STEP 2016 Solutions

Contents

Author's notes:

1. *At time of writing, I am not affiliated with Cambridge Assessment Admissions Testing. I did an undergraduate maths degree at Cambridge, so I sat the STEP II and III papers as an A-level student (in 2015), and I have also been one of a team of markers for the STEP exams (in 2019 and 2020). Any opinions given here are entirely my own, based on my own experiences of STEP.*

2. *These 'solutions' are not intended to be used as any sort of mark scheme. In terms of method, often there will be more than one correct way to answer a STEP question, and it is certainly not the case that the answers presented here are the only correct approaches to these questions. The worked solutions here were typed up after attempting the questions myself, and I have checked them against the official mark schemes published online. However, there is no guarantee that the solutions typed up here would achieve full marks. In particular, I have not provided diagrams for all questions due to the difficulties of typesetting them neatly. Many questions may ask the student to draw a diagram, and in these instances marks are often awarded for this. Another point of consideration is explanation: sometimes marks are awarded for explicitly justifying an assumption used. I have tried to justify these as I think necessary, but there is no guarantee that these solutions justify all assumptions to the standards of the mark schemes.*

3. *If you are preparing to sit the STEP exams, I hope these can be of some help.*

STEP I

Section A: Pure Mathematics

Question 1

(i) Substituting in $n = 1, 2, 3$, expanding and simplifying, for $x \neq -1$ we have

$n = 1$:

$$\begin{aligned} p_1(x) &= (x+1)^2 - 3x \\ &= x^2 - x + 1 \quad , \\ \text{and} \qquad q_1(x) &= \frac{x^3+1}{x+1} = x^2 - x + 1 \quad , \end{aligned}$$

$n = 2$:

$$\begin{aligned} p_2(x) &= (x+1)^4 - 5x(x^2+x+1) \\ &= x^4 + 4x^3 + 6x^2 + 4x + 1 - (5x^3 + 5x^2 + 5x) \\ &= x^4 - x^3 + x^2 - x + 1 \quad , \\ \text{and} \qquad q_2(x) &= \frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1 \quad , \end{aligned}$$

and $n = 3$:

$$\begin{aligned} p_3(x) &= (x+1)^6 - 7x(x^2+x+1)^2 \\ &= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \\ &\qquad\qquad - 7x(x^4 + 2x^3 + 3x^2 + 2x + 1) \\ &= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \\ &\qquad\qquad - (7x^5 + 14x^4 + 21x^3 + 14x^2 + 7x) \\ &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \quad , \\ \text{and} \qquad q_3(x) &= \frac{x^7+1}{x+1} = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \quad . \end{aligned}$$

Hence for $x \neq -1$, we have $p_n(x) \equiv q_n(x)$ for $n = 1, 2, 3$.

In the case $n = 4$, consider $x = 1$:

$$\begin{aligned} p_4(1) &= 2^8 - 9 \cdot 1 \cdot 3^3 \\ &= 256 - 9 \cdot 27 = 256 - 243 \\ &= 13 \quad , \end{aligned}$$

but

$$q_4(1) = \frac{2}{2} = 1 \quad .$$

Hence $p_4(x) \not\equiv q_4(x)$.

(ii) (a) Using $x = 300$, $n = 1$, we have

$$\begin{aligned} & q_1(300) = p_1(300) \\ \implies \quad & \frac{300^3+1}{301} = 301^2 - 3 \cdot 300 \\ & \qquad\quad = 301^2 - 30^2 \\ & \qquad\quad = (301+30)(301-30) \\ \implies \quad & \frac{300^3+1}{301} = 331 \cdot 271 \quad . \end{aligned}$$

(b) Using $x = 7^7$, $n = 3$

$$\begin{aligned} q_3(7^7) &= p_3(7^7) \\ \frac{(7^7)^7+1}{7^7+1} &= (7^7+1)^6 - 7 \cdot 7^7(7^{14}+7^7+1)^2 \\ \frac{7^{49}+1}{7^7+1} &= ((7^7+1)^3)^2 - (7^4(7^{14}+7^7+1))^2 \\ &= ((7^7+1)^3 + 7^4(7^{14}+7^7+1))((7^7+1)^3 - 7^4(7^{14}+7^7+1)) \\ &= (7^{21}+3\cdot 7^{14}+3\cdot 7^7+1+7^{18}+7^{11}+7^4) \\ &\qquad \cdot (7^{21}+3\cdot 7^{14}+3\cdot 7^7+1-7^{18}-7^{11}-7^4) \\ \frac{7^{49}+1}{7^7+1} &= (7^{21}+7^{18}+3\cdot 7^{14}+7^{11}+3\cdot 7^7+7^4+7^0) \\ &\qquad \cdot (7^{21}-7^{18}+3\cdot 7^{14}-7^{11}+3\cdot 7^7-7^4+7^0) \quad . \end{aligned}$$

Question 2

We shall call the given function $f(x)$:

$$f(x) = (ax^2 + bx + c)\ln(x + \sqrt{1+x^2}) + (dx+e)\sqrt{1+x^2} \quad ,$$

and to split this up a little let

$$\begin{aligned} g(x) &= \ln(x + \sqrt{1+x^2}) \quad , \\ \text{and} \qquad h(x) &= (dx+e)\sqrt{1+x^2} \quad . \end{aligned}$$

Then we have

$$f(x) = (ax^2 + bx + c)g(x) + h(x) \quad ,$$

giving

$$f'(x) = (2ax+b)g(x) + (ax^2+bx+c)g'(x) + h'(x) \quad .$$

To find $g'(x)$ we first exponentiate before differentiating:

$$\begin{aligned} e^{g(x)} &= x + \sqrt{1+x^2} \\ \implies \qquad g'(x)e^{g(x)} &= 1 + \frac{x}{\sqrt{1+x^2}} \\ g'(x) &= \frac{1}{e^{g(x)}}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) \\ &= \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \\ &= \frac{\sqrt{1+x^2} + x}{x + \sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \\ g'(x) &= \frac{1}{\sqrt{1+x^2}} \quad . \end{aligned}$$

Differentiating directly we find

$$\begin{aligned} h'(x) &= d\sqrt{1+x^2} + (dx+e)\frac{x}{\sqrt{1+x^2}} \\ &= \frac{d(1+x^2) + (dx+e)x}{\sqrt{1+x^2}} \\ &= \frac{2dx^2 + ex + d}{\sqrt{1+x^2}} \quad . \end{aligned}$$

All together this gives

$$\begin{aligned} f'(x) &= (2ax+b)\ln(x + \sqrt{1+x^2}) + \frac{ax^2+bx+c}{\sqrt{1+x^2}} + \frac{2dx^2+ex+d}{\sqrt{1+x^2}} \\ &= (2ax+b)\ln(x + \sqrt{1+x^2}) + \frac{(a+2d)x^2 + (b+e)x + (c+d)}{\sqrt{1+x^2}} \quad . \end{aligned}$$

(i) If we set $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = -1$, we get

$$f'(x) = \ln(x + \sqrt{1 + x^2}) \quad ,$$

$$\text{and} \qquad f(x) = x \ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} \quad ,$$

thus we have

$$\int \ln(x + \sqrt{1 + x^2})dx = x \ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + k \quad ,$$

where k is an arbitrary constant.

(ii) If we set $a = 0$, $b = 0$, $c = \frac{1}{2}$, $d = \frac{1}{2}$, $e = 0$, we get

$$f'(x) = \frac{x^2 + 1}{\sqrt{1 + x^2}} = \sqrt{1 + x^2} \quad ,$$

$$\text{and} \qquad f(x) = \frac{1}{2} \ln(x + \sqrt{1 + x^2}) + \frac{1}{2} x \sqrt{1 + x^2} \quad ,$$

thus we have

$$\int \sqrt{1 + x^2} dx = \frac{1}{2} \ln(x + \sqrt{1 + x^2}) + \frac{1}{2} x \sqrt{1 + x^2} + k \quad .$$

(iii) If we set $a = \frac{1}{2}$, $b = 0$, $c = \frac{1}{4}$, $d = -\frac{1}{4}$, $e = 0$, we get

$$f'(x) = x \ln(x + \sqrt{1 + x^2}) \quad ,$$

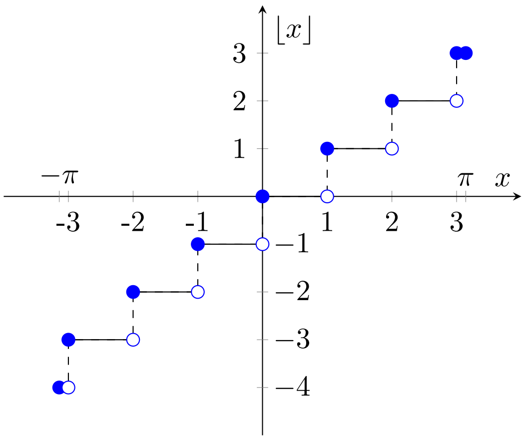
$$\text{and} \qquad f(x) = \left(\frac{1}{2}x^2 + \frac{1}{4}\right) \ln(x + \sqrt{1 + x^2}) - \frac{1}{4} x \sqrt{1 + x^2} \quad ,$$

thus we have

$$\int x \ln(x + \sqrt{1 + x^2}) dx = \left(\frac{1}{2}x^2 + \frac{1}{4}\right) \ln(x + \sqrt{1 + x^2}) - \frac{1}{4} x \sqrt{1 + x^2} + k \quad .$$

Question 3

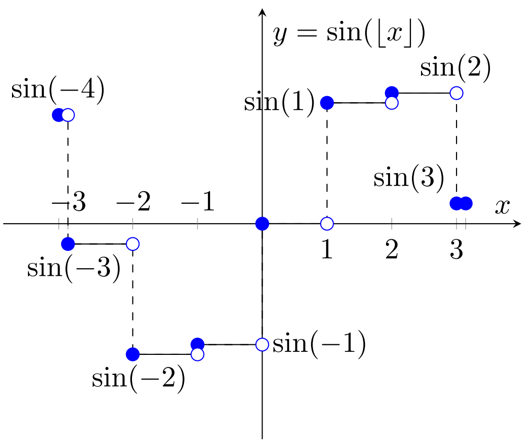
(i) Noting that $\lfloor -\pi \rfloor = -4$, and $\lfloor \pi \rfloor = 3$, we get the following sketch of $y = \lfloor x \rfloor$, $-\pi \leqslant x \leqslant \pi$:



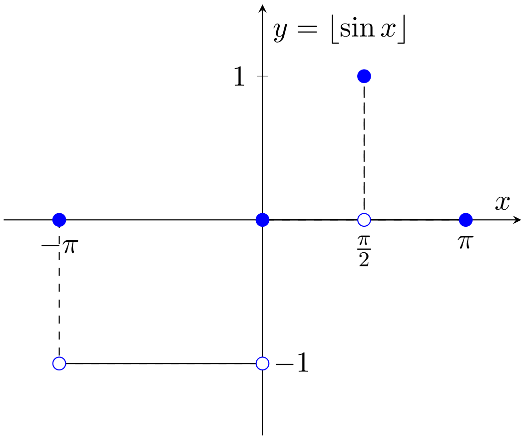
(ii) Since the $\lfloor \cdot \rfloor$ is inside the sin function our second sketch will jump at the same points as $y = \lfloor x \rfloor$. We note in particular that $-2\pi < -4 < -\pi$ so $\sin(-4) > 0$, also $\frac{\pi}{2} \approx 1.57$ is closer to 2 than to 1, hence $\sin(2) > \sin(1)$. Similarly $\pi - 3 < 1$ so $\sin(3)$ is closer to $\sin(\pi) = 0$ than $\sin(1)$ is to $\sin(0) = 0$, so $0 < \sin(3) < \sin(1)$. Finally, we note that π is closer to 3 than it is to 4 so $\sin(3) < \sin(-4)$, and $4 - \pi < 1$, hence $\sin(-4) < \sin(1)$. All together we have

$$\sin(-2) < \sin(-1) < \sin(-3) < 0 < \sin(3) < \sin(-4) < \sin(1) < \sin(2) \quad ,$$

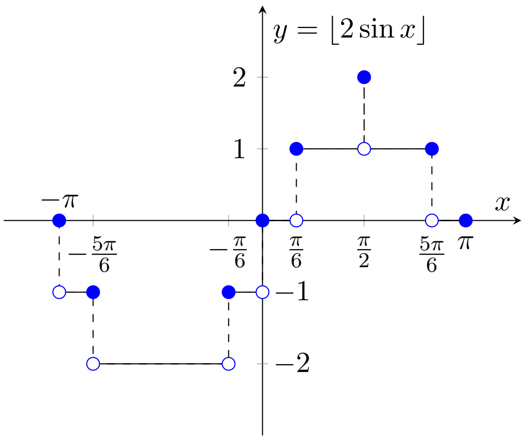
giving us the following sketch:



(iii) Now the $\lfloor\cdot\rfloor$ is applied to the value of $\sin x$; this will give $y = -1$ for $-1 \leqslant \sin(x) < 0$, $y = 0$ for $0 \leqslant \sin(x) < 1$ and a single point $y = 1$ at $\sin(x) = 1$:



(iv) Doubling within the $\lfloor\cdot\rfloor$, we now get possible values $y = -2, -1, 0, 1, 2$ with vertical jumps at $\sin(x) = -\frac{1}{2}, 0, \frac{1}{2}, 1$:



Question 4

(i) We shall call the given function $g(z)$:

$$g(z) = \frac{z}{(1+z^2)^{\frac{1}{2}}} \quad .$$

Taking logs and then differentiating, we get

$$\begin{aligned} \log g(z) &= \log z - \frac{1}{2}\log(1+z^2) \\ \implies \qquad \frac{g'(z)}{g(z)} &= \frac{1}{z} - \frac{z}{1+z^2} \\ &= \frac{1+z^2-z^2}{z(1+z^2)} \\ \implies g'(z) &= g(z) \cdot \frac{1}{z(1+z^2)} \\ g'(z) &= \frac{1}{(1+z^2)^{\frac{3}{2}}} \quad . \end{aligned}$$

(ii) Let $z = \frac{dy}{dx} = f'(x)$, then we have

$$\begin{aligned} \kappa &= \frac{\frac{dz}{dx}}{(1+z^2)^{\frac{3}{2}}} \\ \kappa \frac{dx}{dz} &= \frac{1}{(1+z^2)^{\frac{3}{2}}} \\ \implies \qquad \kappa(x-c_1) &= \frac{z}{(1+z^2)^{\frac{1}{2}}} \quad , \qquad \text{(by the result of (i))} \end{aligned}$$

where c_1 is some arbitrary constant. Rearranging this

$$\begin{aligned} \kappa(x-c_1) &= \frac{z}{(1+z^2)^{\frac{1}{2}}} \\ \kappa^2(x-c_1)^2 &= \frac{z^2}{1+z^2} \\ \kappa^2(x-c_1)^2 z^2 + \kappa^2(x-c_1)^2 &= z^2 \\ (\kappa^2(x-c_1))^2 - 1)z^2 &= -\kappa^2(x-c_1)^2 \\ z^2 &= \frac{\kappa^2(x-c_1)^2}{1-\kappa^2(x-c_1)^2} \\ z &= \pm\frac{\kappa(x-c_1)}{\sqrt{1-\kappa^2(x-c_1)^2}} \\ \frac{dy}{dx} &= \pm\frac{\kappa(x-c_1)}{\sqrt{1-\kappa^2(x-c_1)^2}} \quad . \end{aligned}$$

Integrating this, we use the substitution $u = \kappa(x - c_1)$:

$$\begin{aligned} y &= \pm \int \frac{\kappa(x - c_1)}{\sqrt{1 - \kappa^2(x - c_1)^2}} dx \\ &= \pm \int \frac{u}{\sqrt{1 - u^2}} \frac{1}{\kappa} du \\ \kappa y &= \pm \int \frac{u}{\sqrt{1 - u^2}} du \\ \kappa(y - c_2) &= \pm\sqrt{1 - u^2} \quad , \end{aligned}$$

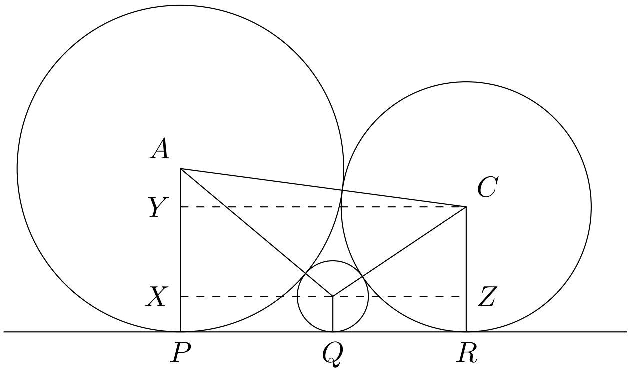
where c_2 is some arbitrary constant. Substituting back in $u = \kappa(x - c_1)$, we find

$$\begin{aligned} \kappa(y - c_2) &= \pm\sqrt{1 - \kappa^2(x - c_1)^2} \\ \kappa^2(y - c_2)^2 &= 1 - \kappa^2(x - c_1)^2 \\ (x - c_1)^2 + (y - c_2)^2 &= \frac{1}{\kappa^2} \quad . \end{aligned}$$

This is the equation of a circle with centre (c_1, c_2) and radius $\frac{1}{|\kappa|}$, that is: the (modulus of the) signed curvature is the reciprocal of the radius.

Question 5

(i)



We shall use A, B, and C to also refer to the centres of the circles and consider the above diagram. Since the radius of a circle to a point on its circumference intersects the tangent at that point at right angles, the lines AB, BC, and CA pass exactly through the touching points of the circles, and the lines AP, BQ, and CR are parallel to each other and perpendicular to PQR. We construct the lines XZ and CY parallel to PQR passing through B and C respectively; they are then perpendicular to AP and CR.

By Pythagoras' theorem then

$$\begin{aligned} |PR| &= |QR| + |PQ| \\ \implies \quad |YC| &= |BZ| + |XB| \\ \implies \quad \sqrt{|AC|^2 - |AY|^2} &= \sqrt{|BC|^2 - |CZ|^2} + \sqrt{|AB|^2 - |AX|^2} \\ \sqrt{(a+c)^2 - (a-c)^2} &= \sqrt{(b+c)^2 - (c-b)^2} + \sqrt{(a+b)^2 - (a-b)^2} \\ \sqrt{4ac} &= \sqrt{4bc} + \sqrt{4ab} \\ \implies \quad \frac{\sqrt{ac}}{\sqrt{abc}} &= \frac{\sqrt{bc}}{\sqrt{abc}} + \frac{\sqrt{ab}}{\sqrt{abc}} \\ \frac{1}{\sqrt{b}} &= \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \quad . \end{aligned}$$

Squaring this result twice, we find

$$\begin{aligned} \frac{1}{b} &= \frac{1}{a} + \frac{1}{c} + \frac{2}{\sqrt{ac}} \\ \frac{1}{b} - \frac{1}{a} - \frac{1}{c} &= \frac{2}{\sqrt{ac}} \\ \left(\frac{1}{b} - \frac{1}{a} - \frac{1}{c}\right)^2 &= \frac{4}{ac} \quad , \end{aligned}$$

and now rearranging

$$\begin{aligned}
\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{bc}+\frac{2}{ac} &= \frac{4}{ac} \\
\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} &= \frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc} \\
2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) &= \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc} \\
2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) &= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \quad .
\end{aligned}$$

(ii) Supposing a, b, c are positive numbers satisfying $(**)$ such that $b < c < a$, let $\alpha = \frac{1}{\sqrt{a}}$, $\beta = \frac{1}{\sqrt{b}}$, $\gamma = \frac{1}{\sqrt{c}}$, then we have $\beta > \gamma > \alpha > 0$ and by reversing the rearrangement above we get

$$\left(\frac{1}{b}-\frac{1}{a}-\frac{1}{c}\right)^2 = \frac{4}{ac} \qquad \Longrightarrow \qquad (\beta^2-\alpha^2-\gamma^2)^2 = 4\alpha^2\gamma^2 \quad .$$

Square rooting both sides

$$\begin{aligned}
\beta^2-\alpha^2-\gamma^2 &= \pm 2\alpha\gamma \\
\beta^2 &= \alpha^2 \pm 2\alpha\gamma + \gamma^2 \\
\beta^2 &= (\alpha \pm \gamma)^2 \quad .
\end{aligned}$$

Since $\beta > 0$ and $\gamma > \alpha$ we cannot have $\beta = \alpha - \gamma$. Similarly, since $\alpha > 0$ and $\beta > \gamma$ we cannot have $\beta = \gamma - \alpha$. Hence we must have a $+$ in the brackets on the right-hand side and square rooting again gives

$$\beta = \pm(\alpha + \gamma) \quad .$$

Now since α, β, and γ are all positive we cannot have $\beta = -(\alpha+\gamma)$ and thus we deduce

$$\beta = \alpha + \gamma \quad ,$$

that is:

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \quad .$$

Hence a, b, c satisfy $(*)$.

