Admissions Testing Service

Sixth Term Examination Papers **9470**

MATHEMATICS 2 Morning

THURSDAY 26 JUNE 2014 Time: 3 hours

2987726248

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 9 printed pages and 3 blank pages.

Section A: Pure Mathematics

1 In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leqslant \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$\big(1 + 2\cos(\alpha+\beta)\big)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0\,. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leqslant \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45^\circ$ and (b) $\alpha = 30^\circ$, $\beta = 90^\circ$, and illustrate each case with a sketch.

2 This question concerns the inequality

$$\int_0^\pi \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x \leqslant \int_0^\pi \big(\mathrm{f}'(x)\big)^2\,\mathrm{d}x\,. \qquad (*)$$

(i) Show that $(*)$ is satisfied in the case $\mathrm{f}(x) = \sin nx$, where n is a positive integer.

Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $\mathrm{f}(0) \neq 0$ or $\mathrm{f}(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $\mathrm{f}(0) = \mathrm{f}(\pi) = 0$.

By setting $\mathrm{f}(x) = ax^2 + bx + c$, where a, b and c are suitably chosen, show that $\pi^2 \leqslant 10$.

By setting $\mathrm{f}(x) = p\sin\frac{1}{2}x + q\cos\frac{1}{2}x + r$, where p, q and r are suitably chosen, obtain another inequality for π.

Which of these inequalities leads to a better estimate for π^2?

3 **(i)** Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geqslant 0$, is $c(m^2+1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C.

Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2\big(1 + (y')^2\big)\,. \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1+m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

4 **(i)** By using the substitution $u = 1/x$, show that for $b > 0$

$$\int_{1/b}^{b} \frac{x \ln x}{(a^2+x^2)(a^2x^2+1)}\,\mathrm{d}x = 0\,.$$

(ii) By using the substitution $u = 1/x$, show that for $b > 0$,

$$\int_{1/b}^{b} \frac{\arctan x}{x}\,\mathrm{d}x = \frac{\pi \ln b}{2}\,.$$

(iii) By using the result $\displaystyle\int_0^\infty \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = k/x$, for suitable k, show that

$$\int_0^\infty \frac{1}{(a^2+x^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3} \qquad (a > 0).$$

5 Given that $y = xu$, where u is a function of x, write down an expression for $\dfrac{dy}{dx}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{dy}{dx} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$.

Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{dy}{dx} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

6 By simplifying $\sin(r + \frac{1}{2})x - \sin(r - \frac{1}{2})x$ or otherwise show that, for $\sin \frac{1}{2}x \neq 0$,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}.$$

The functions S_n, for $n = 1, 2, \ldots$, are defined by

$$S_n(x) = \sum_{r=1}^{n} \frac{1}{r} \sin rx \qquad (0 \leqslant x \leqslant \pi).$$

(i) Find the stationary points of $S_2(x)$ for $0 \leqslant x \leqslant \pi$, and sketch this function.

(ii) Show that if $S_n(x)$ has a stationary point at $x = x_0$, where $0 < x_0 < \pi$, then

$$\sin nx_0 = (1 - \cos nx_0) \tan \tfrac{1}{2}x_0$$

and hence that $S_n(x_0) \geqslant S_{n-1}(x_0)$. Deduce that if $S_{n-1}(x) > 0$ for all x in the interval $0 < x < \pi$, then $S_n(x) > 0$ for all x in this interval.

(iii) Prove that $S_n(x) \geqslant 0$ for $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$.

7 **(i)** The function f is defined by $f(x) = |x - a| + |x - b|$, where $a < b$. Sketch the graph of $f(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $g(x)$, where $g(x) = |2x - a - b|$.

What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $(b, f(b))$ and $(a, f(a))$?

(ii) Show graphically that the equation

$$|x - a| + |x - b| = |x - c|\,,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x - a| + |x - b| = |x - c| + |x - d|\,,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

8 For positive integers n, a and b, the integer c_r $(0 \leqslant r \leqslant n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r, n, a and b.

