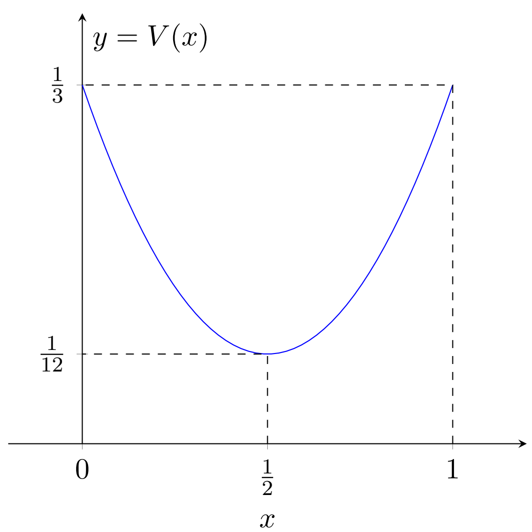
$$\mu = \mathbb{E}(X) = \int_0^1 x dx = \frac{1}{2}$$

$$\text{and} \qquad \sigma^2 = \mathrm{Var}(X) = \int_0^1 x^2 dx - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad ,$$

thus

$$V(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2 = \frac{1}{3} - x + x^2 \quad .$$

Giving a quick sketch, we see that Y can take values $\frac{1}{12} \leqslant y \leqslant \frac{1}{3}$:



To find the probability density function of Y consider that for $\frac{1}{12} \leqslant y \leqslant \frac{1}{3}$:

$$\begin{aligned}
\mathbb{P}(Y \leqslant y) &= \mathbb{P}\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 \leqslant y\right) \\
&= \mathbb{P}\left(\left(X - \frac{1}{2}\right)^2 \leqslant y - \frac{1}{12}\right) \\
&= \mathbb{P}\left(\left|X - \frac{1}{2}\right| \leqslant \sqrt{y - \frac{1}{12}}\right) \\
&= \mathbb{P}\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} \leqslant X \leqslant \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) \\
&= \frac{1}{2} + \sqrt{y - \frac{1}{12}} - \left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) \\
&= 2\sqrt{y - \frac{1}{12}} \quad .
\end{aligned}$$

Differentiating this, we get the probability density function for Y:

$$f(y) = \frac{d}{dy}\left(2\sqrt{y - \frac{1}{12}}\right) = \frac{y - \frac{1}{12}}{\sqrt{y - \frac{1}{12}}} = \frac{1}{\sqrt{y - \frac{1}{12}}} \quad ,$$

where $\frac{1}{12} \leqslant y \leqslant \frac{1}{3}$. Now we can evaluate $\mathbb{E}(Y)$ directly:

$$\begin{aligned}
\mathbb{E}(Y) &= \int_{\frac{1}{12}}^{\frac{1}{3}} y f(y) dy = \int_{\frac{1}{12}}^{\frac{1}{3}} \frac{y}{\sqrt{y - \frac{1}{12}}} dy \\
&= \int_{\frac{1}{12}}^{\frac{1}{3}} \left(\frac{y - \frac{1}{12}}{\sqrt{y - \frac{1}{12}}} + \frac{\frac{1}{12}}{\sqrt{y - \frac{1}{12}}}\right) dy \\
&= \int_{\frac{1}{12}}^{\frac{1}{3}} \left(\left(y - \frac{1}{12}\right)^{\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}\right) dy \quad ,
\end{aligned}$$

let $u = y - \frac{1}{12}$, then

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^{\frac{1}{4}} \left(u^{\frac{1}{2}} + \frac{1}{12}u^{-\frac{1}{2}}\right) du = \left[\frac{2}{3}u^{\frac{3}{2}} + \frac{1}{6}u^{\frac{1}{2}}\right]_0^{\frac{1}{4}} \\
&= \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{6} \cdot \frac{1}{2} - 0 \\
&= \frac{1}{6} \quad ,
\end{aligned}$$

which is equal to $2\sigma^2 = \frac{2}{12} = \frac{1}{6}$ as we found before.