Question 6

Since OX is parallel to OA we can write

$$\boldsymbol{x} - \boldsymbol{0} = m(\boldsymbol{a} - \boldsymbol{0}) \qquad \Longrightarrow \qquad \boldsymbol{x} = m\boldsymbol{a} \quad ,$$

for some positive scalar m. Since X lies between O and A we must have $0 < m < 1$. Similarly, since CB is parallel to OA we can write

$$\begin{aligned} \boldsymbol{b} - \boldsymbol{c} &= \kappa(\boldsymbol{a} - \boldsymbol{0}) \\ \Longrightarrow \qquad \boldsymbol{c} &= -\kappa\boldsymbol{a} + \boldsymbol{b} \quad , \end{aligned}$$

where κ is some positive scalar (positive because the orientation of $\overrightarrow{CB}$ is the same as $\overrightarrow{OA}$). Writing $k = -\kappa$, we get

$$\boldsymbol{c} = k\boldsymbol{a} + \boldsymbol{b} \quad ,$$

where $k < 0$. Let $\boldsymbol{d}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ be the position vectors of D, Y, and Z respectively. Since D lies on both OB and AC we can write

$$\boldsymbol{d} = \alpha\boldsymbol{b} \qquad \text{and} \qquad \boldsymbol{d} = \boldsymbol{a} + \beta(\boldsymbol{c} - \boldsymbol{a}) \quad ,$$

for some scalars α, β. Eliminating $\boldsymbol{d}$ and substituting in our expression for $\boldsymbol{c}$, we find

$$\begin{aligned} \alpha\boldsymbol{b} &= \boldsymbol{a} + \beta(k\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{a}) \\ (\alpha - \beta)\boldsymbol{b} &= (\beta k - \beta + 1)\boldsymbol{a} \quad . \end{aligned}$$

Since B does not lie on OA, the scalar coefficients in this last equation must both equal zero. That is:

$$\begin{aligned} \alpha = \beta \qquad &, \qquad \beta k - \beta + 1 = 0 \\ \Longrightarrow \qquad &\alpha = \beta = \frac{1}{1-k} \quad , \end{aligned}$$

giving $\boldsymbol{d} = \frac{1}{1-k}\boldsymbol{b}$. Following the same process for $\boldsymbol{y}$: Y lies on both XD and BC so

$$\boldsymbol{y} = \boldsymbol{x} + \gamma(\boldsymbol{d} - \boldsymbol{x}) \qquad \text{and} \qquad \boldsymbol{y} = \boldsymbol{c} + \delta(\boldsymbol{b} - \boldsymbol{c}) \quad ,$$

for some scalars γ, δ. Eliminating $\boldsymbol{y}$ and substituting in for $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{x}$:

$$\begin{aligned} \boldsymbol{x} + \gamma(\boldsymbol{d} - \boldsymbol{x}) &= \boldsymbol{c} + \delta(\boldsymbol{b} - \boldsymbol{c}) \\ m\boldsymbol{a} + \gamma\left(\frac{1}{1-k}\boldsymbol{b} - m\boldsymbol{a}\right) &= k\boldsymbol{a} + \boldsymbol{b} + \delta(\boldsymbol{b} - k\boldsymbol{a} - \boldsymbol{b}) \\ (1-\gamma)m\boldsymbol{a} + \frac{\gamma}{1-k}\boldsymbol{b} &= (1-\delta)k\boldsymbol{a} + \boldsymbol{b} \\ \left(\frac{\gamma}{1-k} - 1\right)\boldsymbol{b} &= ((1-\delta)k - (1-\gamma)m)\boldsymbol{a} \quad . \end{aligned}$$

Again these scalar coefficients must both equal zero, hence

$$\begin{aligned} &\gamma = 1-k \quad , \quad (1-\delta)k - km = 0 \\ \implies \quad &\gamma = 1-k \quad , \quad \delta = 1-m \ . \end{aligned}$$

This gives $\boldsymbol{y} = (1-\gamma)\boldsymbol{x} + \gamma\boldsymbol{d} = km\boldsymbol{a} + \boldsymbol{b}$.

Lastly, we repeat this process for $\boldsymbol{z}$: Z lies on both OY and AB hence

$$\boldsymbol{z} = \lambda\boldsymbol{y} \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{a} + \mu(\boldsymbol{b}-\boldsymbol{a}) \ ,$$

for some scalars λ, μ giving

$$\begin{aligned} \lambda(mk\boldsymbol{a} + \boldsymbol{b}) &= (1-\mu)\boldsymbol{a} + \mu\boldsymbol{b} \\ (\lambda-\mu)\boldsymbol{b} &= (1-\mu-\lambda mk)\boldsymbol{a} \ . \end{aligned}$$

Hence

$$\begin{aligned} &\lambda = \mu \quad , \quad 1-\mu-\mu mk = 0 \\ \implies \quad &\lambda = \mu = \frac{1}{mk+1} \ . \end{aligned}$$

Thus we have

$$\boldsymbol{z} = \frac{1}{mk+1}\boldsymbol{y} = \frac{mk\boldsymbol{a} + \boldsymbol{b}}{mk+1} \ .$$

Let $\boldsymbol{t}$ be the position vector of T. Since T lies on both DZ and OA we can write

$$\boldsymbol{t} = \boldsymbol{z} + \rho(\boldsymbol{d}-\boldsymbol{z}) \quad \text{and} \quad \boldsymbol{t} = \sigma\boldsymbol{a} \ ,$$

for some scalars ρ, σ. By the same methods as above we find

$$\begin{aligned} (1-\rho)\boldsymbol{z} + \rho\boldsymbol{d} &= \sigma\boldsymbol{a} \\ \frac{1-\rho}{mk+1}(mk\boldsymbol{a} + \boldsymbol{b}) + \frac{\rho}{1-k}\boldsymbol{b} &= \sigma\boldsymbol{a} \\ (1-\rho)(1-k)(mk\boldsymbol{a} + \boldsymbol{b}) + \rho(mk+1)\boldsymbol{b} &= \sigma(1-k)(mk+1)\boldsymbol{a} \\ (1-\rho-k+\rho k+\rho mk+\rho)\boldsymbol{b} &= (1-k)(\sigma(mk+1) - (1-\rho)mk)\boldsymbol{a} \\ (1-k+\rho(m+1)k)\boldsymbol{b} &= (1-k)(\sigma(mk+1) + \rho mk - mk)\boldsymbol{a} \ , \end{aligned}$$

thus $\rho = \frac{k-1}{(m+1)k}$ and

$$\begin{aligned} \sigma(mk+1) + \frac{m(k-1)}{m+1} - mk &= 0 \\ \sigma(mk+1) &= mk - \frac{m(k-1)}{m+1} \\ &= \frac{m^2k + mk - mk + m}{m+1} = \frac{m^2k+m}{m+1} \\ \implies \quad \sigma &= \frac{1}{mk+1}\frac{m^2k+m}{m+1} = \frac{m}{m+1} \ . \end{aligned}$$

Hence $\boldsymbol{t} = \frac{m}{m+1}\boldsymbol{a}$.

With these results, we have

$$|OT| \cdot |OA| = \frac{m}{m+1}|\boldsymbol{a}| \cdot |\boldsymbol{a}| = \frac{m}{m+1}|\boldsymbol{a}|^2 \quad ,$$

and

$$|OX| \cdot |TA| = m|\boldsymbol{a}| \cdot \left(1 - \frac{m}{m+1}\right)|\boldsymbol{a}| = \frac{m}{m+1}|\boldsymbol{a}|^2 \quad ,$$

that is: $|OT| \cdot |OA| = |OX| \cdot |TA|$. Likewise

$$\frac{1}{|OT|} = \frac{m+1}{m} \cdot \frac{1}{|\boldsymbol{a}|} \quad ,$$

and

$$\frac{1}{|OX|} + \frac{1}{|OA|} = \frac{1}{m} \cdot \frac{1}{|\boldsymbol{a}|} + \frac{1}{|\boldsymbol{a}|} = \frac{m+1}{m} \cdot \frac{1}{|\boldsymbol{a}|} \quad ,$$

that is:

$$\frac{1}{|OT|} = \frac{1}{|OX|} + \frac{1}{|OA|} \quad .$$

Question 7

(i) Since every positive integer that leaves an odd remainder upon division by 4 is necessarily odd, and all positive odd integers will either leave either a remainder of 1 or 3 upon division by 4, the set $S \cup T$ is the set of all odd positive integers. Since no integer can leave a remainder of 1 *and* a remainder of 3 upon division by 4, the set $S \cap T$ is empty.

(ii) Suppose integers a and b are in S, then we must have

$$a = 4n_1 + 1 \qquad \text{and} \qquad b = 4n_2 + 1 \quad ,$$

for some integers $n_1, n_2 \in \{0, 1, 2, \dots\}$. Then

$$\begin{aligned} ab &= 16n_1n_2 + 4(n_1 + n_2) + 1 \\ &= 4(4n_1n_2 + n_1 + n_2) + 1 \\ \implies \qquad ab &\equiv 1 \mod 4 \quad , \end{aligned}$$

hence ab is also in S. Similarly, suppose c and d are in T, then we must have

$$c = 4n_3 + 3 \qquad \text{and} \qquad d = 4n_4 + 3 \quad ,$$

for some integers $n_3, n_4 \in \{0, 1, 2, \dots\}$. Then

$$\begin{aligned} cd &= 16n_3n_4 + 12(n_3 + n_4) + 9 \\ &= 4(4n_3n_4 + 3n_3 + 3n_4 + 2) + 1 \\ \implies \qquad cd &\equiv 1 \mod 4 \quad , \end{aligned}$$

hence cd is not in T, in fact cd is in S.

As an extra remark, if f is in S and g is in T, then we must have

$$f = 4n_5 + 1 \qquad \text{and} \qquad g = 4n_6 + 3 \quad ,$$

for some integers $n_5, n_6 \in \{0, 1, 2, \dots\}$. Then

$$\begin{aligned} fg &= 16n_5n_6 + 4(3n_5 + n_6) + 3 \\ &= 4(4n_5n_6 + 3n_5 + n_6) + 3 \\ \implies \qquad fg &\equiv 3 \mod 4 \quad , \end{aligned}$$

hence fg is in T.

(iii) Let m be an integer in T that is not a prime number. Trivially m must be odd, and so all its prime factors are odd (that is: 2 is not a prime factor of m) so each prime factor of m is either in S or in T. Suppose that m has no prime factors in T; then m is equal to a product of integers in S. By (ii) this implies that $m \in S$, and since $S \cap T$ is empty, this implies $m \notin T$, which is a contradiction. Thus we deduce that m must have at least one prime factor in T.

(iv) (a) Let q be an integer in T and suppose q is not T-prime, so we can write q as a product of integers all in T. Each of these factors is either T-prime or can be decomposed further as a product of integers that are all in T. Iterating this, we can eventually write q as a product of integers that are all T-prime. Suppose the number of T-primes in this factorisation is r. If r is even then by (ii) we could pair up the factors and multiply each pair together to give an integer in S. Multiplying all the factors together would then be a product of integers in S giving $q \in S$ – a contradiction. Hence r must be odd. That is: either q is T-prime, or q can be written as the product of an odd number of T-prime integers.

(b) Consider the elements of S

$$S = \{1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, ...\} \quad .$$

We want to find two distinct pairs of S-prime integers that have the same product (by (ii) this product is necessarily in S). By definition, 1 is not S-prime, so as the next smallest integer in S, 5 *is* S-prime. By (ii) any other multiple of 5 that is in S must be the product of 5 with another integer in S, hence in searching for S-prime integers, we eliminate all greater multiples of 5. This leaves that 9 is S-prime (and again, all greater multiples of 9 in S must not be S-prime). Iterating this we find the first few S-prime integers:

$$S_{\text{prime}} = \{5, 9, 13, 17, 21, 29, 33, 37, 41, 49, 53, 57, 61, 69, 73, 77, ...\} \quad .$$

From these we can pick out, for example

$$\begin{aligned} 9 \cdot 77 &= 3^2 \cdot 7 \cdot 11 \\ &= 3 \cdot 7 \cdot 3 \cdot 11 \\ &= 21 \cdot 33 \quad , \end{aligned}$$

where 9, 21, 33, and 77 are all S-prime integers. Our example is thus $9 \cdot 77 = 21 \cdot 33 = 693$ (which as noted must be in S by the result in (ii)).

Question 8

(i) The binomial series of $(1-x)^{-2}$ is $1+2x+3x^2+4x^3+\dots$, hence setting $u_n = n$ gives

$$\begin{aligned} f(x) &= 0 + x + 2x^2 + 3x^3 + \dots \\ &= x(1 + 2x + 3x^2 + \dots) \\ &= x(1-x)^{-2} \quad . \end{aligned}$$

By using the binomial series, if we instead set $f(x) = x(1-x)^{-3}$ we find

$$\begin{aligned} f(x) = x(1-x)^{-3} &= x\sum_{n=0}^{\infty} \frac{(n+2)!}{n! \cdot 2!} x^n \\ &= x\sum_{n=0}^{\infty} \frac{1}{2}(n+1)(n+2)x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(n+1)(n+2)x^{n+1} \\ &= \sum_{n=1}^{\infty} \frac{1}{2}n(n+1)x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2}n(n+1)x^n \quad , \end{aligned}$$

thus the sequence $u_n = \frac{1}{2}n(n+1)$ has generating function $f(x) = x(1-x)^{-3}$. For the sequence $u_n = n^2$, we can then write

$$\begin{aligned} \sum_{n=0}^{\infty} n^2 x^n &= \sum_{n=0}^{\infty} \left(2 \cdot \frac{1}{2}n(n+1) - n \right) x^n \\ &= 2\sum_{n=0}^{\infty} \frac{1}{2}n(n+1)x^n - \sum_{n=0}^{\infty} nx^n \\ &= 2x(1-x)^{-3} - x(1-x)^{-2} \\ &= x(1-x)^{-3}(2-(1-x)) \\ &= x(1+x)(1-x)^{-3} \quad . \end{aligned}$$

That is: the sequence $u_n = n^2$ has generating function $f(x) = x(1+x)(1-x)^{-3}$.

(ii) (a) Summing the identity $u_n x^n = k u_{n-1} x^n$ over n we find

$$\begin{aligned}
\sum_{n=1}^{\infty} u_n x^n &= \sum_{n=1}^{\infty} k u_{n-1} x^n \\
\sum_{n=0}^{\infty} u_n x^n &= u_0 + k \sum_{n=1}^{\infty} u_{n-1} x^n \\
&= a + kx \sum_{n=1}^{\infty} u_{n-1} x^{n-1} \\
\sum_{n=0}^{\infty} u_n x^n &= a + kx \sum_{n=0}^{\infty} u_n x^n \\
\implies f(x) &= a + kx f(x) \quad .
\end{aligned}$$

Rearranging this we have

$$\begin{aligned}
(1 - kx) f(x) &= a \\
f(x) &= \frac{a}{1 - kx} \quad .
\end{aligned}$$

(b) Summing the identity $u_n x^n = u_{n-1} x^n + u_{n-2} x^n$ over n we find

$$\begin{aligned}
\sum_{n=2}^{\infty} u_n x^n &= \sum_{n=2}^{\infty} u_{n-1} x^n + \sum_{n=2}^{\infty} u_{n-2} x^n \\
\sum_{n=0}^{\infty} u_n x^n &= u_0 + u_1 x + x \sum_{n=2}^{\infty} u_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} u_{n-2} x^{n-2} \\
&= 0 + x + x \sum_{n=1}^{\infty} u_n x^n + x^2 \sum_{n=0}^{\infty} u_n x^n \\
\sum_{n=0}^{\infty} u_n x^n &= x + x \sum_{n=0}^{\infty} u_n x^n + x^2 \sum_{n=0}^{\infty} u_n x^n \\
f(x) &= x + (x + x^2) f(x) \\
(1 - x - x^2) f(x) &= x \\
f(x) &= \frac{x}{1 - x - x^2} \quad .
\end{aligned}$$

Section B: Mechanics

Question 9

Since $a\sin\theta < d$ the centre of mass of the rod is to the left of the rail. Let the weight of the rod be W. Let F_r and F_w be the frictional forces on the rod from the rail and the wall respectively. Similarly let R_r and R_w be the normal reaction forces on the rod from the rail and the wall respectively. In limiting equilibrium with the rod on the point of slipping at both the wall and the rail we have $F_r = \lambda R_r$ and $F_w = \mu R_w$, with F_r directed towards B and F_w directed vertically upwards.

Resolving forces on the rod horizontally, we get

$$\begin{aligned} R_w + F_r \sin\theta &= R_r \cos\theta \\ \implies \qquad R_w &= R_r(\cos\theta - \lambda\sin\theta) \quad , \end{aligned} \tag{$*$}$$

and resolving forces on the rod vertically, we get

$$\begin{aligned} W &= F_w + R_r\sin\theta + F_r\cos\theta \\ \implies \qquad W &= \mu R_w + R_r(\sin\theta + \lambda\cos\theta) \quad . \end{aligned} \tag{$**$}$$

Taking moments about A, we get

$$\begin{aligned} Wa\sin\theta &= R_r\frac{d}{\sin\theta} \\ W &= \frac{R_r d}{a\sin^2\theta} \quad . \end{aligned}$$

Eliminating W from ($**$) we get

$$\frac{R_r d}{a\sin^2\theta} = \mu R_w + R_r(\sin\theta + \lambda\cos\theta) \quad ,$$

and then using ($*$), we can eliminate R_w and cancel factors of R_r:

$$\begin{aligned} \frac{R_r d}{a\sin^2\theta} &= \mu R_r(\cos\theta - \lambda\sin\theta) + R_r(\sin\theta + \lambda\cos\theta) \\ d\,\text{cosec}^2\theta &= a\left(\mu(\cos\theta - \lambda\sin\theta) + \sin\theta + \lambda\cos\theta\right) \\ d\,\text{cosec}^2\theta &= a\left((\lambda+\mu)\cos\theta + (1-\lambda\mu)\sin\theta\right) \quad . \end{aligned}$$

If instead $a\sin\theta > d$ the centre of mass of the rod is to the right of the rail. In this case, at the limiting equilibrium with the rod on the point of slipping at both the rod and the wall the frictional forces F_r and F_w act in the opposite directions to before. Accordingly, by changing the signs of F_r and F_w in the above (equivalent to changing the signs of λ and μ), we get the corresponding result in this case:

$$d\,\text{cosec}^2\theta = a\left((-\lambda-\mu)\cos\theta + (1-\lambda\mu)\sin\theta\right) \quad .$$

Question 10

(i) In the collision between A and B, let v_a, v_b be the particle's velocities after the collision. By conservation of momentum

$$\begin{aligned} \lambda m v_a + m v_b &= \lambda m u \\ \implies \qquad \lambda v_a + v_b &= \lambda u \quad , \end{aligned}$$

and by the law of restitution

$$v_b - v_a = eu \quad .$$

Solving for v_b, we have

$$\begin{aligned} \lambda v_b - \lambda v_a &= \lambda e u \\ \lambda v_b + v_b &= \lambda e u + \lambda u \\ v_b &= \frac{\lambda(1+e)}{\lambda+1} u \quad . \end{aligned}$$

Similarly, by setting $\lambda = 1$ in this, we can see that after the collision between C and D, particle C is travelling towards B with speed v_c given by

$$v_c = \frac{1}{2}(1+e)u \quad ,$$

and using $v_c - v_d = eu$, we find that particle D is travelling (in the same direction as it was initially projected) with speed

$$v_d = v_c - eu = \frac{1}{2}(1-e)u \quad .$$

Then in the collision between B and C, we are told B comes to rest and we let w_c be the final velocity of C (back towards D). By conservation of momentum

$$\begin{aligned} w_c m &= \frac{\lambda(1+e)}{\lambda+1} um - \frac{1}{2}(1+e)um \\ w_c &= (1+e)\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right) u \\ w_c &= (1+e)\frac{\lambda-1}{2(\lambda+1)} u \quad , \end{aligned}$$

and then by the law of restitution

$$\begin{aligned} w_c &= e(v_b + v_c) \\ &= e(1+e)\left(\frac{\lambda}{\lambda+1} + \frac{1}{2}\right) u \\ \implies \qquad (1+e)\frac{\lambda-1}{2(\lambda+1)} u &= e(1+e)\left(\frac{\lambda}{\lambda+1} + \frac{1}{2}\right) u \\ \frac{\lambda-1}{2(\lambda+1)} &= e\frac{3\lambda+1}{2(\lambda+1)} \\ \implies \qquad e &= \frac{\lambda-1}{3\lambda+1} \quad . \end{aligned}$$

Rearranging this a little

$$\begin{aligned} e &= \frac{\frac{1}{3}(3\lambda+1) - \frac{4}{3}}{3\lambda+1} \\ &= \frac{1}{3} - \frac{4}{3} \cdot \frac{1}{3\lambda+1} \quad , \end{aligned}$$

and so since $\lambda > 0$, we deduce that $e < \frac{1}{3}$.