For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \geqslant c_r$ for $0 \leqslant r \leqslant n$).

Show that

$$\frac{b(n+1)}{a+b} - 1 \leqslant m \leqslant \frac{b(n+1)}{a+b}\,.$$

Deduce that m is either a unique integer or one of two consecutive integers.

Let $G(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

(i) Evaluate $G(9, 1, 3)$ and $G(9, 2, 3)$.

(ii) For any positive integer k, find $G(2k, a, a)$ and $G(2k - 1, a, a)$ in terms of k.

(iii) For fixed n and b, determine a value of a for which $G(n, a, b)$ is greatest.

(iv) For fixed n, find the greatest possible value of $G(n, 1, b)$. For which values of b is this greatest value achieved?

Section B: Mechanics

9 A uniform rectangular lamina $ABCD$ rests in equilibrium in a vertical plane with the corner A in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side AB at a distance d from A. The other end of the string is attached to a point on the wall above A where it makes an acute angle θ with the downwards vertical. The side AB makes an acute angle ϕ with the upwards vertical at A. The sides BC and AB have lengths $2a$ and $2b$ respectively. The coefficient of friction between the lamina and the wall is μ.

(i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d\sin(\theta+\phi) = (\cos\theta + \mu\sin\theta)(a\cos\phi + b\sin\phi)\,. \qquad (*)$$

(ii) How should $(*)$ be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

(iii) Find a condition on d, in terms of a, b, $\tan\theta$ and $\tan\phi$, which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if $b(2\tan\theta + \tan\phi) < a$.

10 A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O. Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g.

Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u.

Hence find the furthest distance from O that can be achieved by such a projectile.

11 A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O, of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion.

Let x be the distance between O and the ring. Taking the y-axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x, L and α.

(i) By considering the vertical component of the equation of motion of P, show that

$$km\ddot{x}\cos\alpha = T\cos\alpha - kmg\,,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R.

(ii) Show that $\dfrac{\sin\alpha}{(1-\sin\alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k.

(iii) Show that $\ddot{x} = -g\tan\alpha\,.$

Section C: Probability and Statistics

12 The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $\mathrm{f}(t)$ and cumulative distribution function $\mathrm{F}(t)$. The *hazard function*, $\mathrm{h}(t)$, is defined, for $\mathrm{F}(t) < 1$, by

$$\mathrm{h}(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)}\,.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $\mathrm{h}(t)\,\delta t$ for small values of δt.

(ii) Find the hazard function in the case $\mathrm{F}(t) = t/a$ for $0 < t < a$. Sketch $\mathrm{f}(t)$ and $\mathrm{h}(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $\mathrm{h}(t)$ is constant for $t > b$ and zero otherwise if and only if $\mathrm{f}(t) = k\mathrm{e}^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$\mathrm{h}(t) = \left(\frac{\lambda}{\theta^{\lambda}}\right) t^{\lambda - 1}\,,$$

where λ and θ are positive constants. Find the probability density function for T.

13 A random number generator prints out a sequence of integers I_1, I_2, I_3, ... Each integer is independently equally likely to be any one of 1, 2, ..., n, where n is fixed. The random variable X takes the value r, where I_r is the first integer which is a repeat of some earlier integer.

Write down an expression for $\mathrm{P}(X = 4)$.

(i) Find an expression for $\mathrm{P}(X = r)$, where $2 \leqslant r \leqslant n+1$. Hence show that, for any positive integer n,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots \;=\; 1\,.$$

(ii) Write down an expression for $\mathrm{E}(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $\mathrm{P}(X \geqslant k)$.

(iv) Show that, for any discrete random variable Y taking the values 1, 2, ..., N,

$$\mathrm{E}(Y) = \sum_{k=1}^{N} \mathrm{P}(Y \geqslant k)\,.$$

Hence show that, for any positive integer n,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right) + \cdots \;=\; 0.$$

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