(ii) Given that C and D are then moving towards each other with the same speed, we equate

$$\begin{aligned} w_c &= v_d \\ \Longrightarrow \qquad (1+e)\frac{\lambda-1}{2(\lambda+1)}u &= \frac{1}{2}(1-e)u \\ \Longrightarrow \qquad \left(1 + \frac{\lambda-1}{3\lambda+1}\right)\frac{\lambda-1}{2(\lambda+1)} &= \frac{1}{2}\left(1 - \frac{\lambda-1}{3\lambda+1}\right) \\ (3\lambda+1+\lambda-1)(\lambda-1) &= (\lambda+1)(3\lambda+1-\lambda+1) \\ 4\lambda(\lambda-1) &= (\lambda+1)(2\lambda+2) \\ 2\lambda^2 - 8\lambda - 2 &= 0 \\ \lambda^2 - 4\lambda - 1 &= 0 \\ \Longrightarrow \qquad \lambda &= 2+\sqrt{5} \quad , \end{aligned}$$

where we have taken the positive square root, because $2-\sqrt{5} < 0$ but we are given that $\lambda > 1$. Substituting this value of λ into the expression for e, we find

$$\begin{aligned} e &= \frac{1+\sqrt{5}}{7+3\sqrt{5}} = \frac{(1+\sqrt{5})(7-3\sqrt{5})}{49-45} \\ &= \frac{7-15+4\sqrt{5}}{4} \\ e &= \sqrt{5}-2 \quad . \end{aligned}$$

Question 11

Parametrically, the trajectory of the particle is given by

$$x(t) = u\cos\alpha\, t \qquad , \qquad y(t) = u\sin\alpha\, t - \tfrac{1}{2}gt^2 \quad .$$

When the particle hits the plain, we have $y = -h$ giving

$$\begin{aligned} -h &= u\sin\alpha\, t - \tfrac{1}{2}gt^2 \\ gt^2 &= 2h + 2u\sin\alpha\, t \quad . \end{aligned}$$

Using the equation for the x-coordinate to substitute for t, we get

$$\begin{aligned} g\left(\frac{x}{u\cos\alpha}\right)^2 &= 2h + 2u\sin\alpha\frac{x}{u\cos\alpha} \\ \frac{gx^2}{u^2} &= 2h\cos^2\alpha + 2\sin\alpha\cos\alpha\, x \\ &= 2h\cos^2\alpha + x\sin 2\alpha \\ \frac{gx^2}{u^2} &= h(1+\cos 2\alpha) + x\sin 2\alpha \quad . \end{aligned}$$

Differentiating this with respect to α

$$\frac{2gx}{u^2}\frac{dx}{d\alpha} = -2h\sin 2\alpha + \sin 2\alpha\frac{dx}{d\alpha} + 2x\cos 2\alpha \quad ,$$

hence when $\frac{dx}{d\alpha} = 0$ we get

$$\begin{aligned} -2h\sin 2\alpha + 2x\cos 2\alpha &= 0 \\ x &= h\tan 2\alpha \quad . \end{aligned}$$

Substituting in this value for x we get

$$\begin{aligned} \frac{gh^2\tan^2 2\alpha}{u^2} &= h(1+\cos 2\alpha) + h\frac{\sin^2 2\alpha}{\cos 2\alpha} \\ gh\sin^2 2\alpha &= u^2(1+\cos 2\alpha)\cos^2 2\alpha + u^2\sin^2 2\alpha\cos 2\alpha \\ gh(1-\cos^2 2\alpha) &= u^2(1+\cos 2\alpha)\cos^2 2\alpha + u^2(1-\cos^2 2\alpha)\cos 2\alpha \\ gh - gh\cos^2 2\alpha &= u^2\cos^2 2\alpha + u^2\cos 2\alpha \\ (u^2+gh)\cos^2 2\alpha + u^2\cos 2\alpha - gh &= 0 \\ ((u^2+gh)\cos 2\alpha - gh)(\cos 2\alpha + 1) &= 0 \quad . \end{aligned}$$

We ignore the root $\cos 2\alpha = -1$, since this gives $\alpha = \frac{1}{2}\pi$ and the particle would be projected vertically upwards, hence we have

$$\begin{aligned} (u^2+gh)\cos 2\alpha - gh &= 0 \\ \cos 2\alpha &= \frac{gh}{u^2+gh} \quad . \end{aligned}$$

The greatest achievable distance d between O and the landing point thus satisfies

$$\begin{aligned} d^2 &= x^2 + h^2 = h^2 \tan^2 2\alpha + h^2 \\ &= h^2(\tan^2 2\alpha + 1) \\ &= h^2 \sec^2 2\alpha \\ &= h^2 \cdot \frac{(u^2 + gh)^2}{g^2 h^2} \\ &= \left(\frac{u^2}{g} + h \right)^2 \quad , \end{aligned}$$

and square rooting both sides, since $d > 0$ and $\frac{u^2}{g} + h > 0$ we deduce

$$d = \frac{u^2}{g} + h \quad .$$

Section C: Probability and Statistics

Question 12

Let A be the number of heads that Alice gets and B be the number of heads that Bob gets.

(i) By considering the possible values of A and B such that $B > A$, we get

$$\begin{aligned}\mathbb{P}\{B > A\} &= \mathbb{P}\{B = 3\} + \mathbb{P}\{B = 2\} \cdot \mathbb{P}\{A \neq 2\} + \mathbb{P}\{B = 1\} \cdot \mathbb{P}\{A = 0\} \\ &= \frac{1}{8} + \left(3 \cdot \frac{1}{8}\right)\left(1 - \frac{1}{4}\right) + \left(3 \cdot \frac{1}{8}\right)\left(\frac{1}{4}\right) \\ &= \frac{4}{32} + \frac{9}{32} + \frac{3}{32} = \frac{16}{32} \\ &= \frac{1}{2} \quad .\end{aligned}$$

(ii) Again by considering the possible values of A and B, we get

$$\begin{aligned}\mathbb{P}\{B > A\} &= \mathbb{P}\{B = 4\} + \mathbb{P}\{B = 3\} \cdot \mathbb{P}\{A \neq 3\} \\ &\qquad + \mathbb{P}\{B = 2\} \cdot (\mathbb{P}\{A = 1\} + \mathbb{P}\{A = 0\}) + \mathbb{P}\{B = 1\} \cdot \mathbb{P}\{A = 0\} \\ &= \frac{1}{16} + \frac{4}{16} \cdot \left(1 - \frac{1}{8}\right) + \frac{6}{16} \cdot \left(\frac{3}{8} + \frac{1}{8}\right) + \frac{4}{16} \cdot \frac{1}{8} \\ &= \frac{8}{128} + \frac{28}{128} + \frac{24}{128} + \frac{4}{128} = \frac{64}{128} \\ &= \frac{1}{2} \quad .\end{aligned}$$

(iii) By considering the number of heads Bob gets on the first n tosses, the probability that Bob gets more heads than Alice after the $(n+1)$-th toss is

$$\mathbb{P}\{B > A\} = p_1 \cdot \mathbb{P}\{\text{head on } (n+1)\text{-th toss}\} + p_2 = \frac{1}{2}p_1 + p_2 \quad .$$

Considering only the first n tosses, by symmetry the probability that Bob gets more heads than Alice is the same as the probability that Alice gets more heads than Bob. Since either Bob gets more heads than Alice or Alice gets more than Bob or they get the same number, we have

$$\begin{aligned}p_2 + p_2 + p_1 &= 1 \\ 2p_2 + p_1 &= 1 \quad ,\end{aligned}$$

hence our probability that Bob gets more heads in $n+1$ tosses than Alice does in n tosses is

$$\mathbb{P}\{B > A\} = \frac{1}{2}p_1 + p_2 = \frac{1}{2}(2p_1 + p_2) = \frac{1}{2} \quad ,$$

and this holds for any value of $n \geqslant 1$.

Question 13

Let $X_1, \ldots, X_n$ be the arrival times of the n emails. We note that these are independent. For each of these arrival times for $0 \leqslant t < \infty$ we have

$$\mathbb{P}\{X_i \leqslant t\} = \int_0^t \lambda e^{-\lambda\tau} d\tau = \Big[-e^{-\lambda\tau}\Big]_0^t = 1 - e^{-\lambda t} \quad ,$$

hence also

$$\mathbb{P}\{X_i \geqslant t\} = e^{-\lambda t} \quad .$$

(i) We have $T = \min\{X_1, \ldots, X_n\}$ and so for $0 \leqslant t < \infty$ we get

$$\begin{aligned}
\mathbb{P}\{T \leqslant t\} &= 1 - \mathbb{P}\{T \geqslant t\} \\
&= 1 - \mathbb{P}\{\min\{X_1, \ldots, X_n\} \geqslant t\} \\
&= 1 - \mathbb{P}\{X_1 \geqslant t,\, X_2 \geqslant t,\, \ldots,\, X_n \geqslant t\} \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}\{X_i \geqslant t\} \\
&= 1 - \left(e^{-\lambda t}\right)^n \\
&= 1 - e^{-n\lambda t} \quad .
\end{aligned}$$

Differentiating this, we find the probability density function of T:

$$f_T(t) = \frac{d}{dt}\left(1 - e^{-n\lambda t}\right) = n\lambda e^{-n\lambda t} \quad .$$

From this we can compute the expectation of T

$$\begin{aligned}
\mathbb{E}(T) &= \int_0^\infty n\lambda t e^{-n\lambda t} dt \\
&= \Big[-te^{-n\lambda t}\Big]_0^\infty + \int_0^\infty e^{-n\lambda t} dt \\
&= 0 + \left[-\frac{1}{n\lambda} e^{-n\lambda t}\right]_0^\infty \\
&= \frac{1}{n\lambda} \quad .
\end{aligned}$$

(ii) Let T_2 be the arrival time of the second email to arrive at its destination. By considering the possible arrival times of the first email, we can write

$$\mathbb{P}\{T_2 \geqslant t\} = \mathbb{P}\{T \geqslant t\} + \mathbb{P}\{T \leqslant t,\, T_2 \geqslant t\} \quad .$$

By the above, we have $\mathbb{P}\{T \geqslant t\} = e^{-n\lambda t}$. For the second probability; the condition that $T \leqslant t$ and $T_2 \geqslant t$ is the condition that exactly one of the n emails arrives before time t and all the rest arrive after time t. There are n distinct ways this can occur (determined by which of the n emails it is that arrives first), each with equal probability, thus we can find

$$\begin{aligned}\mathbb{P}\{T \leqslant t,\, T_2 \geqslant t\} &= n \cdot \mathbb{P}\{X_1 \leqslant t\} \prod_{i=2}^{n} \mathbb{P}\{X_i \geqslant t\} \\ &= n\left(1 - e^{-\lambda t}\right)\left(e^{-\lambda t}\right)^{n-1} \\ &= ne^{-(n-1)\lambda t}(1 - e^{-\lambda t}) \\ &= ne^{-(n-1)\lambda t} - ne^{-n\lambda t}\end{aligned}$$

Hence the probability that the second email to arrive at its destination arrives after time t is

$$\mathbb{P}\{T_2 \geqslant t\} = e^{-n\lambda t} + ne^{-(n-1)\lambda t} - ne^{-n\lambda t} \quad .$$

This gives

$$\mathbb{P}\{T_2 \leqslant t\} = 1 - e^{-n\lambda t} - ne^{-(n-1)\lambda t} + ne^{-n\lambda t} \quad ,$$

and differentiating this we get the probability density for T_2:

$$\begin{aligned}f_{T_2}(t) &= \frac{d}{dt}\left(1 - e^{-n\lambda t} - ne^{-(n-1)\lambda t} + ne^{-n\lambda t}\right) \\ &= n\lambda e^{-n\lambda t} + n(n-1)\lambda e^{-(n-1)\lambda t} - n^2\lambda e^{-n\lambda t} \\ &= n\lambda e^{-(n-1)\lambda t}\left(e^{-\lambda t} + (n-1) - ne^{-\lambda t}\right) \\ &= n(n-1)\lambda e^{-(n-1)\lambda t}\left(1 - e^{-\lambda t}\right) \quad , \qquad 0 \leqslant t < \infty \quad .\end{aligned}$$

The expectation of the time of arrival of the second email is thus

$$\begin{aligned}\mathbb{E}(T_2) &= \int_0^\infty t f_{T_2}(t)dt = \int_0^\infty n(n-1)\lambda t e^{-(n-1)\lambda t}\left(1 - e^{-\lambda t}\right)dt \\ &= n\int_0^\infty (n-1)\lambda t e^{-(n-1)\lambda t}dt - (n-1)\int_0^\infty n\lambda t e^{-n\lambda t}dt \\ &= n \cdot \frac{1}{(n-1)\lambda} - (n-1) \cdot \frac{1}{n\lambda} \\ &= \frac{1}{\lambda}\left(\frac{n}{n-1} - \frac{n-1}{n}\right) \\ &= \frac{1}{\lambda}\left(1 + \frac{1}{n-1} - 1 + \frac{1}{n}\right) \\ &= \frac{1}{\lambda}\left(\frac{1}{n-1} + \frac{1}{n}\right) \quad .\end{aligned}$$

STEP II

Section A: Pure Mathematics

Question 1

Let P be the point on the curve with parameter p and Q be the point on the curve with parameter q. Thus the gradient of the line OP is $\frac{p^3}{p^2} = p$ and similarly the gradient of OQ is q. Since $\angle POQ$ is a right angle we thus deduce that $pq = -1$.

Differentiating, the gradient of C_1 at the point with parameter t is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \quad ,$$

hence we find the equation of the tangent to C_1 at P:

$$\begin{aligned} y - p^3 &= \frac{3}{2}p(x - p^2) \\ y &= \frac{3}{2}px - \frac{1}{2}p^3 \\ y &= \frac{1}{2}p(3x - p^2) \quad . \end{aligned}$$

Similarly, the tangent to C_1 at Q is

$$y = \frac{1}{2}q(3x - q^2) \quad .$$

Combining these, at the intersection of the two tangents we have

$$\begin{aligned} \frac{1}{2}p(3x - p^2) &= \frac{1}{2}q(3x - q^2) \\ 3x(p - q) &= p^3 - q^3 \\ 3x &= p^2 + pq + q^2 \qquad\qquad (p \neq q) \\ x &= \frac{1}{3}(p^2 + pq + q^2) \quad , \end{aligned}$$

giving

$$\begin{aligned} y &= \frac{1}{2}p(pq + q^2) \\ &= \frac{1}{2}pq(p + q) \quad . \end{aligned}$$

Using $pq = -1$ we can rearrange these:

$$x = \frac{1}{3}((p + q)^2 - pq) = \frac{1}{3}((p + q)^2 + 1)$$

$$\text{and} \qquad y = -\frac{1}{2}(p + q) \quad .$$

Eliminating $p+q$, we see that the point of intersection satisfies

$$\begin{aligned} x &= \frac{1}{3}((-2y)^2+1) \\ 3x &= 4y^2+1 \\ \implies \qquad 4y^2 &= 3x-1 \quad , \end{aligned}$$

the equation given for C_2.

We can write the equation for C_1 non-parametrically:

$$x = t^2 \quad , \quad y = t^3 \qquad \implies \qquad y^2 = x^3 \quad ,$$

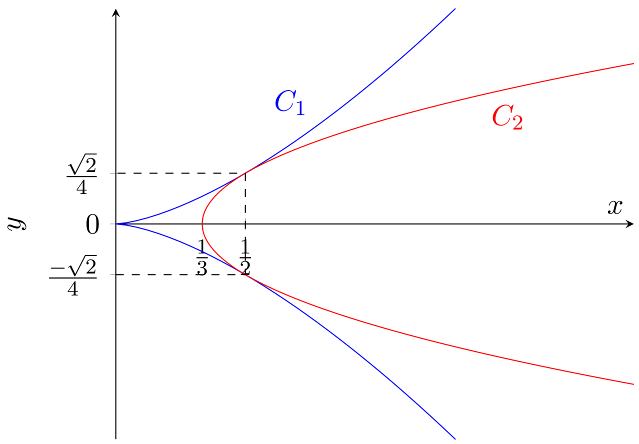
which is a bi-cubic in $x \geqslant 0$. To see if C_1 and C_2 intersect then, we attempt to solve these simultaneously for x and y:

$$\begin{aligned} 4(x^3) &= 3x-1 \\ 4x^3-3x+1 &= 0 \\ (x+1)(4x^2-4x+1) &= 0 \\ (x+1)(2x-1)^2 &= 0 \quad . \end{aligned}$$

We ignore the root $x=-1$, since C_1 lies entirely in the right-half plane, the remaining root $x=\frac{1}{2}$ gives two solutions:

$$x = \frac{1}{2} \qquad \implies \qquad y^2 = \frac{1}{8} \qquad \implies \qquad y = \pm\frac{\sqrt{2}}{4} \quad .$$

So the curves C_1 and C_2 intersect at exactly two points. Our sketch is as follows.



Question 2

If $a+b-c=0$, then we have $c=a+b$ and

$$\begin{aligned}(a+b+c)^3 &- 6(a+b+c)(a^2+b^2+c^2)+8(a^3+b^3+c^3)\\ &= (2a+2b)^3-6(2a+2b)(a^2+b^2+(a+b)^2)+8(a^3+b^3+(a+b)^3)\\ &= 8(a+b)^3-12(a+b)(2(a+b)^2-2ab)+8(2(a+b)^3-3(a^2b+ab^2))\\ &= (8-24+16)(a+b)^3+24ab(a+b)-24(a^2b+ab^2)\\ &= 0 \quad ,\end{aligned}$$

hence $a+b-c$ is a factor of $(*)$.

By symmetry, $a-b+c$ and $-a+b+c$ are also factors of $(*)$. Since these are three distinct factors and $(*)$ is cubic in a, b, c these are the only factors and we have

$$\begin{aligned}(a+b+c)^3-6(a+b+c)(a^2+b^2+c^2)+8(a^3+b^3+c^3)&\\ = k(a+b-c)(a-b+c)(-a+b+c) \quad ,&\end{aligned}$$

for some constant k. The coefficient of a^3 in $(*)$ is $1-6+8=3$, hence $k=-3$, and

$$\begin{aligned}(a+b+c)^3-6(a+b+c)(a^2+b^2+c^2)+8(a^3+b^3+c^3)&\\ = 3(a+b-c)(a-b+c)(a-b-c) \quad .&\end{aligned}$$

(i) To link the given cubic in x with $(*)$ we want to choose a, b, c to satisfy

$$a+b+c=x+1 \quad , \quad a^2+b^2+c^2=x^2+\frac{5}{2} \quad , \quad a^3+b^3+c^3=x^3+\frac{13}{4} \quad .$$

We choose $a=x$ arbitrarily, then we need b and c to satisfy

$$b+c=1 \quad , \quad b^2+c^2=\frac{5}{2} \quad , \quad b^3+c^3=\frac{13}{4} \quad . \qquad (**)$$

Manipulating these equations we find

$$\begin{aligned} & b^3+c^3=(b+c)(b^2-bc+c^2)=(b+c)((b+c)^2-3bc)\\ \implies \quad & \frac{13}{4}=1-3bc\\ & bc=-\frac{3}{4} \quad ,\end{aligned}$$

and then

$$\begin{aligned} & b^2+c^2=(b-c)^2+2bc\\ \implies & \frac{5}{2}=(b-c)^2-\frac{3}{2}\\ & (b-c)^2=4 \quad .\end{aligned}$$

Suppose (without loss of generality) that $b \geqslant c$, so we may take square roots of this final result to get $b - c = 2$. Then we have

$$b + c = 1 \quad , \quad b - c = 2 \qquad \Longrightarrow \qquad b = \frac{3}{2} \quad , \quad c = -\frac{1}{2} \quad .$$

Since $(**)$ is a set of three simultaneous equations in two variables, we check that these values of b and c do in fact satisfy all three equations:

$$\frac{3}{2} - \frac{1}{2} = 1 \quad , \quad \left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{10}{4} = \frac{5}{2} \quad , \quad \left(\frac{3}{2}\right)^3 + \left(-\frac{1}{2}\right)^3 = \frac{26}{8} = \frac{13}{4} \quad .$$

Now our factorisation of $(*)$ gives that

$$\begin{aligned}(x+1)^3 - 3(x+1)(2x^2+5) + 2(4x^3+13)& \\ &= 3\left(x + \frac{3}{2} + \frac{1}{2}\right)\left(x - \frac{3}{2} - \frac{1}{2}\right)\left(x - \frac{3}{2} + \frac{1}{2}\right) \\ &= 3\,(x+2)\,(x-2)\,(x-1) \quad ,\end{aligned}$$

So the solutions are $x = 2$, $x = 1$, $x = -2$.

(ii) Setting $c = d + e$ in $(*)$ we have that $a + b - d - e$ is a factor of

$$\begin{aligned}&(a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+(d+e)^2) + 8(a^3+b^3+(d+e)^3) \\ &\quad = (a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3) \\ &\qquad\qquad - 6(a+b+d+e)(2de) + 8(3d^2e+3de^2) \\ &\quad = (a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3) \\ &\qquad\qquad - 12de(a+b+d+e) + 12de(2d+2e) \\ &\quad = (a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3) \\ &\qquad\qquad - 12de(a+b-d-e) \quad .\end{aligned}$$

Since $a + b - d - e$ is clearly a factor of the final term here, we therefore deduce that it is also a factor of

$$(a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3) \quad .$$

Again by symmetry, we know that the other factors are $a - b - d + e$ and $a + b - d + e$, and by considering the coefficients of a^3 we again have a constant factor of $k = 3$, that is:

$$\begin{aligned}(a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3)& \\ = 3(a+b-d-e)(a-b+d-e)(a-b-d+e) \quad .&\end{aligned}$$

Now to link the given cubic in x with this factorised expression we choose $a = x$ arbitrarily, and we want to find b, d, e, which satisfy

$$b + d + e = 6 \quad , \quad b^2 + d^2 + e^2 = 14 \quad , \quad b^3 + d^3 + e^3 = 36 \quad .$$

We can spot that $b = 1$, $d = 2$, $e = 3$ is a solution, and thus

$$\begin{aligned}(x+6)^3 - 6(x+6)(x^2+14) + 8(x^3+36) \\ &= 3(x+1-2-3)(x-1+2-3)(x-1-2+3) \\ &= 3(x-4)(x-2)x \quad ,\end{aligned}$$

hence the solutions are $x = 4$, $x = 2$, $x = 0$.

Or, for a more systematic solution of this final part of the question, consider

$$\begin{aligned} 2(bd+de+be) &= (b+d+e)^2 - (b^2+d^2+e^2) \\ \implies \qquad bd+de+be &= \frac{1}{2}\left(6^2-14\right) = 11 \quad , \end{aligned}$$

and then

$$\begin{aligned} -3bde &= (b+d+e)^3 - (b^3+d^3+e^3) - 3(b+d+e)(bd+de+be) \\ \implies \qquad bde &= -\frac{1}{3}\left(6^3-36-3\cdot 6\cdot 11\right) \\ &= -\frac{1}{3}(216-36-198) \\ &= 6 \quad , \end{aligned}$$

then we can eliminate d and e to see that b satisfies

$$\begin{aligned} bd+de+be &= 11 \\ b(b+d+e-b) + \frac{bde}{b} &= 11 \\ \implies \qquad b(6-b) + \frac{6}{b} &= 11 \\ b^2(6-b) + 6 &= 11b \\ b^3 - 6b^2 + 11b - 6 &= 0 \\ (b-1)(b-2)(b-3) &= 0 \quad . \end{aligned}$$

By symmetry, d and e must solve the same cubic equation. Since $bde = 6$ we deduce that one of b, d, e, takes each value 1, 2, 3, that is: $\{b, d, e\} = \{1, 2, 3\}$, in any order.

Question 3

(i) Differentiating, for $n \geqslant 1$ we have

$$\begin{aligned} f_n'(x) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} \\ &= f_{n-1}(x) \quad . \end{aligned}$$

(ii) Suppose $a \geqslant 0$, then

$$\begin{aligned} & f_n(a) = 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!} \\ \Longrightarrow \quad & f_n(a) \geqslant 1 + 0 + 0 + \cdots + 0 \\ & f_n(a) \geqslant 1 \quad . \end{aligned}$$

Therefore if $f_n(a) = 0$, then we must have $a < 0$.

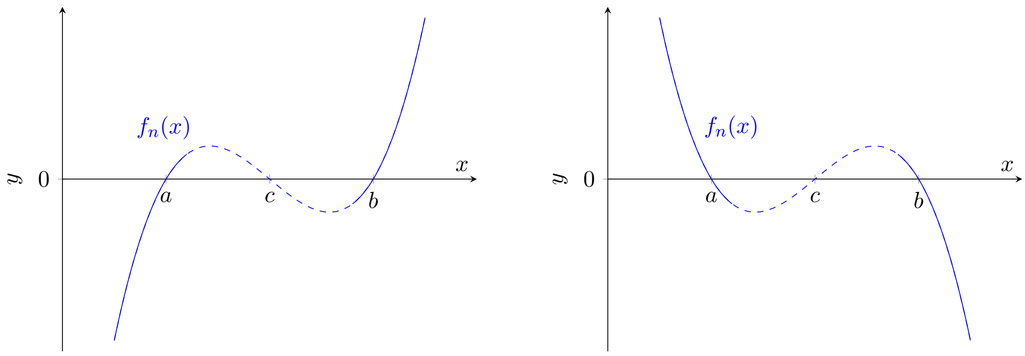
(iii) Note that for all $n \geqslant 1$ we have $f_n(x) \equiv f_{n-1}(x) + \frac{x^n}{n!}$. If $f_n(a) = f_n(b) = 0$ for some $n \geqslant 2$ then we have

$$\begin{aligned} f_n'(a)f_n'(b) &= f_{n-1}(a)f_{n-1}(b) \\ &= \left(f_n(a) - \frac{a^n}{n!}\right)\left(f_n(b) - \frac{b^n}{n!}\right) \\ &= \left(0 - \frac{a^n}{n!}\right)\left(0 - \frac{b^n}{n!}\right) \\ &= \frac{(ab)^n}{(n!)^2} \quad . \end{aligned}$$

By (ii) we must have $a < 0$ and $b < 0$, thus $ab > 0$ and so

$$f_n'(a)f_n'(b) = \frac{(ab)^n}{(n!)^2} > 0 \quad .$$

From this we deduce that either $f_n'(a)$ and $f_n'(b)$ are both positive, or they are both negative. Suppose (without loss of generality) that $a < b$. If $f_n'(a) > 0$ and $f_n'(b) > 0$ then at these two roots the graph $y = f_n(x)$ is increasing from below $y = 0$ to above $y = 0$ as x increases. Provided that $f_n(x)$ is continuous, then there must be some intermediate value c at which the graph crosses from $y > 0$ to $y < 0$. We sketch this below. The case $f_n'(a) < 0$, $f_n'(b) < 0$ is similar.



If we suppose that $(*)$ has two distinct roots a, b such that $a < b$ (without loss of generality), then we can therefore find another root c such that $a < c < b$. By repeating the above arguments then we can find another root d such that $a < d < c$. Continuing this, $(*)$ must have infinitely many distinct roots. Since f_n is a polynomial of order n, it can have at most n distinct real roots. Thus, by contradiction, $f_n(x) = 0$ has at most one real root.

If n is odd, $f_n(x) = 0$ must have at least one real root since $f_n(x) \to -\infty$ as $x \to -\infty$ and $f_n(x) \to \infty$ as $x \to \infty$. Thus $(*)$ has exactly one real root in this case.

If n is even, $f_n(x) \to \infty$ as $x \to \pm\infty$, so $f_n(x) = 0$ could have no real roots or could have exactly one repeated real root. We have

$$\begin{aligned} & f_n(x) = f_{n-1}(x) + \frac{x^n}{n!} \\ \implies \qquad & f_n(x) = f_n'(x) + \frac{x^n}{n!} \quad . \end{aligned}$$

If $f_n(x) = 0$ has a repeated root a, then $f_n(a) = f_n'(a) = 0$, which by the above equation requires $\frac{a^n}{n!} = 0$. Since $a = 0$ cannot be a root (by the result in part (ii)) we conclude that $f_n(x) = 0$ cannot have a repeated root. Thus $(*)$ has no real roots in the case that n is even.

Question 4

(i) We have

$$\begin{aligned} y\cos\theta - \sin\theta &= \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}\cos\theta - \sin\theta \\ &= \frac{(x^2 + x\sin\theta + 1)\cos\theta - (x^2 + x\cos\theta + 1)\sin\theta}{x^2 + x\cos\theta + 1} \\ &= \frac{(\cos\theta - \sin\theta)x^2 + (\cos\theta - \sin\theta)}{x^2 + x\cos\theta + 1} \\ &= (\cos\theta - \sin\theta)\frac{x^2 + 1}{x^2 + x\cos\theta + 1} \quad , \end{aligned}$$

and

$$\begin{aligned} y - 1 &= \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1} - 1 \\ &= \frac{x^2 + x\sin\theta + 1 - (x^2 + x\cos\theta + 1)}{x^2 + x\cos\theta + 1} \\ &= \frac{-(\cos\theta - \sin\theta)x}{x^2 + x\cos\theta + 1} \quad , \end{aligned}$$

hence

$$\begin{aligned} (y\cos\theta - \sin\theta)^2 &= (\cos\theta - \sin\theta)^2\frac{(x^2+1)^2}{(x^2 + x\cos\theta + 1)^2} \\ &= \frac{(x^2+1)^2}{x^2}\frac{(\cos\theta - \sin\theta)^2x^2}{(x^2 + x\cos\theta + 1)^2} \\ &= \frac{(x^2+1)^2}{x^2}(y-1)^2 \quad . \end{aligned}$$

Consider the function $f(x) = \frac{(x^2+1)^2}{x^2}$:

$$f(x) = \frac{(x^2+1)^2}{x^2} = \frac{x^4 + 2x^2 + 1}{x^2} = x^2 + 2 + x^{-2} \quad .$$

Note $f(x)$ is strictly positive for all x and $f(x) \to \infty$ as $x \to 0$ or $x \to \pm\infty$. Differentiating, we get

$$\begin{aligned} f'(x) = 2x - 2x^{-3} = 2x^{-3}(x^4 - 1) &= 2x^{-3}(x^2+1)(x^2-1) \quad , \\ \text{and} \qquad f''(x) &= 2 + 6x^{-4} \quad . \end{aligned}$$

Hence $f(x)$ has stationary points at $x = \pm 1$, giving $f(\pm 1) = \frac{(1+1)^2}{1} = 4$, and $f''(1) = 2 + 6 = 8 > 0$ hence these are minima. We deduce that $f(x) \geqslant 4$ for all real x, and thus

$$\begin{aligned} & (y\cos\theta - \sin\theta)^2 = \frac{(x^2+1)^2}{x^2}(y-1)^2 \\ \implies \qquad & (y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2 \quad . \end{aligned}$$

We can rewrite

$$
\begin{aligned}
y\cos\theta - \sin\theta &= \sqrt{y^2+1}\left(\frac{y}{\sqrt{y^2+1}}\cos\theta - \frac{1}{\sqrt{y^2+1}}\sin\theta\right) \\
&= \sqrt{y^2+1}\cos(\theta+\alpha) \quad ,
\end{aligned}
$$

where α is the angle in $0 \leqslant \alpha \leqslant \pi$ satisfying

$$
\cos\alpha = \frac{y}{\sqrt{y^2+1}} \qquad , \qquad \sin\alpha = \frac{1}{\sqrt{y^2+1}} \quad .
$$

We note that this gives $\tan\alpha = \frac{1}{y}$. Substituting this form into the inequality, we have

$$
\begin{aligned}
\left(\sqrt{y^2+1}\cos(\theta+\alpha)\right)^2 &\geqslant 4(y-1)^2 \\
(y^2+1)\cos^2(\theta+\alpha) &\geqslant 4(y-1)^2 \\
\implies \qquad y^2+1 &\geqslant 4(y-1)^2 \quad ,
\end{aligned}
$$

since $\cos^2(\theta+\alpha) \leqslant 1$.

Rearranging this inequality, it is equivalent to

$$
\begin{aligned}
y^2+1 &\geqslant 4y^2-8y+4 \\
\iff \qquad 3y^2-8y+3 &\leqslant 0 \quad .
\end{aligned}
$$

This holds with equality at

$$
y = \frac{8 \pm \sqrt{8^2-4\cdot 9}}{6} = \frac{8 \pm \sqrt{28}}{6} = \frac{4 \pm \sqrt{7}}{3} \quad ,
$$

and for the quadratic expression in y to be less than zero, y must lie between these two roots (since the coefficient of y^2 is positive). Hence we deduce that

$$
\frac{4-\sqrt{7}}{3} \leqslant y \leqslant \frac{4+\sqrt{7}}{3} \quad .
$$

(ii) In the case $y = \frac{4+\sqrt{7}}{3}$, the inequalities $(y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2$ and $y^2+1 \geqslant 4(y-1)^2$ hold with equality, thus we have

$$
y^2+1 = 4(y-1)^2 \quad .
$$

Since the given value of y is greater than 1, taking square roots of this gives

$$
\sqrt{y^2+1} = 2(y-1) \quad .
$$

Since $(y\cos\theta - \sin\theta)^2 = y^2 + 1 = 4(y-1)^2$, we have

$$y\cos\theta - \sin\theta = \pm\sqrt{y^2+1} = \pm 2(y-1) \quad .$$

Hence we find

$$\begin{aligned}
y &= \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1} \\
y(x^2 + x\cos\theta + 1) &= x^2 + x\sin\theta + 1 \\
(y-1)x^2 + (y\cos\theta - \sin\theta)x + (y-1) &= 0 \\
(y-1)x^2 \pm 2(y-1)x + (y-1) &= 0 \\
(y-1)(x\pm 1)^2 &= 0 \quad ,
\end{aligned}$$

thus $x = \pm 1$, and

$$\begin{aligned}
(y\cos\theta - \sin\theta)^2 &= y^2 + 1 \\
y^2\cos^2\theta - 2y\cos\theta\sin\theta + \sin^2\theta &= y^2 + 1 \\
y^2(\cos^2\theta - 1) - 2y\cos\theta\sin\theta + \sin^2\theta - 1 &= 0 \\
-y^2\sin^2\theta - 2y\cos\theta\sin\theta - \cos^2\theta &= 0 \\
y^2\tan^2\theta + 2y\tan\theta + 1 &= 0 \\
(y\tan\theta + 1)^2 &= 0 \\
\tan\theta &= -\frac{1}{y} = -\frac{3}{4+\sqrt{7}} = -\frac{3(4-\sqrt{7})}{16-7} \\
\tan\theta &= -\frac{4-\sqrt{7}}{3} \quad .
\end{aligned}$$

Question 5

(i) The coefficient of x^n in the binomial expansion of $(1-x)^{-N}$ is

$$\begin{aligned}(-1)^n \frac{-N(-N-1)\cdots(-N-n+1)}{n!} &= \frac{N(N+1)\cdots(N+n-1)}{n!} \\ &= \frac{(N+n-1)!}{n!(N-1)!} = \binom{N+n-1}{n} \quad .\end{aligned}$$

The full series expansion is thus

$$(1-x)^{-N} = \sum_{k=0}^{\infty} \binom{N+k-1}{k} x^k \quad .$$

Considering $(1-x)^{-1}(1-x)^{-N}$ where N is a positive integer, we have

$$\begin{aligned} & (1-x)^{-1}(1-x)^{-N} = (1-x)^{-(N+1)} \\ \Longrightarrow \quad & \left(\sum_{k=0}^{\infty} \binom{k}{k} x^k\right) \left(\sum_{k=0}^{\infty} \binom{N+k-1}{k} x^k\right) = \left(\sum_{k=0}^{\infty} \binom{N+k}{k} x^k\right) \\ & \left(\sum_{k=0}^{\infty} x^k\right) \left(\sum_{k=0}^{\infty} \binom{N+k-1}{k} x^k\right) = \left(\sum_{k=0}^{\infty} \binom{N+k}{k} x^k\right) \quad .\end{aligned}$$

Matching the terms of x^n on each side here, we have

$$\begin{aligned} & \sum_{j=0}^{n} \left(x^{n-j}\right) \left(\binom{N+j-1}{j} x^j\right) = \binom{N+n}{n} x^n \\ \Longrightarrow \quad & \sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n} \quad .\end{aligned}$$

(ii) Consider $(1+x)^{m+n}$, where m and n are positive integers, and suppose $n \leqslant m$: by using the binomial expansion of $(1+x)^N$ we have

$$\begin{aligned} & (1+x)^{m+n} = (1+x)^m (1+x)^n \\ \Longrightarrow \quad & \left(\sum_{k=0}^{m+n} \binom{m+n}{k} x^k\right) = \left(\sum_{k=0}^{m} \binom{m}{k} x^k\right) \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \quad .\end{aligned}$$

For $r \leqslant m+n$, matching the terms of x^r on each side we have

$$\begin{aligned} & \binom{m+n}{r} x^r = \sum_{j=0}^{r} \left(\binom{m}{j} x^j\right) \left(\binom{n}{r-j} x^{r-j}\right) \\ \Longrightarrow \quad & \binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j} \binom{n}{r-j} \quad .\end{aligned}$$

(iii) Consider $(1-x)^{N+m}$, where m and N are positive integers. We have

$$(1-x)^{N+m}(1-x)^{-m} = (1-x)^N$$

$$\left(\sum_{k=0}^{N+m}\binom{N+m}{k}(-1)^k x^k\right)\left(\sum_{k=0}^{\infty}\binom{m+k-1}{k}x^k\right) = \left(\sum_{k=0}^{N}\binom{N}{k}(-1)^k x^k\right) .$$

Now matching the terms of x^n on each side we have

$$\sum_{j=0}^{n}\left(\left(\binom{N+m}{n-j}(-1)^{n-j}x^{n-j}\right)\left(\binom{m+j-1}{j}x^j\right)\right) = \binom{N}{n}(-1)^n x^n$$

$$\Longrightarrow \qquad \sum_{j=0}^{n}(-1)^{n-j}\binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}(-1)^n$$

$$\sum_{j=0}^{n}(-1)^{-j}\binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n} .$$

Question 6

(i) Substituting $k = 1$ into $(*)$ we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 + y^2 = 1 \quad .$$

Substituting $y(x) = x$ into the left-hand side we get

$$\begin{aligned}(1-x^2)\left(\frac{dy}{dx}\right)^2 + y^2 &= (1-x^2) + x^2 \\ &= 1 \quad ,\end{aligned}$$

so $y(x) = x$ is a solution to $(*)$ with $k = 1$. We note that $y(x) = x$ satisfies $y(1) = 1$, thus by the uniqueness condition we deduce that $y_1(x) = x$.

(ii) Substituting $k = 2$ into $(*)$ we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 + 4y^2 = 4 \quad .$$

Substituting $y(x) = 2x^2 - 1$ into the left-hand side we get

$$\begin{aligned}(1-x^2)\left(\frac{dy}{dx}\right)^2 + 4y^2 &= (1-x^2)(4x)^2 + 4(2x^2-1)^2 \\ &= 16x^2 - 16x^4 + 4(4x^4 - 4x^2 + 1) \\ &= 4 \quad ,\end{aligned}$$

so $y(x) = 2x^2 - 1$ is a solution to $(*)$ with $k = 2$. We note that $y(x) = 2x^2 - 1$ again satisfies $y(1) = 1$, thus by the uniqueness condition we deduce that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2(y_n(x))^2 - 1$, then we have

$$\begin{aligned}(1-x^2)\left(\frac{dz}{dx}\right)^2 &= (1-x^2)\left(4y_n\frac{dy_n}{dx}\right)^2 \\ &= 16y_n^2(1-x^2)\left(\frac{dy_n}{dx}\right)^2 \\ &= 16y_n^2\left(n^2 - n^2y_n^2\right) \qquad \text{(by $(*)$, with $k = n$)} \\ &= 4n^2\left(4y_n^2 - 4y_n^4\right) \\ &= 4n^2\left(-1 + 4y_n^2 - 4y_n^4\right) + 4n^2 \\ &= -4n^2z^2 + 4n^2 \quad .\end{aligned}$$

Thus

$$
\begin{aligned}
(1-x^2)\left(\frac{dz}{dx}\right)^2 &= -4n^2z^2+4n^2 \\
\implies \qquad (1-x^2)\left(\frac{dz}{dx}\right)^2 + 4n^2z^2 &= 4n^2 \quad .
\end{aligned}
$$

We see that $z(x)$ satisfies $(*)$ with $k=2n$. We note that $z(1)=2(y_n(1))^2-1=1$, thus by the uniqueness condition we deduce that $z(x)=y_{2n}(x)$, that is:

$$
y_{2n}(x)=2(y_n(x))^2-1 \quad .
$$

(iv) Let $v(x)=y_n(y_m(x))$ and let $u(x)=y_m(x)$. We have

$$
\begin{aligned}
(1-x^2)\left(\frac{dv}{dx}\right)^2 &= (1-x^2)\left(\frac{dv}{du}\frac{du}{dx}\right)^2 = (1-x^2)\left(\frac{du}{dx}\right)^2\left(\frac{dv}{du}\right)^2 \\
&= \left(m^2-m^2u^2\right)\left(\frac{dv}{du}\right)^2 \qquad (u(x)=y_m(x) \text{ satsifies } (*) \text{ with } k=m) \\
&= m^2\left(1-u^2\right)\left(\frac{dv}{du}\right)^2 \\
&= m^2\left(n^2-n^2v^2\right) \qquad (v=y_n(u) \text{ satsifies } (*) \text{ with } k=n) \\
&= m^2n^2(1-v^2) \quad .
\end{aligned}
$$

Hence

$$
\begin{aligned}
(1-x^2)\left(\frac{dv}{dx}\right)^2 &= m^2n^2(1-v^2) \\
\implies \qquad (1-x^2)\left(\frac{dv}{dx}\right)^2 + (mn)^2v^2 &= (mn)^2 \quad .
\end{aligned}
$$

We see that $v(x)$ satisfies $(*)$ with $k=mn$ and we note that

$$
v(1)=y_n(y_m(1))=y_n(1)=1 \quad ,
$$

thus by the uniqueness condition we deduce that $v(x)=y_{mn}(x)$.

Question 7

Using the substitution $u = a - x$, we have

$$\int_0^a f(x)dx = \int_a^0 f(a-u)(-1)du = \int_0^a f(a-u)du \quad ,$$

that is: replacing $u = x$ in the final expression

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad ,$$

for any function $f(x)$ such that the left-hand integral exists.

(i) Using $(*)$ with $a = \frac{1}{2}\pi$, $f(x) = \frac{\sin x}{\cos x + \sin x}$, we get

$$\begin{aligned}
\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x} dx &= \int_0^{\frac{1}{2}\pi} \frac{\sin\left(\frac{1}{2}\pi - x\right)}{\cos\left(\frac{1}{2}\pi - x\right) + \sin\left(\frac{1}{2}\pi - x\right)} dx \\
&= \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\sin x + \cos x} dx \\
\implies \quad 2\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x} dx &= \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\sin x + \cos x} dx + \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x} dx \\
&= \int_0^{\frac{1}{2}\pi} \frac{\cos x + \sin x}{\cos x + \sin x} dx \\
&= \int_0^{\frac{1}{2}\pi} 1\, dx \\
&= \frac{1}{2}\pi \\
\implies \quad \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x} dx &= \frac{1}{4}\pi \quad .
\end{aligned}$$

(ii) Now with $a = \frac{1}{4}\pi$

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x} dx = \int_0^{\frac{1}{4}\pi} \frac{\sin\left(\frac{1}{4}\pi - x\right)}{\cos\left(\frac{1}{4}\pi - x\right) + \sin\left(\frac{1}{4}\pi - x\right)} dx \quad ,$$

and by the addition formulae

$$\begin{aligned}
\sin\left(\tfrac{1}{4}\pi - x\right) &= \sin\tfrac{1}{4}\pi\cos x - \cos\tfrac{1}{4}\pi\sin x = \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x \\
\cos\left(\tfrac{1}{4}\pi - x\right) &= \cos\tfrac{1}{4}\pi\cos x + \sin\tfrac{1}{4}\pi\sin x = \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x \quad ,
\end{aligned}$$

we thus get

$$\begin{aligned}
\int_0^{\frac14\pi} \frac{\sin x}{\cos x+\sin x}dx &= \int_0^{\frac14\pi} \frac{\sin\left(\frac14\pi - x\right)}{\cos\left(\frac14\pi-x\right)+\sin\left(\frac14\pi-x\right)}dx \\
&= \int_0^{\frac14\pi} \frac{\frac{1}{\sqrt2}\cos x - \frac{1}{\sqrt2}\sin x}{\frac{1}{\sqrt2}\cos x + \frac{1}{\sqrt2}\sin x + \frac{1}{\sqrt2}\cos x - \frac{1}{\sqrt2}\sin x}dx \\
&= \frac12\int_0^{\frac14\pi} \frac{\cos x - \sin x}{\cos x}dx \\
&= \frac12\int_0^{\frac14\pi} (1-\tan x)dx \\
&= \frac12\left[x+\ln|\cos x|\right]_0^{\frac14\pi} \\
&= \frac12\left(\frac{\pi}{4}+\ln\left(\frac{1}{\sqrt2}\right)\right) \\
&= \frac12\left(\frac{\pi}{4}-\frac12\ln 2\right) \\
&= \frac18\pi - \frac14\ln 2 \quad .
\end{aligned}$$

(iii) Using $(*)$ with $a = \frac14\pi$, $f(x) = \ln(1+\tan x)$, we get

$$\begin{aligned}
\int_0^{\frac14\pi} \ln(1+\tan x)dx &= \int_0^{\frac14\pi} \ln\left(1+\tan\left(\tfrac14\pi - x\right)\right)dx \\
&= \int_0^{\frac14\pi} \ln\left(1+\frac{\tan\frac14\pi - \tan x}{1+\tan\frac14\pi\tan x}\right)dx \\
&= \int_0^{\frac14\pi} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx \\
&= \int_0^{\frac14\pi} \ln\left(\frac{2}{1+\tan x}\right)dx \\
&= \int_0^{\frac14\pi} \ln 2\,dx - \int_0^{\frac14\pi} \ln\left(1+\tan x\right)dx \\
\implies \qquad 2\int_0^{\frac14\pi} \ln(1+\tan x)dx &= \int_0^{\frac14\pi} \ln 2\,dx \\
&= \frac14\pi\ln 2 \\
\int_0^{\frac14\pi} \ln(1+\tan x)dx &= \frac18\pi\ln 2 \quad .
\end{aligned}$$

(iv) Lastly, using $(*)$ again

$$\begin{aligned}
\int_0^{\frac{1}{4}\pi} & \frac{x}{\cos x(\cos x+\sin x)} dx \\
&= \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi - x}{\cos\left(\frac{1}{4}\pi - x\right)\left(\cos\left(\frac{1}{4}\pi - x\right)+\sin\left(\frac{1}{4}\pi - x\right)\right)} dx \\
&= \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi - x}{\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)\frac{2}{\sqrt{2}}\cos x} dx \\
&= \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi - x}{\cos x(\cos x+\sin x)} dx \quad ,
\end{aligned}$$

and adding the original integral to both sides

$$\begin{aligned}
2\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x(\cos x+\sin x)} dx &= \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi}{\cos x(\cos x+\sin x)} dx \\
&= \frac{1}{4}\pi \int_0^{\frac{1}{4}\pi} \frac{1}{\cos x(\cos x+\sin x)} dx \\
&= \frac{1}{4}\pi \int_0^{\frac{1}{4}\pi} \frac{\sec^2 x}{1+\tan x} dx \\
&= \frac{1}{4}\pi \left[\ln|1+\tan x|\right]_0^{\frac{1}{4}\pi} \\
&= \frac{1}{4}\pi \ln 2 \\
\implies \qquad \int_0^{\frac{1}{4}\pi} \frac{x}{\cos x(\cos x+\sin x)} dx &= \frac{1}{8}\pi \ln 2 \quad .
\end{aligned}$$

Question 8

For $m > \frac{1}{2}$, we can evaluate

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{m-\frac{1}{2}}^{\infty} = \frac{1}{m-\frac{1}{2}} = \frac{2}{2m-1} \quad .$$

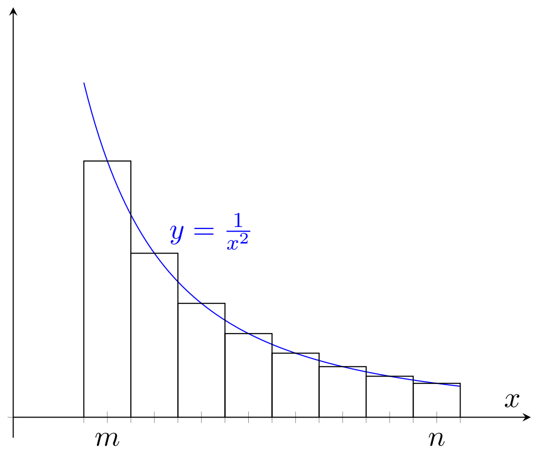
If we approximate

$$\frac{1}{r^2} \approx \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2} dx \quad , \qquad (**)$$

for positive integer r, and sum this approximation for $m \leqslant r \leqslant n$ we get

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2} dx \quad , \qquad (*)$$

for positive integers $m < n$. The approximation $(**)$ is a rectangle approximation for the integral $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2} dx$, approximating the integral by the area of one rectangle centred at $x = r$. In a sketch, summing this approximation over $m \leqslant r \leqslant n$ looks as follows, where we are approximating the area under the curve $y = \frac{1}{x^2}$ with the total area of the rectangles each of width 1.



(i) Using the integral evaluated above, in the limit $n \to \infty$, $(*)$ gives

$$\sum_{r=m}^{\infty} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx = \frac{2}{2m-1} \quad .$$

Setting $m = 1$,

$$E = \sum_{r=1}^{\infty} \frac{1}{r^2} \approx \frac{2}{2-1}$$
$$\implies \qquad E \approx 2 \quad .$$

Instead setting $m = 2$,

$$\begin{aligned} E = \frac{1}{1^2} + \sum_{r=2}^{\infty} \frac{1}{r^2} &\approx 1 + \frac{2}{2 \cdot 2 - 1} \\ \implies \qquad E &\approx 1 + \frac{2}{3} \\ E &\approx \frac{5}{3} \quad . \end{aligned}$$

Instead setting $m = 3$,

$$\begin{aligned} E = \frac{1}{1^2} + \frac{1}{2^2} + \sum_{r=3}^{\infty} \frac{1}{r^2} &\approx 1 + \frac{1}{4} + \frac{2}{2 \cdot 3 - 1} \\ \implies \qquad E &\approx 1 + \frac{1}{4} + \frac{2}{5} \\ E &\approx \frac{33}{20} \quad . \end{aligned}$$

(ii) The error in the approximation $(**)$ is

$$\begin{aligned} \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2} dx - \frac{1}{r^2} &= \left(\int_{r-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx - \int_{r+\frac{1}{2}}^{\infty} \frac{1}{x^2} dx \right) - \frac{1}{r^2} \\ &= \left(\frac{2}{2r-1} - \frac{2}{2r+1} \right) - \frac{1}{r^2} \\ &= \frac{4}{4r^2 - 1} - \frac{1}{r^2} \\ &= \frac{4r^2 - (4r^2 - 1)}{4r^4 - r^2} = \frac{1}{4r^4 - r^2} \quad . \end{aligned}$$

When r is large, we have $r^4 \gg r^2$ and so $4r^4 - r^2 \approx 4r^4$, and the error is approximately

$$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2} dx - \frac{1}{r^2} = \frac{1}{4r^4 - r^2} \approx \frac{1}{4r^4} \quad .$$

Summing this approximation for $r \geqslant 3$, we have

$$\begin{aligned} \frac{1}{4} \sum_{r=3}^{\infty} \frac{1}{r^4} &\approx \int_{3-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx - \sum_{r=3}^{\infty} \frac{1}{r^2} \\ \frac{1}{4} \sum_{r=3}^{\infty} \frac{1}{r^4} &\approx \frac{2}{2 \cdot 3 - 1} - \left(E - 1 - \frac{1}{4} \right) \\ \sum_{r=3}^{\infty} \frac{1}{r^4} &\approx \frac{8}{5} - 4E + 5 \quad . \end{aligned}$$

Adding $\frac{1}{1^4} + \frac{1}{2^4}$ to both sides we get

$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1 + \frac{1}{16} + \frac{8}{5} - 4E + 5$$
$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 6 + 0.0625 + 1.6 - 4E$$
$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 7.6625 - 4E \quad .$$

Given that $E \approx 1.645$, we have

$$4E \approx 6.58 \quad ,$$

and thus

$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 7.6625 - 6.58 = 1.0825$$
$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08 \quad .$$

Section B: Mechanics

Question 9

(i) The initial total kinetic energy of the bullet and the block is $\frac{1}{2}mu^2$. This is reduced to zero over a distance a by the constant resistance R to the bullet's motion, hence

$$\frac{1}{2}mu^2 = aR \qquad \Longrightarrow \qquad a = \frac{mu^2}{2R} \quad .$$

(ii) Note that the bullet moves a total distance of $b + c$ relative to the table before it comes to rest relative to the block. Let v be the final velocity of the bullet and the block relative to the table, then the final total kinetic energy of the bullet and the block is $\frac{1}{2}(m + M)v^2$. By conservation of momentum, we have

$$mu = (m + M)v \qquad \Longrightarrow \qquad v = \frac{mu}{m + M} \quad .$$

From part (i), the deceleration of the bullet due to the resistance from the block is $-\frac{R}{m} = -\frac{u^2}{2a}$, thus considering the motion of the bullet

$$\begin{aligned} v^2 &= u^2 + 2\left(-\frac{u^2}{2a}\right)(b + c) \\ \frac{m^2u^2}{(m + M)^2} &= u^2\left(1 - \frac{1}{a}(b + c)\right) \\ b + c &= a\left(1 - \frac{m^2}{(m + M)^2}\right) \\ b + c &= \frac{a(2mM + M^2)}{(m + M)^2} \quad . \end{aligned}$$

Correspondingly, the acceleration on the block is $\frac{R}{M} = \frac{mu^2}{2aM}$, thus considering the motion of the block

$$\begin{aligned} v^2 &= 0^2 + 2\left(\frac{mu^2}{2aM}\right)c \\ \frac{m^2u^2}{(m + M)^2} &= \frac{mu^2}{aM}c \\ c &= \frac{amM}{(m + M)^2} \quad . \end{aligned}$$

This then gives

$$b = \frac{a(2mM + M^2)}{(m + M)^2} - \frac{amM}{(m + M)^2} = \frac{a(mM + M^2)}{(m + M)^2} = \frac{aM}{m + M} \quad .$$

Question 10

Let x be the perpendicular distance of the centre of mass of the triangular frame from the side BC. By considering the weighted average of the masses of each side we have

$$(2a + 2b)x = 2a \cdot 0 + b \cdot \frac{1}{2} b \cos\theta + b \cdot \frac{1}{2} b \cos\theta$$
$$x = \frac{b^2 \cos\theta}{2(a+b)} \quad .$$

Let y be the distance between the mid point of BC and the point of contact of the wire frame with the peg, so $0 \leqslant y \leqslant a$. When the triangle rests in equilibrium, its centre of mass lies vertically below the peg. Let α be the angle between the side BC and the horizontal when the triangle rests at this equilibrium, then we have $\tan\alpha = \frac{y}{x}$. Let W be the weight of the wire frame, let R be the normal reaction force on the wire from the peg, and let F be the frictional force on the wire from the peg.

By resolving forces on the wire frame parallel to BC, we have

$$F = W \sin\alpha \quad ,$$

and by resolving forces on the wire frame perpendicular to BC, we have

$$R = W \cos\alpha \quad .$$

Combining these we have

$$\frac{F}{R} = \tan\alpha = \frac{y}{x} \qquad \Longrightarrow \qquad F = \frac{2y(a+b)}{b^2 \cos\theta} R \quad .$$

For the triangle to rest in equilibrium, it must not slip on the peg, hence $F \leqslant \mu R$:

$$\frac{2y(a+b)}{b^2 \cos\theta} R \leqslant \mu R \qquad \Longleftrightarrow \qquad \mu \geqslant \frac{2y(a+b)}{b^2 \cos\theta} \quad .$$

The triangle can rest in equilibrium with the peg in contact with *any* point on BC if this inequality holds for all y in $0 \leqslant y \leqslant a$, since $0 < 2\theta < \pi$ (as an interior angle of a triangle), we have $0 < \theta < \frac{1}{2}\pi$ and so $\cos\theta > 0$. Thus the inequality holds for all $0 \leqslant y \leqslant a$ if and only if it holds for $y = a$, the maximum value of y. That is:

$$\mu \geqslant \frac{2a(a+b)}{b^2 \cos\theta} \quad .$$

We have $\sin\theta = \frac{a}{b}$, and using this we deduce

$$\mu \geqslant \frac{2(b\sin\theta)(b\sin\theta + b)}{b^2 \cos\theta}$$
$$\mu \geqslant \frac{2\sin\theta(\sin\theta + 1)}{\cos\theta}$$
$$\mu \geqslant 2\tan\theta(\sin\theta + 1) \quad .$$

Question 11

(i) Equating the coordinates, the particles collide at time t satisfying

$$\begin{aligned} a + ut\cos\alpha &= vt\cos\beta \\ \text{and} \qquad ut\sin\alpha &= b + vt\sin\beta \quad , \end{aligned}$$

giving

$$t = \frac{a}{v\cos\beta - u\cos\alpha} = \frac{b}{u\sin\alpha - v\sin\beta} \quad .$$

Thus

$$\begin{aligned} \frac{a}{v\cos\beta - u\cos\alpha} &= \frac{b}{u\sin\alpha - v\sin\beta} \\ a(u\sin\alpha - v\sin\beta) &= b(v\cos\beta - u\cos\alpha) \\ u(a\sin\alpha + b\cos\alpha) &= v(a\sin\beta + b\cos\beta) \\ u\left(\frac{a}{\sqrt{a^2+b^2}}\sin\alpha + \frac{b}{\sqrt{a^2+b^2}}\cos\alpha\right) &= v\left(\frac{a}{\sqrt{a^2+b^2}}\sin\beta + \frac{b}{\sqrt{a^2+b^2}}\cos\beta\right) \\ u\sin(\alpha+\theta) &= v\sin(\beta+\theta) \quad , \end{aligned}$$

where $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$, so $\tan\theta = \frac{b}{a}$.

(ii) Fixing the origin to be the base of the tower, the position (in Cartesian coordinates) of the bullet at time t is $(a + ut\cos\alpha, ut\sin\alpha - \frac{1}{2}gt^2)$ and the position of the target at time t is $(vt\cos\beta, b + vt\sin\beta - \frac{1}{2}gt^2)$. The bullet hits the target at time t satisfying

$$\begin{aligned} a + ut\cos\alpha &= vt\cos\beta \\ \text{and} \qquad ut\sin\alpha - \frac{1}{2}gt^2 &= b + vt\sin\beta - \frac{1}{2}gt^2 \qquad (\implies \quad ut\sin\alpha = b + vt\sin\beta) \quad , \end{aligned}$$

which are the same equations as in (i), thus

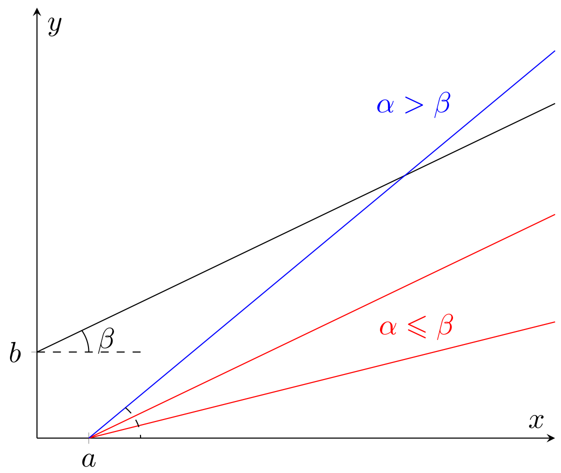
$$t = \frac{b}{u\sin\alpha - v\sin\beta} \quad .$$

By considering the vertical position of the target at this time, the collision happens before the target hits the ground, hence

$$\begin{aligned} ut\sin\alpha - \frac{1}{2}gt^2 &> 0 \\ \implies \qquad u\sin\alpha &> \frac{1}{2}gt \qquad\qquad (since t ¿ 0) \\ u\sin\alpha &> \frac{1}{2}g\cdot\frac{b}{u\sin\alpha - v\sin\beta} \\ 2u\sin\alpha(u\sin\alpha - v\sin\beta) &> bg \quad , \end{aligned}$$

where $t > 0$ and $b > 0$, so we must have $u\sin\alpha - v\sin\beta > 0$.

As shown, the conditions for the bullet to hit the target are equivalent for there to be a collision between the two particles in (i). Sketching the trajectories of the particles in (i) we can see that a collision is only possible if $\alpha > \beta$: if $\alpha = \beta$ the two trajectories are parallel, and if $\alpha < \beta$ they do not intersect for any positive t.



Section C: Probability and Statistics

Question 12

Replacing B in the given identity with $B \cup C$ we get

$$\begin{aligned} P(A \cup (B \cup C)) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ P(A \cup B \cup C) &= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C)) \quad . \end{aligned}$$

Now using the root identity to expand $P(B \cup C)$ and $P((A \cap B) \cup (A \cap C))$, we get

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) \\ &\qquad - P(A \cap C) + P((A \cap B) \cap (A \cap C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) \\ &\qquad - P(A \cap C) + P(A \cap B \cap C) \quad . \end{aligned}$$

The corresponding result for four events A, B, C, and D is

$$\begin{aligned} P(A \cup B \cup C \cup D) = P(A) &+ P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) \\ &- P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cup D) \\ &+ P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) \\ &+ P(B \cap C \cap D) - P(A \cap B \cap C \cap D) \quad . \end{aligned}$$

(i) Any given card will end up in each position with equal probability, hence

$$P(E_i) = \frac{1}{n} \quad .$$

(ii) There are $n!$ total different arrangements, each occurring with equal probability. If cards i and j ($j \neq i$) are in the ith and jth positions respectively, then there are $(n-2)!$ ways to arrange the remaining cards, hence

$$P(E_i \cap E_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)} \quad ,$$

for $i \neq j$.

(iii) By the same reasoning as for (ii)

$$P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} \quad ,$$

for $i \neq j$, $j \neq k$, $k \neq i$.

The probability that at least one card is in the same position as the number it bears is thus

$$\begin{aligned}
P(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{i=1}^{n} P(E_i) - \sum_{i \neq j} P(E_i \cap E_j) + \sum_{i \neq j, j \neq k, k \neq i} P(E_i \cap E_j \cap E_k) \\
&\qquad - \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n) \\
&= \frac{n}{n} - \binom{n}{2} \frac{(n-2)!}{n!} + \binom{n}{3} \frac{(n-3)!}{n!} - \cdots + (-1)^{n+1} \frac{1}{n!} \\
&= 1 - \frac{n!}{2!(n-2)!} \frac{(n-2)!}{n!} + \frac{n!}{3!(n-3)!} \frac{(n-3)!}{n!} - \cdots + (-1)^{n+1} \frac{1}{n!} \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!} \quad .
\end{aligned}$$

By symmetry, the probability that exactly one card is in the same position as the number it bears is n times the probability that the nth card is in position n and of the first $n-1$ cards none are in the same position as the number they bear, that is:

$$\begin{aligned}
P &= n \cdot \frac{1}{n} \cdot \left(1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{(n-1)!}\right)\right) \\
&= 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{(n-1)!}\right) \\
&= \frac{1}{2!} - \frac{1}{3!} + \cdots - (-1)^n \frac{1}{(n-1)!} \\
&= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{(n-1)!} \quad .
\end{aligned}$$

Question 13

(i) The random variable $X \sim \text{Binomial}(16, \frac{1}{2})$ has mean $np = 8$ and variance $np(1-p) = 4$, hence for large n we can approximate $X \approx 8 + \sqrt{4}Z$, where $Z \sim N(0,1)$ is a standard normal random variable. This gives the approximation

$$\begin{aligned}
& P(X=8) \approx P\left(8-\tfrac{1}{2} \leqslant 8+2Z \leqslant 8+\tfrac{1}{2}\right) = P\left(-\tfrac{1}{4} \leqslant Z \leqslant \tfrac{1}{4}\right) \\
\implies \quad & P(X=8) \approx \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \\
& \qquad\qquad \approx \left(\frac{1}{4} - \left(-\frac{1}{4}\right)\right)\frac{1}{\sqrt{2\pi}} \\
& P(X=8) \approx \frac{1}{2\sqrt{2\pi}} \quad ,
\end{aligned}$$

where we have approximated the integral by a rectangle approximation using one rectangle centred at $x = 0$.

(ii) Similarly, the random variable $Y \sim \text{Binomial}(2n, \frac{1}{2})$ has mean n and variance $\frac{1}{2}n$, hence for large n we can approximate $Y \approx n + \sqrt{\frac{1}{2}n}Z$, where $Z \sim N(0,1)$ is a standard normal random variable. This gives the approximation

$$\begin{aligned}
& P(Y=n) \approx P\left(n-\tfrac{1}{2} \leqslant n+\sqrt{\tfrac{1}{2}n}Z \leqslant n+\tfrac{1}{2}\right) = P\left(-\tfrac{1}{\sqrt{2n}} \leqslant Z \leqslant \tfrac{1}{\sqrt{2n}}\right) \\
\implies \quad & P(Y=n) \approx \int_{-\frac{1}{\sqrt{2n}}}^{\frac{1}{\sqrt{2n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \\
& \qquad\qquad \approx \left(\frac{1}{\sqrt{2n}} - \left(-\frac{1}{\sqrt{2n}}\right)\right)\frac{1}{\sqrt{2\pi}} \\
& P(Y=n) \approx \frac{1}{\sqrt{n\pi}} \quad ,
\end{aligned}$$

where we again approximated the integral by a rectangle approximation using one rectangle centred at $x = 0$. We can compute exactly that

$$P(Y=n) = \binom{2n}{n} \cdot \frac{1}{2^n} \cdot \frac{1}{2^n} = \frac{(2n)!}{(n!)^2} \cdot 2^{-2n} \quad ,$$

hence

$$\begin{aligned}
& \frac{(2n)!}{(n!)^2} \cdot 2^{-2n} \approx \frac{1}{\sqrt{n\pi}} \\
\implies \quad & (2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}} \quad .
\end{aligned}$$

(iii) The random variable $U \sim \text{Poi}(n)$ has mean n and variance n, hence for large n we can approximate $U \approx n + \sqrt{n}Z$, where $Z \sim N(0,1)$ is a standard normal random variable. This gives the approximation

$$\begin{aligned}
& P(U = n) \approx P\left(n - \tfrac{1}{2} \leqslant n + \sqrt{n}Z \leqslant n + \tfrac{1}{2}\right) = P\left(-\tfrac{1}{2\sqrt{n}} \leqslant Z \leqslant \tfrac{1}{2\sqrt{n}}\right) \\
\implies \quad & P(U = n) \approx \int_{-\frac{1}{2\sqrt{n}}}^{\frac{1}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \\
& \qquad\qquad \approx \left(\frac{1}{2\sqrt{n}} - \left(-\frac{1}{2\sqrt{n}}\right)\right) \frac{1}{\sqrt{2\pi}} \\
& P(U = n) \approx \frac{1}{\sqrt{2n\pi}} \quad ,
\end{aligned}$$

where we again approximated the integral by a rectangle approximation using one rectangle centred at $x = 0$. We can compute exactly that

$$P(U = n) = \frac{e^{-n} n^n}{n!} \quad ,$$

hence

$$\begin{aligned}
& \frac{1}{\sqrt{2n\pi}} \approx \frac{e^{-n} n^n}{n!} \\
\implies \quad & n! \approx \sqrt{2\pi n} e^{-n} n^n \quad .
\end{aligned}$$

STEP III

Section A: Pure Mathematics

Question 1

(i) Using the given substitution, we have

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} (x^2 + 2ax + b)^{-1} dx \\
&= \int_{-\infty}^{\infty} ((x+a)^2 + (b - a^2))^{-1} dx \\
&= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} ((b-a^2)\tan^2 u + (b-a^2))^{-1} \cdot \sqrt{b-a^2}\sec^2 u \, du \\
&= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2 u}{\tan^2 u + 1} du \\
&= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 1 \, du \\
&= \frac{\pi}{\sqrt{b-a^2}} \quad .
\end{aligned}$$

(ii) Integrating by parts, we find

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty} \frac{1}{(x^2+2ax+b)^n} dx \\
&= \left[\frac{x}{(x^2+2ax+b)^n} \right]_{-\infty}^{\infty} + n \int_{-\infty}^{\infty} \frac{x(2x+2a)}{(x^2+2ax+b)^{n+1}} dx \\
&= 0 + n \int_{-\infty}^{\infty} \frac{2(x^2+2ax+b) - (2ax+2b)}{(x^2+2ax+b)^{n+1}} dx \\
&= 2n \int_{-\infty}^{\infty} \frac{1}{(x^2+2ax+b)^n} dx - n \int_{-\infty}^{\infty} \frac{2ax+2b}{(x^2+2ax+b)^{n+1}} dx \\
&= 2nI_n - n \int_{-\infty}^{\infty} \frac{a(2x+2a) + 2(b-a^2)}{(x^2+2ax+b)^{n+1}} dx \\
&= 2nI_n - an \int_{-\infty}^{\infty} \frac{2x+2a}{(x^2+2ax+b)^{n+1}} dx \\
&\qquad\qquad - 2n(b-a^2) \int_{-\infty}^{\infty} \frac{1}{(x^2+2ax+b)^{n+1}} dx \\
&= 2nI_n + a \left[\frac{1}{(x^2+2ax+b)^n} \right]_{-\infty}^{\infty} - 2n(b-a^2) I_{n+1} \\
&= 2nI_n - 2n(b-a^2) I_{n+1} \quad ,
\end{aligned}$$

that is:

$$\begin{aligned} I_n &= 2nI_n - 2n(b-a^2)I_{n+1} \\ \implies \qquad 2n(b-a^2)I_{n+1} &= (2n-1)I_n \quad . \end{aligned}$$

(iii) We thus have

$$I_{n+1} = \frac{2n-1}{2n}(b-a^2)^{-1}I_n \quad ,$$

for $n \geqslant 1$. Substituting $n = 1$ into the given formula

$$\frac{\pi}{2^{2-2}(b-a^2)^{1-\frac{1}{2}}}\binom{2-2}{1-1} = \frac{\pi}{\sqrt{b-a^2}} \quad ,$$

hence the result holds for $n = 1$. Now supposing the result holds for $n = k$ for some $k \geqslant 1$, we have

$$\begin{aligned} I_{k+1} &= \frac{2k-1}{2k}(b-a^2)^{-1}I_k \\ &= \frac{2k-1}{2k}(b-a^2)^{-1}\frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}}\binom{2k-2}{k-1} \\ &= \frac{\pi}{2^{2(k+1)-2}(b-a^2)^{(k+1)-\frac{1}{2}}}\frac{2(2k-1)}{k}\binom{2k-2}{k-1} \\ &= \frac{\pi}{2^{2(k+1)-2}(b-a^2)^{(k+1)-\frac{1}{2}}}\frac{2k(2k-1)}{k^2}\frac{(2k-2)!}{(k-1)!(k-1)!} \\ &= \frac{\pi}{2^{2(k+1)-2}(b-a^2)^{(k+1)-\frac{1}{2}}}\frac{(2k)!}{k!k!} \\ &= \frac{\pi}{2^{2(k+1)-2}(b-a^2)^{(k+1)-\frac{1}{2}}}\binom{2(k+1)-2}{(k+1)-1} \quad , \end{aligned}$$

hence the result holds for $n = k + 1$, and so it holds for all $n \geqslant 1$ by induction.

Question 2

(i) We write the parabola parametrically as $(x, y) = (at^2, 2at)$. The gradient of the curve at the point with parameter t is then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} \quad ,$$

and hence the gradient of the normal to the parabola at this point is $-t$. Thus the normal to the parabola at Q is given by

$$y - 2aq = -q(x - aq^2) \quad ,$$

and similarly, the equation of the normal at R is

$$y - 2ar = -r(x - ar^2) \quad .$$

At the intersection of these two normals we have

$$\begin{aligned} 2aq - q(x - aq^2) &= 2ar - r(x - ar^2) \\ 2a(q - r) &= (q - r)x - a(q^3 - r^3) \\ 2a &= x - a(q^2 + qr + r^2) \\ x &= a(q^2 + qr + r^2 + 2) \quad , \end{aligned}$$

where we have used the fact that Q and R are distinct points, and so $q \neq r$. This then gives

$$\begin{aligned} y &= 2aq - q(x - aq^2) \\ &= 2aq - aq(qr + r^2 + 2) \\ &= -aq(qr + r^2) \\ y &= -aqr(q + r) \quad . \end{aligned}$$

We are given that the intersection of these two normals is the point P, thus

$$\begin{aligned} && ap^2 &= a(q^2 + qr + r^2 + 2) \\ \Longrightarrow && p^2 &= q^2 + qr + r^2 + 2 \quad , \end{aligned}$$

and

$$\begin{aligned} 2ap &= -aqr(q + r) \\ 2p &= -qr(q + r) \quad . \end{aligned}$$

From the first of these equations we can write

$$r(q + r) = p^2 - q^2 - 2 \quad ,$$

then using the second equation we find

$$\begin{aligned}
-qr(q+r) &= -q(p^2-q^2-2) \\
\implies \qquad 2p &= -qp^2+q^3+2q \\
-q^3+qp^2+2(p-q) &= 0 \\
(p-q)(q^2+qp+2) &= 0 \\
q^2+qp+2 &= 0 \quad ,
\end{aligned}$$

where we have used the fact that $p \neq q$ since P and Q are distinct points.

(ii) By symmetry, we know also that $r^2+rp+2=0$. Subtracting one of these identities from the other, we have

$$\begin{aligned}
q^2-r^2+(q-r)p &= 0 \\
(q-r)(q+r+p) &= 0 \\
\implies \qquad q+r+p &= 0 \quad ,
\end{aligned}$$

since $q \neq r$. Hence $q+r=-p$. We can also eliminate p as follows:

$$\begin{aligned}
r(q^2+qp+2)-q(r^2+rp+2) &= 0 \\
rq^2-qr^2+2(r-q) &= 0 \\
(qr-2)(q-r) &= 0 \\
\implies \qquad qr &= 2 \quad ,
\end{aligned}$$

since $q \neq r$. We can now find that the line QR is given by

$$\begin{aligned}
\frac{y-2aq}{x-aq^2} &= \frac{2ar-2aq}{ar^2-aq^2} \\
&= \frac{2}{r+q} \\
(r+q)(y-2aq) &= 2x-2aq^2 \\
(r+q)y-2aqr &= 2x \\
-py &= 2x+4a \quad .
\end{aligned}$$

Substituting $y=0$, we get $x=-2a$. That is: the line QR passes through the point $(-2a,0)$ independent of the value of P.

(iii) The line OP is given by

$$\begin{aligned}
\frac{y}{x} &= \frac{2ap}{ap^2} \\
\implies \qquad py &= 2x \quad ,
\end{aligned}$$

hence at the intersection with QR

$$-2x = 2x + 4a \qquad \Longrightarrow \qquad x = -a \quad ,$$

that is: T lies on the line $x = -a$, independent of the value of P.

The coordinates of T are $(-a, \frac{-2a}{p})$, hence the distance z from the x-axis to T is

$$z = \left|\frac{2a}{p}\right| \quad .$$

Consider $z^2 = \frac{4a^2}{p^2}$. By the above, we have

$$q^2 + qp + 2 = 0 \qquad \Longleftrightarrow \qquad p = -\frac{q^2+2}{q} \quad ,$$

hence

$$\begin{aligned} z^2 &= 4a^2 \frac{q^2}{(q^2+2)^2} \\ \Longrightarrow \qquad \frac{d}{dq}(z^2) &= 4a^2 \frac{(q^2+2)^2 \cdot 2q - q^2 \cdot 4q(q^2+2)}{(q^2+2)^4} \quad , \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dq}(z^2) = 0 \qquad \Longleftrightarrow \qquad 2q(q^2+2)^2 &= 4q^3(q^2+2) \\ q(q^2+2) &= 2q^3 \\ q(q^2 - 2q^2 + 2) &= 0 \\ q(2 - q^2) &= 0 \quad . \end{aligned}$$

$q = 0$ gives a minimum $z^2 = 0$, while $q^2 = 2$ gives a maximum

$$q^2 = 2 \qquad \Longrightarrow \qquad z^2 = 4a^2 \frac{2}{(2+2)^2} = \frac{a^2}{2} \quad .$$

We cannot achieve $q^2 = 2$, since this would then require $q^2 = qr$ and $q = \pm\sqrt{2} \neq 0$ giving $q = r$ (a contradiction). Hence $z = \frac{a}{\sqrt{2}}$ is a strict upper bound, and the distance from the x-axis to T is always strictly less than $\frac{a}{\sqrt{2}}$.

Question 3

(i) Differentiating the given equation with respect to x, we have

$$\frac{x^3-2}{(x+1)^2}e^x = \frac{P(x)}{Q(x)}e^x + \frac{Q(x)P'(x)-P(x)Q'(x)}{Q(x)^2}e^x$$

$$\implies \qquad (x^3-2)Q(x)^2 = (x+1)^2\left(P(x)Q(x)+Q(x)P'(x)-P(x)Q'(x)\right) \quad . \quad (*)$$

Now substituting in $x=-1$, we get

$$-3Q(-1)^2 = 0 \qquad \implies \qquad Q(-1)=0 \quad ,$$

hence, by the factor theorem, $Q(x)$ has a factor of $x+1$.

Let p be the degree of polynomial $P(x)$ and q be the degree of $Q(x)$, then $P(x)Q(x)$ has degree $p+q$, and $Q(x)P'(x)$ and $P(x)Q'(x)$ each have degree $p+q-1$. The left-hand side of $(*)$ thus has degree $3+2q$ (from the term $x^3Q(x)^2$) and the right-hand side has degree $2+p+q$ (from the term $(x+1)^2P(x)Q(x)$), and we have

$$3+2q = 2+p+q \qquad \implies \qquad p=q+1 \quad ,$$

that is: the degree of $P(x)$ is exactly one more than the degree of $Q(x)$. In the case $Q(x)=x+1$, we know $P(x)$ has degree 2. Substituting $Q(x)=x+1$, $P(x)=ax^2+bx+c$ into $(*)$ we get

$$\begin{aligned}(x^3-2)(x+1)^2 &= (x+1)^2\big((ax^2+bx+c)(x+1)+(x+1)(2ax+b)\\ &\qquad\qquad\qquad -(ax^2+bx+c)\big)\\ x^3-2 &= (ax^2+bx+c)x+(x+1)(2ax+b)\\ &= ax^3+(2a+b)x^2+(2a+b+c)x+b \quad ,\end{aligned}$$

hence $a=1$, $b=-2$, and $c=0$:

$$P(x) = x^2-2x \quad .$$

(ii) Suppose that the given formula holds for some polynomials $P(x)$, $Q(x)$ with no common factors. In this case, differentiating as before gives

$$Q(x)^2 = (x+1)\left(P(x)Q(x)+Q(x)P'(x)-P(x)Q'(x)\right) \quad , \qquad (**)$$

and substituting in $x=-1$ again gives $Q(-1)=0$, so we deduce that $Q(x)$ has a factor of $x+1$. Let $Q(x)=(x+1)^nR(x)$ for some integer $n\geqslant 1$, where $R(x)$ does not have a factor of $x+1$. Now $(**)$ gives

$$\begin{aligned}(x+1)^{2n}R(x)^2 &= (x+1)\big((x+1)^nP(x)R(x)+(x+1)^nR(x)P'(x)\\ &\qquad -n(x+1)^{n-1}P(x)R(x)-(x+1)^nP(x)R'(x)\big)\\ (x+1)^nR(x)^2 &= (x+1)P(x)R(x)+(x+1)R(x)P'(x)\\ &\qquad -nP(x)R(x)-(x+1)P(x)R'(x) \quad ,\end{aligned}$$

now substituting in $x = -1$ we get

$$0 = 0 + 0 - nP(-1)R(-1) - 0 \qquad \Longrightarrow \qquad nP(-1)R(-1) = 0 \ .$$

Since $R(x)$ does not have a factor of $x + 1$, and $n \geqslant 1$, this requires $P(-1) = 0$, and so $P(x)$ must have a factor of $x + 1$, which contradicts the supposition that P and Q have no common factors. Thus there are no polynomials $P(x)$ and $Q(x)$ such that the given formula holds.

Question 4

(i) For $|x| \neq 1$, we have

$$\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}} = \frac{x^{r+1}-x^r}{(1+x^r)(1+x^{r+1})} = (x-1)\frac{x^r}{(1+x^r)(1+x^{r+1})} \quad ,$$

hence

$$\begin{aligned}\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})} &= \frac{1}{x-1}\sum_{r=1}^{N}\left(\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}}\right) \\ &= \frac{1}{x-1}\left(\frac{1}{1+x^1} - \frac{1}{1+x^{N+1}}\right) \\ &= \frac{1}{1-x}\left(\frac{1}{1+x^{N+1}} - \frac{1}{1+x}\right) \quad .\end{aligned}$$

For $|x| < 1$ we have $\lim_{N\to\infty} |x|^{N+1} = 0$ and so for in the limit $N \to \infty$ we find

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} &= \frac{1}{1-x}\left(\frac{1}{1+0} - \frac{1}{1+x}\right) \\ &= \frac{1}{1-x}\left(1 - \frac{1}{1+x}\right) \\ &= \frac{1}{1-x}\cdot\frac{x}{1+x} \\ &= \frac{x}{1-x^2} \quad ,\end{aligned}$$

for $|x| < 1$.

(ii) We have

$$\begin{aligned}\operatorname{sech}(ry)\operatorname{sech}((r+1)y) &= \frac{2}{e^{ry}+e^{-ry}}\cdot\frac{2}{e^{(r+1)y}+e^{-(r+1)y}} \\ &= \frac{4e^{-ry}e^{-(r+1)y}}{(1+e^{-2ry})(1+e^{-2(r+1)y})} \\ &= 4e^{-y}\frac{(e^{-2y})^r}{(1+(e^{-2y})^r)(1+(e^{-2y})^{r+1})} \quad ,\end{aligned}$$

hence if we let $x = e^{-2y}$, we have $|x| < 1 \iff y > 0$ and the result of (i) gives

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{(e^{-2y})^r}{(1+(e^{-2y})^r)(1+(e^{-2y})^{r+1})} &= \frac{e^{-2y}}{1-(e^{-2y})^2} \\ \implies \qquad \sum_{r=1}^{\infty} \frac{1}{4}e^{y}\operatorname{sech}(ry)\operatorname{sech}((r+1)y) &= \frac{1}{e^{2y}-e^{-2y}} \\ \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) &= 2e^{-y}\frac{2}{e^{2y}-e^{-2y}} = 2e^{-y}\operatorname{cosech}(2y) \quad .\end{aligned}$$

We then have

$$\begin{aligned}
\sum_{r=-\infty}^{\infty} & \operatorname{sech}(ry)\operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \operatorname{sech}(0)\operatorname{sech}(y) \\
&\qquad + \operatorname{sech}(-y)\operatorname{sech}(0) + \sum_{r=1}^{\infty}\operatorname{sech}(-ry)\operatorname{sech}(-(r+1)y) \\
&= 2\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}((r+1)y) + 2\operatorname{sech}(y) \\
&= 4e^{-y}\operatorname{cosech}(2y) + 2\operatorname{sech}(y) \\
&= 2\left(\frac{2e^{-y}}{\sinh(2y)} + \frac{1}{\cosh y}\right) \\
&= 2\left(\frac{e^{-y}}{\sinh y} + 1\right)\frac{1}{\cosh y} \\
&= 2\left(\frac{2e^{-y}}{e^{y}-e^{-y}} + \frac{e^{y}-e^{-y}}{e^{y}-e^{-y}}\right)\frac{2}{e^{y}+e^{-y}} \\
&= 2\cdot\frac{e^{y}+e^{-y}}{e^{y}-e^{-y}}\cdot\frac{2}{e^{y}+e^{-y}} \\
&= 2\cdot\frac{2}{e^{y}-e^{-y}} \\
&= 2\operatorname{cosech} y \quad .
\end{aligned}$$

Question 5

(i) For any positive integer m, the binomial expansion of $(1+x)^{2m+1}$ is

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k \quad ,$$

and if we substitute $x = 1$ we get

$$\begin{aligned} 2^{2m+1} &= \sum_{k=0}^{2m+1} \binom{2m+1}{k} \\ &= \sum_{k=0}^{m} \binom{2m+1}{k} + \sum_{k=m+1}^{2m+1} \binom{2m+1}{k} \\ &= \sum_{k=0}^{m} \binom{2m+1}{k} + \sum_{k=m+1}^{2m+1} \binom{2m+1}{2m+1-k} \\ &= \sum_{k=0}^{m} \binom{2m+1}{k} + \sum_{k=0}^{m} \binom{2m+1}{k} \\ &= 2\sum_{k=0}^{m} \binom{2m+1}{k} \quad . \end{aligned}$$

All of the terms in the sum are strictly positive, and if so if we consider only the final term ($k = m$) we get the inequality

$$\begin{aligned} 2^{2m+1} &> 2\binom{2m+1}{m} \\ \implies \qquad \binom{2m+1}{m} &< 2^{2m} \quad . \end{aligned}$$

(ii) Let p be a prime such that $m+1 < p \leqslant 2m+1$, then we must have that p divides $(2m+1)!$ and p does not divide $m!$ or $(m+1)!$, thus p divides

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!} \quad ,$$

since this is necessarily an integer, and p divides the numerator of the fraction and does not divide the denominator. Since $P_{m+1,2m+1}$ is the product of all such primes we deduce that $P_{m+1,2m+1}$ divides $\binom{2m+1}{m}$.

Since $P_{m+1,2m+1}$ divides $\binom{2m+1}{m}$ we deduce

$$\begin{aligned} P_{m+1,2m+1} &\leqslant \binom{2m+1}{m} < 2^{2m} \\ \implies \qquad P_{m+1,2m+1} &< 2^{2m} \quad . \end{aligned}$$

(iii) If $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$ then we have

$$\begin{aligned} P_{1,2m+1} &= P_{1,m+1} P_{m+1,2m+1} \\ &< 4^{m+1} \cdot 2^{2m} \\ \implies \quad P_{1,2m+1} &< 4^{m+1} \cdot 4^m \\ P_{1,2m+1} &< 4^{2m+1} \quad , \end{aligned}$$

where we have used the given result for $k = m+1$ and the inequality from (ii).

(iv) We also have

$$P_{1,2m+2} = P_{1,m+1} P_{m+1,2m+2} = P_{1,m+1} P_{m+1,2m+1} \quad ,$$

since $2m+2$ cannot be prime (it is even). Thus if $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$ then we have

$$\begin{aligned} P_{1,2m+2} &< 4^{2m+1} \\ \implies \quad P_{1,2m+2} &< 4^{2m+2} \quad . \end{aligned}$$

By induction then, if the result holds for $k = 2$ (that is: if $P_{1,2} < 4^2$), then we have $P_{1,n} < 4^n$ for all $n \geqslant 2$. We can easily compute that

$$P_{1,2} = 2 \qquad \text{and} \qquad 4^2 = 16$$
$$\implies \qquad P_{1,2} < 4^2 \quad .$$

and so we do indeed have $P_{1,n} < 4^n$ for all $n \geqslant 2$.

Question 6

Expanding, we have

$$R\cosh(x+\gamma) = R\cosh\gamma\cosh x + R\sinh\gamma\sinh x \quad ,$$

hence we can write $A\sinh x + B\cosh x = R\cosh(x+\gamma)$ provided we can satisfy

$$\begin{cases} R\cosh\gamma = B \\ R\sinh\gamma = A \end{cases} \quad\Longrightarrow\quad \begin{cases} R^2 = B^2 - A^2 \\ \tanh\gamma = \frac{A}{B} \end{cases} \quad .$$

Since $B > A > 0$ we have $B^2 - A^2 > 0$ and $1 > \frac{A}{B} > 0$, hence we may take

$$R = \sqrt{B^2 - A^2} \qquad \text{and} \qquad \gamma = \operatorname{arctanh}\frac{A}{B} \quad .$$

Likewise, if $B < -A$ then we have $B^2 - A^2 > 0$ and $-1 < \frac{A}{B} < 0$, hence we may write $A\sinh x + B\cosh x = R\cosh(x+\gamma)$ where

$$R = -\sqrt{B^2 - A^2} \qquad \text{and} \qquad \gamma = \operatorname{arctanh}\frac{A}{B} \quad ,$$

and we must take the negative root for R, since $R\cosh\gamma = B$ and $\cosh\gamma$ is strictly positive, while B is negative.

If $B = A$, then we have

$$A\sinh x + B\cosh x = A(\sinh x + \cosh x) = Ae^{x} \quad ,$$

and if $B = -A$, then we have

$$A\sinh x + B\cosh x = A(\sinh x - \cosh x) = -Ae^{-x} \quad ,$$

neither of which may be written in terms of a single hyperbolic function.

If $-A < B < A$, then expanding $R\sinh(x+\gamma)$ we have

$$R\sinh(x+\gamma) = R\sinh\gamma\cosh x + R\cosh\gamma\sinh x \quad ,$$

hence we may write $A\sinh x + B\cosh x = R\sinh(x+\gamma)$ provided we can satisfy

$$\begin{cases} R\cosh\gamma = A \\ R\sinh\gamma = B \end{cases} \quad\Longrightarrow\quad \begin{cases} R^2 = A^2 - B^2 \\ \tanh\gamma = \frac{B}{A} \end{cases} \quad .$$

Since $|B| < A$ we have $A^2 - B^2 > 0$ and $-1 < \frac{B}{A} < 1$, and hence we may take

$$R = \sqrt{A^2 - B^2} \qquad \text{and} \qquad \gamma = \operatorname{arctanh}\frac{B}{A} \quad .$$

In this last case, we do not need to change the sign of R dependent on the sign of B, since $R\sinh\gamma = B$, where sinh is an odd function and the sign of γ will change with the sign of B.

(i) Given the curves intersect, the x-coordinate at intersection satisfies

$$\begin{aligned} a\tanh x + b &= \text{sech} x \\ \iff \quad a\sinh x + b\cosh x &= 1 \\ \iff \quad \sqrt{b^2-a^2}\cosh\left(x + \text{arctanh}\,\frac{a}{b}\right) &= 1 \\ \cosh\left(x + \text{arctanh}\,\frac{a}{b}\right) &= \frac{1}{\sqrt{b^2-a^2}} \quad , \end{aligned}$$

since $b > a > 0$. Taking arcosh of both sides we get two roots, since cosh is an even function. Thus

$$\begin{aligned} x + \text{arctanh}\,\frac{a}{b} &= \pm\text{arcosh}\left(\frac{1}{\sqrt{b^2-a^2}}\right) \\ x &= \pm\text{arcosh}\left(\frac{1}{\sqrt{b^2-a^2}}\right) - \text{arctanh}\,\frac{a}{b} \quad . \end{aligned}$$

(ii) In the case $a > b > 0$ we have

$$\begin{aligned} a\sinh x + b\cosh x &= 1 \\ \iff \quad \sqrt{a^2-b^2}\sinh\left(x + \text{arctanh}\,\frac{b}{a}\right) &= 1 \\ x + \text{arctanh}\,\frac{b}{a} &= \text{arcsinh}\left(\frac{1}{\sqrt{a^2-b^2}}\right) \\ x &= \text{arcsinh}\left(\frac{1}{\sqrt{a^2-b^2}}\right) - \text{arctanh}\,\frac{b}{a} \quad . \end{aligned}$$

We note that in this case we have only one point of intersection.

(iii) The curve $y = \text{sech}\, x$ lies entirely in $0 < y \leqslant 1$ and is increasing for $x < 0$ and decreasing for $x > 0$. Since $a > 0$, the curve $y = a\tanh x + b$ is strictly increasing for all x. If $b \leqslant 0$ then this second curve lies in $y \leqslant 0$ for $x \leqslant 0$, and there can be at most one intersection in $x > 0$ (one curve is strictly increasing over this domain, while the other is strictly decreasing). Thus a necessary condition for two distinct points of intersection is $b > 0$.

By the result of (ii), a further necessary condition is that $b \geqslant a$ (since for $0 < b < a$ we have exactly one intersection). If $b = a$ we find

$$\begin{aligned} a\sinh x + a\cosh x &= 1 \\ \iff ae^x &= 1 \\ x &= -\ln a \quad , \end{aligned}$$

so again we have only one point of intersection, and a necessary condition for two distinct points of intersection is that $b > a$.

By the result of (i) if $b > a$ and the curves *do* intersect then there will be two distinct points of intersection. The sufficient conditions for the curves to intersect at two distinct points are thus $b > a$ and $\frac{1}{\sqrt{b^2-a^2}} > 1$ (for the arcosh to be well-defined and to give two distinct roots). That is: the necessary and sufficient conditions for two distinct points of intersection are

$$\begin{aligned} & b > a \quad \text{and} \quad \sqrt{b^2 - a^2} < 1 \\ \iff & \quad a < b < \sqrt{a^2+1} \ . \end{aligned}$$

(iv) The necessary and sufficient conditions for the curves to 'touch' (that is: intersect tangentially to each other), is that two distinct points of intersection become coincident, that is:

$$\begin{aligned} & b > a \quad \text{and} \quad \sqrt{b^2 - a^2} = 1 \\ \iff & b > a \quad \text{and} \quad b = \sqrt{a^2+1} \\ \iff & \quad b = \sqrt{a^2+1} \ . \end{aligned}$$

In this case, the point at which they touch is given by

$$\begin{aligned} & x = -\text{arctanh}\,\frac{a}{b} = -\text{arctanh}\,\frac{a}{\sqrt{a^2+1}} \ , \\ \implies \quad & y = a\tanh\left(-\text{arctanh}\,\frac{a}{\sqrt{a^2+1}}\right) + b \\ & = -a \cdot \frac{a}{\sqrt{a^2+1}} + \sqrt{a^2+1} \\ & = -\frac{a^2+1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{a^2+1}} + \sqrt{a^2+1} \\ & = \frac{1}{\sqrt{a^2+1}} \ . \end{aligned}$$

Question 7

Consider $\omega^k = e^{\frac{2\pi ik}{n}}$ for $k = 0, 1, \ldots, n-1$. These are all distinct (for example, consider $\arg(\omega^k) = \frac{2k\pi}{n} = k \arg \omega$), and we have

$$(\omega^k)^n = (e^{\frac{2\pi ik}{n}})^n = e^{2\pi ik} = 1 \quad ,$$

for each $k = 0, 1, \ldots, n-1$. Thus these are the n (distinct) complex roots of the equation $z^n = 1$, and so we may factorise

$$\begin{aligned} z^n - 1 &= (z - \omega^0)(z - \omega^1) \cdots (z - \omega^{n-1}) \\ \implies z^n - 1 &= (z-1)(z-\omega) \cdots (z - \omega^{n-1}) \quad . \end{aligned}$$

Without loss of generality, we may rotate the polygon such that the points X_0, X_1, ..., X_{n-1} lie at the points represented by the complex numbers 1, ω, ..., ω^{n-1}.

(i) Taking moduli of the given factorisation, we have that for any given complex number z

$$\begin{aligned} |z^n - 1| &= \big|(z-1)(z-\omega) \cdots (z-\omega^{n-1})\big| \\ &= |z-1| \cdot |z-\omega| \cdot \cdots \cdot |z - \omega^{n-1}| \quad . \end{aligned}$$

Let p be the complex number representing the point P. Given that P is equidistant from X_0 and X_1, either $\arg p = \frac{\pi}{n}$ or $\arg p = \pi + \frac{\pi}{n}$; so $p = \pm re^{\frac{i\pi}{n}}$ for some $r \geqslant 0$. Then we have

$$\begin{aligned} |PX_0| \cdot |PX_1| \cdot \cdots \cdot |PX_{n-1}| &= |p-1| \cdot |p-\omega| \cdot \cdots \cdot |p - \omega^{n-1}| \\ &= |p^n - 1| \\ &= |(\pm re^{\frac{i\pi}{n}})^n - 1| \\ &= |e^{i\pi}(\pm r)^n - 1| \\ &= |-(\pm r)^n - 1| \quad . \end{aligned}$$

If n is even, we have $(\pm r)^n = r^n$, giving

$$\begin{aligned} |PX_0| \cdot |PX_1| \cdot \cdots \cdot |PX_{n-1}| &= |-r^n - 1| \\ &= r^n + 1 \\ &= |OP|^n + 1 \quad . \end{aligned}$$

When n is odd, we have $(\pm r)^n = \pm(r^n)$, giving

$$\begin{aligned} |PX_0| \cdot |PX_1| \cdot \cdots \cdot |PX_{n-1}| &= |-(\pm r^n) - 1| \\ &= |\mp r^n - 1| \\ &= \begin{cases} r^n + 1 & \text{if } p = re^{\frac{i\pi}{n}} \\ r^n - 1 & \text{if } p = -re^{\frac{i\pi}{n}} \text{ and } r \geqslant 1 \\ 1 - r^n & \text{if } p = -re^{\frac{i\pi}{n}} \text{ and } r < 1 \end{cases} \quad . \end{aligned}$$

hence when n is odd

$$|PX_0| \cdot |PX_1| \cdot \dots \cdot |PX_{n-1}| = \begin{cases} |OP|^n + 1 & \text{if } p = re^{\frac{i\pi}{n}} \\ |OP|^n - 1 & \text{if } p = -re^{\frac{i\pi}{n}} \text{ and } r \geqslant 1 \\ 1 - |OP|^n & \text{if } p = -re^{\frac{i\pi}{n}} \text{ and } r < 1 \end{cases} \quad .$$

(ii) We have

$$\begin{aligned} |X_0X_1| \cdot |X_0X_2| \cdot \dots \cdot |X_0X_{n-1}| &= |1-\omega| \cdot |1-\omega^2| \cdot \dots \cdot |1-\omega^{n-1}| \\ &= |(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})| \quad . \end{aligned}$$

Consider

$$\begin{aligned} (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) &= z^n - 1 \\ \implies \qquad (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) &= \frac{z^n-1}{z-1} \\ (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) &= z^{n-1} + z^{n-2} + \dots + z + 1 \quad . \end{aligned}$$

Evaluating this at $z = 1$, we find

$$\begin{aligned} (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) &= n \\ \implies \qquad |X_0X_1| \cdot |X_0X_2| \cdot \dots \cdot |X_0X_{n-1}| &= n \quad . \end{aligned}$$

Question 8

(i) Starting from $f(y) + (1-y)f(-y) = y^2$, we substitute $y = -x$ to find

$$f(-x) + (1+x)f(x) = x^2 \quad .$$

This gives that

$$\begin{aligned}(1-x)f(-x) + (1-x^2)f(x) &= (1-x)x^2 \\ (1-x)f(-x) &= (1-x)x^2 + (x^2-1)f(x) \quad ,\end{aligned}$$

and substituting this into $f(x) + (1-x)f(-x) = x^2$ we get

$$\begin{aligned}f(x) + ((1-x)x^2 + (x^2-1)f(x)) &= x^2 \\ x^2 f(x) &= x^3 \\ f(x) &= x \quad .\end{aligned}$$

We verify that setting $f(x) = x$ gives

$$f(x) + (1-x)f(-x) = x - (1-x)x = x^2 \quad ,$$

as required.

(ii) Substituting directly, for $x \neq 1$ we have

$$\begin{aligned}K(K(x)) = \frac{K(x)+1}{K(x)-1} &= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \\ &= \frac{x+1+x-1}{x+1-x+1} \\ &= \frac{2x}{2} \\ &= x \quad .\end{aligned}$$

We have

$$g(x) + xg(K(x)) = x \quad , \qquad \text{for } x \neq 1 \quad .$$

Starting from $g(y) + yg(K(y)) = y$ and substituting $y = K(x)$ we find

$$\begin{aligned}g(K(x)) + K(x)g(K(K(x))) &= K(x) \\ g(K(x)) + K(x)g(x) &= K(x) \\ g(K(x)) &= K(x) - K(x)g(x) \quad .\end{aligned}$$

Substituting this back into the original equation we get

$$\begin{aligned}g(x) + x\left(K(x) - K(x)g(x)\right) &= x \\ (1 - xK(x))g(x) &= x - xK(x) \\ (x - 1 - x(x+1))g(x) &= x(x-1) - x(x+1) \\ (-1-x^2)g(x) &= -2x \\ g(x) &= \frac{2x}{x^2+1} \quad .\end{aligned}$$

(iii) We have that for $x \neq 0$, $x \neq 1$,

$$h(x) + h\left(\frac{1}{1-x}\right) = 1 - x - \frac{1}{1-x} \quad . \tag{1}$$

Substituting in $y = \frac{1}{1-x}$ we get

$$\begin{aligned} h\left(\frac{1}{1-x}\right) + h\left(\frac{1}{1-\frac{1}{1-x}}\right) &= 1 - \frac{1}{1-x} - \frac{1}{1-\frac{1}{1-x}} \\ h\left(\frac{1}{1-x}\right) + h\left(\frac{1-x}{-x}\right) &= 1 - \frac{1}{1-x} - \frac{1-x}{-x} \\ h\left(\frac{1}{1-x}\right) + h\left(1-\frac{1}{x}\right) &= -\frac{1}{1-x} + \frac{1}{x} \quad . \end{aligned} \tag{2}$$

Instead substituting in $y = 1 - \frac{1}{x}$ we get

$$\begin{aligned} h\left(1-\frac{1}{x}\right) + h\left(\frac{1}{1-1+\frac{1}{x}}\right) &= 1 - 1 + \frac{1}{x} - \frac{1}{1-1+\frac{1}{x}} \\ h\left(1-\frac{1}{x}\right) + h(x) &= \frac{1}{x} - x \quad . \end{aligned} \tag{3}$$

Now summing equations (1) − (2) + (3), we find

$$\begin{aligned} 2h(x) &= 1 - 2x \\ h(x) &= \frac{1}{2} - x \quad . \end{aligned}$$

Section B: Mechanics

Question 9

By symmetry, the particle rests at equilibrium at the centroid of the triangle. At this point, by the cosine rule, the length of each spring $|PX| = |QX| = |RX|$ satisfies

$$\begin{aligned} 4a^2 &= 2|PX|^2\left(1-\cos\left(\tfrac{2\pi}{3}\right)\right) \\ &= 2\cdot\frac{3}{2}|PX|^2 \\ \implies \qquad |PX|^2 &= \frac{4a^2}{3} \\ |PX| &= \frac{2a}{\sqrt{3}} \quad , \end{aligned}$$

hence the extension in each spring is $\frac{2a}{\sqrt{3}} - l$.

By Pythagoras' theorem, the perpendicular distance d from the the side QR to the equilibrium point is

$$d = \sqrt{(2a)^2 - a^2} - \frac{2a}{\sqrt{3}} = \left(\sqrt{3} - \frac{2}{\sqrt{3}}\right)a = \frac{a}{\sqrt{3}} \quad .$$

Then after the particle is pulled the small distance x towards P, the length of the spring RX is

$$\begin{aligned} |RX| = \sqrt{a^2 + (d+x)^2} &= \sqrt{a^2 + \left(\frac{a}{\sqrt{3}} + x\right)^2} \\ &= \sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} \quad . \end{aligned}$$

Accordingly, the extension in the spring RX is $\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l$, giving the tension

$$T = \lambda \cdot \frac{\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l}{l} = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right) \quad .$$

The tension in the spring PX is $\frac{\lambda}{l}\left(\frac{2a}{\sqrt{3}} - x - l\right)$. By symmetry the tension in the spring QX is equal to the tension in the spring RX. Let φ be the acute angle between the spring RX and the side QR, then resolving perpendicular to QR, the particle's equation of motion is

$$m\ddot{x} = \frac{\lambda}{l}\left(\frac{2a}{\sqrt{3}} - x - l - 2\sin\varphi\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right)\right) \quad ,$$

where

$$\sin\varphi = \frac{\frac{a}{\sqrt{3}} + x}{\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}} \quad ,$$

hence

$$\begin{aligned} m\ddot{x} &= \frac{\lambda}{l}\left(\frac{2a}{\sqrt{3}} - x - l - 2\left(\frac{a}{\sqrt{3}} + x\right) + 2l\sin\varphi\right) \\ &= \frac{\lambda}{l}\left(-3x - l + 2l\sin\varphi\right) \quad . \end{aligned}$$

Since x is small, we may use a Taylor series to approximate

$$\begin{aligned} \sin\varphi &= \frac{\frac{a}{\sqrt{3}} + x}{\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}} = \frac{\frac{a}{\sqrt{3}} + x}{\frac{2a}{\sqrt{3}}\sqrt{1 + \frac{\sqrt{3}x}{2a} + \frac{3}{4a^2}x^2}} \\ &= \frac{\sqrt{3}}{2a}\left(\frac{a}{\sqrt{3}} + x\right)\left(1 + \frac{\sqrt{3}x}{2a} + \frac{3x^2}{4a^2}\right)^{-\frac{1}{2}} \\ &\approx \frac{\sqrt{3}}{2a}\left(\frac{a}{\sqrt{3}} + x\right)\left(1 - \frac{\sqrt{3}x}{4a}\right) \\ &\approx \frac{\sqrt{3}}{2a}\left(\frac{a}{\sqrt{3}} + \left(1 - \frac{1}{4}\right)x\right) \\ \sin\varphi &\approx \frac{1}{2} + \frac{3\sqrt{3}}{8a}x \quad , \end{aligned}$$

where we have neglected all terms in x^2 and higher powers of x, since these are even smaller. This approximation then gives

$$\begin{aligned} m\ddot{x} &\approx \frac{\lambda}{l}\left(-3x - l + 2l\left(\frac{1}{2} + \frac{3\sqrt{3}}{8a}x\right)\right) = \frac{\lambda}{l}\left(-3x - l + l + \frac{3\sqrt{3}l}{4a}x\right) \\ m\ddot{x} &\approx \frac{\lambda}{l}\left(-3 + \frac{3\sqrt{3}l}{4a}\right)x = \frac{-3\lambda}{4la}\left(4a - \sqrt{3}l\right)x \\ \Longrightarrow \qquad \ddot{x} &\approx \frac{-3(4a - \sqrt{3}l)\lambda}{4mla}x \quad . \end{aligned}$$

This is (approximately) the equation for simple harmonic motion with frequency

$$\sqrt{\frac{3(4a - \sqrt{3}l)\lambda}{4mla}} \quad ,$$

hence the period is

$$\frac{2\pi}{\sqrt{\frac{3(4a-\sqrt{3}l)\lambda}{4mla}}} = 2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}} \quad .$$

Question 10

Let T be the tension in the string and R be the normal reaction force from the plane on the particle. Due to the initial impulse, the particle gets a radial acceleration $\frac{u^2}{r}$ towards point A, where $r = a\cos\beta$ is the radius of the circular locus of a point on the plane at a constant distance a from point A. Resolving in the plane up the line of greatest slope we get

$$m\frac{u^2}{a\cos\beta} = T\cos\beta - mg\sin\alpha \quad ,$$

and resolving perpendicular to the plane we have

$$R + T\sin\beta = mg\cos\alpha \quad .$$

Eliminating T we can find an expression for R:

$$\begin{aligned} R &= mg\cos\alpha - T\sin\beta = mg\cos\alpha - (T\cos\beta)\tan\beta \\ &= mg\cos\alpha - \left(\frac{mu^2}{a\cos\beta} + mg\sin\alpha\right)\tan\beta \\ &= mg\cos\alpha - \frac{mu^2\sin\beta}{a\cos^2\beta} + mg\sin\alpha\tan\beta \quad . \end{aligned}$$

The particle does not immediately leave the plane if R remains positive, which occurs if

$$\begin{aligned} mg\cos\alpha &> \frac{mu^2\sin\beta}{a\cos^2\beta} + mg\sin\alpha\tan\beta \\ ag\cos\alpha\cos^2\beta &> u^2\sin\beta + ag\sin\alpha\sin\alpha\cos\beta \\ ag(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\cos\beta &> u^2\sin\beta \\ ag\cos(\alpha+\beta) &> u^2\tan\beta \quad . \end{aligned}$$

A necessary condition for the particle to perform a complete circle whilst in contact with the plane is for the string to remain under tension as the particle passes the highest point in the circle. The height difference between this highest point and the particle's initial position is

$$2r\sin\alpha = 2a\sin\alpha\cos\beta \quad .$$

By conservation of energy (kinetic and gravitational potential), the velocity v of the particle at the highest point on the circle is given by

$$\begin{aligned} \frac{1}{2}mu^2 &= \frac{1}{2}mv^2 + mg\cdot 2a\sin\alpha\cos\beta \\ v^2 &= u^2 - 4ag\sin\alpha\cos\beta \quad . \end{aligned}$$

Let T' be the tension on the string as the particle passes through the highest point on the circle. Resolving in the plane up the line of greatest slope we get

$$\begin{aligned} \frac{mv^2}{a\cos\beta} &= T'\cos\beta + mg\sin\alpha \\ T'\cos\beta &= \frac{m}{a\cos\beta}\left(v^2 - ag\sin\alpha\cos\beta\right) \quad . \end{aligned}$$

Thus, a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $T' > 0$, which occurs if

$$\begin{aligned} v^2 &> ag\sin\alpha\cos\beta \\ \iff \qquad u^2 - ag\sin\alpha\cos\beta &> 4ag\sin\alpha\cos\beta \\ u^2 &> 5ag\sin\alpha\cos\beta \\ u^2\tan\beta &> 5ag\sin\alpha\sin\beta \quad . \end{aligned}$$

Combining this with the previous inequality that $ag\cos(\alpha+\beta) > u^2\tan\beta$, we require

$$\begin{aligned} ag\cos(\alpha+\beta) &> 5ag\sin\alpha\sin\beta \\ \cos\alpha\cos\beta - \sin\alpha\sin\beta &> 5\sin\alpha\sin\beta \\ \cos\alpha\cos\beta &> 6\sin\alpha\sin\beta \\ 1 &> 6\tan\alpha\tan\beta \quad . \end{aligned}$$

Question 11

Let $a(t)$ be the car's acceleration resulting from the driving force of the engine and the resistance to motion, we have

$$ma = \frac{P}{v} - R \quad ,$$

where $v(t)$ is the car's velocity. We note that

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \quad ,$$

where $x(t)$ is the distance travelled by the car. Thus

$$mv\frac{dv}{dx} = \frac{P}{v} - R \quad .$$

(i) Given that $R = kv$ for some constant k such that $a = 0$ at $v = 4U$, we have

$$0 = \frac{P}{4U} - k \cdot 4U \qquad \Longrightarrow \qquad k = \frac{P}{16U^2} \quad ,$$

and thus

$$\begin{aligned} mv\frac{dv}{dx} &= P\left(\frac{1}{v} - \frac{1}{16U^2}v\right) \\ \frac{v^2}{1 - \frac{1}{16U^2}v^2}\frac{dv}{dx} &= \frac{P}{m} \quad . \end{aligned}$$

Integrating this from $v = U$ to $v = 2U$, we are given that the distance travelled is X_1, and thus

$$\begin{aligned} \int_{v=U}^{v=2U} \frac{P}{m} dx &= \int_{v=U}^{v=2U} \frac{v^2}{1 - \frac{1}{16U^2}v^2} dv \\ \frac{P}{m}X_1 &= \int_U^{2U} \frac{16U^2v^2}{16U^2 - v^2} dv \\ \Longrightarrow \qquad \lambda X_1 &= \frac{1}{U}\int_U^{2U} \frac{v^2}{16U^2 - v^2} dv \\ &= \frac{1}{U}\int_U^{2U} \left(\frac{16U^2}{16U^2 - v^2} - 1\right) dv \\ &= \frac{1}{U}\left(\int_U^{2U} \left(\frac{2U}{4U - v} + \frac{2U}{4U + v}\right) dv - U\right) \\ &= 2\left[-\log(4U - v) + \log(4U + v)\right]_U^{2U} - 1 \\ &= 2\left[\log\left(\frac{4U + v}{4U - v}\right)\right]_U^{2U} - 1 \\ &= 2\left(\log 3 - \log \tfrac{5}{3}\right) - 1 \\ &= 2\log \tfrac{9}{5} - 1 \quad . \end{aligned}$$

(ii) Instead given that $R = kv^2$ for some constant k such that $a = 0$ at $v = 4U$, we have

$$0 = \frac{P}{4U} - k \cdot 16U^2 \qquad \Longrightarrow \qquad k = \frac{P}{64U^3} \quad ,$$

and thus

$$\begin{aligned} mv\frac{dv}{dx} &= P\left(\frac{1}{v} - \frac{1}{64U^3}v^2\right) \\ \frac{v^2}{1 - \frac{1}{64U^3}v^3}\frac{dv}{dx} &= \frac{P}{m} \quad . \end{aligned}$$

Integrating this from $v = U$ to $v = 2U$, we are given that the distance travelled is X_2, and thus

$$\begin{aligned} \int_{v=U}^{v=2U} \frac{P}{m}dx &= \int_{v=U}^{v=2U} \frac{v^2}{1 - \frac{1}{64U^3}v^3}dv \\ \frac{P}{m}X_2 &= \int_{U}^{2U} \frac{v^2}{1 - \frac{1}{64U^3}v^3}dv \\ \Longrightarrow \qquad \lambda X_2 &= 4\int_{U}^{2U} \frac{\frac{1}{64U^3}v^2}{1 - \frac{1}{64U^3}v^3}dv \\ &= -\frac{4}{3}\int_{U}^{2U} \frac{\frac{-3}{64U^3}v^2}{1 - \frac{1}{64U^3}v^3}dv \\ &= -\frac{4}{3}\left[\log\left(1 - \frac{v^3}{64U^3}\right)\right]_{U}^{2U} \\ &= -\frac{4}{3}\left(\log\left(1 - \frac{1}{8}\right) - \log\left(1 - \frac{1}{64}\right)\right) \\ &= \frac{4}{3}\left(\log\tfrac{63}{64} - \log\tfrac{7}{8}\right) \\ &= \frac{4}{3}\log\tfrac{9}{8} \quad . \end{aligned}$$

(iii) Consider

$$\begin{aligned} \lambda X_1 - \lambda X_2 &= 2\log\tfrac{9}{5} - 1 - \frac{4}{3}\log\tfrac{9}{8} \\ \lambda(X_1 - X_2) &= 2\log 9 - 2\log 5 - 1 - \frac{4}{3}\log 9 + \frac{4}{3}\log 8 \\ &= 4\log 3 - 2\log 5 - 1 - \frac{8}{3}\log 3 + \frac{4}{3}\log 8 \\ &= \frac{4}{3}\log 3 - 2\log 5 - 1 + \frac{4}{3}\log 8 \\ &= \frac{4}{3}\log 24 - 2\log 5 - 1 \quad . \end{aligned}$$

By the given bounds on $\log 24$ and $\log 5$ we have

$$\begin{aligned}
\lambda(X_1 - X_2) &> \frac{4}{3} \cdot 3.17 - 2 \cdot 1.61 - 1 = \frac{1}{3}\left(4 \cdot 3.17 - 6 \cdot 1.61 - 3\right) \\
3\lambda(X_1 - X_2) &> 12.68 - 9.66 - 3 \\
3\lambda(X_1 - X_2) &> 0.02 \\
\implies \qquad X_1 - X_2 &> 0 \quad .
\end{aligned}$$

Thus we deduce $X_1 > X_2$.

Section C: Probability and Statistics

Question 12

(i) Let X be the number of times the coin turns up heads, so $X \sim \text{Bin}(100n, 0.2)$ with expectation $\mu = 100n \cdot 0.2 = 20n$ and variance $\sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n$ (giving $\sigma = 4\sqrt{n}$). We have

$$\begin{aligned} \alpha &= \mathbb{P}\left(16n \leqslant X \leqslant 24n\right) \\ &= \mathbb{P}\left(|X - 20n| \leqslant 4n\right) \\ &= 1 - \mathbb{P}\left(|X - 20n| > \sqrt{n} \cdot 4\sqrt{n}\right) \quad . \end{aligned}$$

By Chebyshev's inequality with $k = \sqrt{n}$, we thus have

$$\begin{aligned} & 1 - \alpha = \mathbb{P}\left(|X - \mu| > \sqrt{n}\sigma\right) \\ \Longrightarrow \qquad & 1 - \alpha \leqslant \frac{1}{n} \\ & \alpha \geqslant 1 - \frac{1}{n} \quad . \end{aligned}$$

(ii) Let X be a Poisson random variable with parameter n: $X \sim \text{Poi}(n)$. This has expectation $\mu = n$ and variance $\sigma^2 = n$ (giving $\sigma = \sqrt{n}$). Let β be the probability that $0 \leqslant X \leqslant 2n$, we have

$$\begin{aligned} \beta &= \mathbb{P}\left(0 \leqslant X \leqslant 2n\right) = \mathbb{P}\left(|X - n| \leqslant n\right) \\ &= 1 - \mathbb{P}\left(|X - n| > \sqrt{n} \cdot \sqrt{n}\right) \quad . \end{aligned}$$

By Chebyshev's inequality with $k = \sqrt{n}$, we thus have

$$\begin{aligned} & 1 - \beta = \mathbb{P}\left(|X - \mu| > \sqrt{n}\sigma\right) \\ \Longrightarrow \qquad & 1 - \beta \leqslant \frac{1}{n} \\ & \beta \geqslant 1 - \frac{1}{n} \quad . \end{aligned}$$

From the probability mass function for the Poisson distribution we get

$$\beta = \sum_{r=0}^{2n} \mathbb{P}(X = r) = \sum_{r=0}^{2n} e^{-n} \frac{n^r}{r!} \quad ,$$

thus

$$\begin{aligned} \sum_{r=0}^{2n} e^{-n} \frac{n^r}{r!} &\geqslant 1 - \frac{1}{n} \\ \sum_{r=0}^{2n} \frac{n^r}{r!} &\geqslant \left(1 - \frac{1}{n}\right) e^n \\ 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} &\geqslant \left(1 - \frac{1}{n}\right) e^n \quad . \end{aligned}$$

Question 13

Let $K(Z)$ denote the kurtosis of a random variable Z, that is

$$K(Z) = \frac{\mathbb{E}((Z - \mathbb{E}(Z))^4)}{(\text{Var}(Z))^2} - 3 \quad .$$

If $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$, then we have

$$\mathbb{E}(X - a) = \mathbb{E}(X) - a = \mu - a \quad ,$$

and

$$\begin{aligned}\text{Var}(X - a) &= \mathbb{E}((X - a)^2) - (\mu - a)^2 \\ &= \mathbb{E}(X^2) - 2a\mathbb{E}(X) + a^2 - \mu^2 + 2a\mu - a^2 \\ &= \mathbb{E}(X^2) - \mu^2 - 2a\mu + 2a\mu \\ &= \sigma^2 \quad .\end{aligned}$$

Hence

$$\begin{aligned}K(X - a) &= \frac{\mathbb{E}(X - a - (\mu - a))}{(\text{Var}(X - a))^2} - 3 \\ &= \frac{\mathbb{E}(X - \mu)}{(\sigma^2)^2} - 3 \\ &= \frac{\mathbb{E}(X - \mu)}{\sigma^4} - 3 \\ &= K(X) \quad .\end{aligned}$$

(i) Let $X \sim N(0, \sigma^2)$ be a Normally distributed random variable with mean 0; this has probability density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}$, hence expressing $\mathbb{E}(X^4)$ as an integral gives

$$\begin{aligned}K(X) &= \frac{\mathbb{E}(X^4)}{\sigma^4} - 3 \\ &= \frac{1}{\sigma^4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^4 e^{-\frac{1}{2\sigma^2}x^2} dx - 3 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} u^4 e^{-\frac{1}{2}u^2} \sigma du - 3 \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^4 e^{-\frac{1}{2}u^2} du - 3 \quad ,\end{aligned}$$

where we have used the substitution $u = \frac{x}{\sigma}$.

Now integrating by parts

$$\begin{aligned}
K(X) &= \frac{1}{\sqrt{2\pi}}\left(\left[-u^3 e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2} du\right) - 3 \\
&= \frac{1}{\sqrt{2\pi}}\left(0 + 3\left[-u e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du\right) - 3 \\
&= \frac{1}{\sqrt{2\pi}}\left(0 + 0 + 3\sqrt{2\pi}\right) - 3 \\
&= 0 \quad .
\end{aligned}$$

(ii) Consider the expansion of $T^4 = (\sum_{r=1}^{n} Y_r)^4$. We will get terms in the following forms:

$$Y_r^4 \quad , \quad Y_r^3 Y_s \quad , \quad Y_r^2 Y_s^2 \quad , \quad Y_r^2 Y_s Y_t \quad , \quad Y_r Y_s Y_t Y_u \quad ,$$

where r, s, t, u are all distinct indices. When we take expectations of these, by independence we get

$$\begin{aligned}
\mathbb{E}(Y_r^3 Y_s) = \mathbb{E}(Y_r^3)(Y_s) = 0 \quad , \quad \mathbb{E}(Y_r^2 Y_s^2) &= \mathbb{E}(Y_r^2)\mathbb{E}(Y_s^2) \quad , \\
\mathbb{E}(Y_r^2 Y_s Y_t) = \mathbb{E}(Y_r^2)\mathbb{E}(Y_s)\mathbb{E}(Y_t) &= 0 \quad , \\
\text{and} \qquad \mathbb{E}(Y_r Y_s Y_t Y_u) = \mathbb{E}(Y_r)\mathbb{E}(Y_s)\mathbb{E}(Y_t)\mathbb{E}(Y_u) &= 0 \quad ,
\end{aligned}$$

since $\mathbb{E}(Y_i) = 0$. Thus we have only two different terms in $\mathbb{E}(T^4)$: those of $\mathbb{E}(Y_r^4)$, and $\mathbb{E}(Y_r^2)\mathbb{E}(Y_s^2)$ where $s \neq r$. Our full expansion is

$$\begin{aligned}
T^4 &= \sum\left(\frac{4!}{4!}Y_r^4 + \frac{4!}{3!1!}Y_r^3 Y_s + \frac{4!}{2!2!}Y_r^2 Y_s^2 + \frac{4!}{2!1!1!}Y_r^2 Y_s Y_t + \frac{4!}{1!1!1!1!}Y_r Y_s Y_t Y_u\right) \\
&= \sum\left(Y_r^4 + 4Y_r^3 Y_s + 6Y_r^2 Y_s^2 + 12 Y_r^2 Y_s Y_t + 24 Y_r Y_s Y_t Y_u\right) \quad ,
\end{aligned}$$

where the summation is over each of the indices r, s, t, u (all distinct) without repetition. We thus have

$$\mathbb{E}(T^4) = \sum\left(\mathbb{E}(Y_r^4) + 6\mathbb{E}(Y_r^2)\mathbb{E}(Y_s^2)\right) = \sum_{r=1}^{n}\mathbb{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\mathbb{E}(Y_r^2)\mathbb{E}(Y_s^2) \quad .$$

(iii) Let $\mathbb{E}(X_i) = \mu$, and let $Y_i = X_i - \mu$ for each $i = 1, 2, \ldots, n$. Let $S = \sum_{i=1}^{n} X_i$ and $T = \sum_{i=1}^{n} Y_i = S - n\mu$. Since each Y_i has mean 0, we have $\mathbb{E}(T) = 0$ and by the root result we have $K(S) = K(T + n\mu) = K(T)$, that is:

$$K(S) = K(T) = \frac{\mathbb{E}(T^4)}{(\mathrm{Var}(T))^2} - 3 \quad .$$

By the starting results, we know that $\mathrm{Var}(Y_i) = \mathrm{Var}(X_i - \mu) = \mathrm{Var}(X_i) = \sigma^2$, that is: $\mathbb{E}(Y_i^2) = \sigma^2$, since $\mathbb{E}(Y_i) = 0$, which then gives

$$\mathbb{E}(Y_i)^4 = (K(Y_i) + 3)\sigma^4 = (K(X_i - \mu) + 3)\sigma^4 = (K(X_i) + 3)\sigma^4 = (\kappa + 3)\sigma^4 \quad .$$

By the fact that X_1, ..., X_n are independent we know that Y_1, ..., Y_n are independent, so

$$\mathrm{Var}(T) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathrm{Var}\left(Y_i\right) = n\sigma^2 \quad .$$

Thus all together we have

$$\begin{aligned} K(S) &= \frac{\mathbb{E}(T^4)}{(\mathrm{Var}(T))^2} - 3 = \frac{\mathbb{E}(T^4)}{(n\sigma^2)^2} - 3 \\ &= \frac{\sum_{r=1}^{n} \mathbb{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathbb{E}(Y_r^2)\mathbb{E}(Y_s^2)}{n^2\sigma^4} - 3 \\ &= \frac{n(\kappa+3)\sigma^4 + 3n(n-1)\sigma^2 \cdot \sigma^2}{n^2\sigma^4} - 3 \\ &= \frac{\kappa + 3 + 3(n-1)}{n} - 3 \\ &= \frac{\kappa + 3n}{n} - 3 \\ &= \frac{\kappa}{n} \quad . \end{aligned}$$